ON THE PARITY OF THE FOURIER COEFFICIENTS OF j -FUNCTION

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ABSTRACT. Klein's modular j -function is defined to be

$$j(z) = E_4^3/\Delta(z) = \frac{1}{q} + 744 + \sum_{n=1}^{\infty} c(n)q^n$$

where $z \in \mathbb{C}$ with $\Im(z) > 0$, $q = \exp(2i\pi z)$, $E_4(z)$ denotes the normalized Eisenstein series of weight 4 and $\Delta(z)$ is the Ramanujan's Delta function. In this short note, we show that for each integer $a \geq 1$, the interval $(a, 4a(a+1))$ (respectively, the interval $(16a-1, (4a+1)^2)$) contains an integer n with $n \equiv 7 \pmod{8}$ such that $c(n)$ is odd (respectively, $c(n)$ is even).

1. INTRODUCTION

Let z be a complex number with $\Im(z) > 0$ and $q = e^{2\pi iz}$. The modular invariant j -function defined as

$$(1.1) \quad j(z) = \frac{E_4^3(z)}{\Delta(z)}$$

where

$$(1.2) \quad \Delta(z) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}$$

is the Ramanujan's Delta function and

$$(1.3) \quad E_4(z) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^n$$

is the normalized Eisenstein series of weight 4. The Fourier expansion for $j(z)$ is

$$(1.4) \quad j(z) = \frac{1}{q} + 744 + \sum_{n=1}^{\infty} c(n)q^n$$

where $c(n)$ are integers.

It is well known that $c(n)$ is even whenever $n \not\equiv 7 \pmod{8}$. Indeed, a result of J. P. Serre implies that for almost all integers $n \not\equiv 7 \pmod{8}$, one has $c(n) \equiv 0 \pmod{2^t}$ for any integer $t \geq 1$. Later, Ono and Taguchi [4] proved that for any

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$t \geq 1$, there is a positive integer ℓ such that for every set of distinct odd primes $p_1, p_2, \dots, p_\ell$, one has

$$c(p_1 p_2 \cdots p_\ell m) \equiv 0 \pmod{2^t}$$

whenever $m \geq 1$ is coprime to $p_1 p_2 \cdots p_\ell$ and $p_1 p_2 \cdots p_\ell m \not\equiv 7 \pmod{8}$. Also, recently, Ono and Ramsey [3], extending the work of Alfes [1], proved that for any $D \equiv 7 \pmod{8}$, there are infinitely many n such that $c(Dn^2)$ is even.

Regarding the odd parity of $c(n)$, using the mod p analogue of Atkin-Lehner's theorem and using the generalized Borchers product, Ono and Ramsey [3] proved that for any $D \equiv 7 \pmod{8}$, if there exists one odd integer n such that $c(Dn^2)$ is odd, then there are infinitely many odd integers m such that $c(Dm^2)$ is odd. In particular, it follows that there are infinitely many odd integers $m \equiv 7 \pmod{8}$ such that $c(m)$ is odd. This can be seen by taking $D = 7$ and noting that $c(7)$ is odd.

In this short note, we shall prove the following theorems, in the spirit of O. Kolberg's [2] proof of parity of partition function. Moreover, the following theorems predict a range in which a suitable $n \equiv 7 \pmod{8}$ can be chosen such that $c(n)$ is odd (respectively, even). In particular, our theorem gives an elementary proof of the infinitude of n 's with $n \equiv 7 \pmod{8}$ for which $c(n)$ is odd (respectively, even).

Theorem 1.1. *For every $a \geq 1$, there exists an integer $n \in (a, 4a(a+1) - 1]$ with $n \equiv 7 \pmod{8}$, such that $c(n)$ is an odd integer. In particular, there are infinitely many odd integers $n \equiv 7 \pmod{8}$ for which $c(n)$ is an odd integer.*

Note that when $a = 1$ in Theorem 1.1, we get that the interval $[1, 7]$ contains an integer $n \equiv 7 \pmod{8}$ such that $c(n)$ is odd. This must be $n = 7$. Indeed, $c(7) = 44656994071935$, which is an odd integer.

Corollary 1.2. *For all $x \geq 8$, we have*

$$\begin{aligned} \{1 \leq n \leq x : c(n) \text{ is odd}\} &= \{n \leq x : n \equiv 7 \pmod{8} \text{ and } c(n) \text{ is odd}\} \\ &\geq c_0 \log \log x, \end{aligned}$$

for some positive constant c_0 .

Theorem 1.3. *For all $a \geq 1$, there exists an integer $n \in [16a - 1, (4a + 1)^2 - 1]$ with $n \equiv 7 \pmod{8}$ such that $c(n)$ is even. In particular, there exist infinitely many integers $n \equiv 7 \pmod{8}$ for which $c(n)$ is even.*

When $a = 1$ in Theorem 1.3, we get that 15 and 23 lie in the interval $[15, 24]$. Note that $c(15)$ and $c(23)$ are even integers.

Corollary 1.4. *For all $x \geq 15$, we have*

$$\#\{1 \leq n \leq x : n \equiv 7 \pmod{8} \text{ and } c(n) \text{ is even}\} \geq c_1 \log \log x,$$

for some positive constant c_1 .

Corollary 1.5. *For a given residue class $\epsilon \pmod{2}$, there exist infinitely many n such that $c(n) \equiv \epsilon \pmod{2}$.*

In their paper, Ono and Ramsey [3] mention that it is expected that for half of the $n \equiv 7 \pmod{8}$, we should have $c(n)$ odd.

2. PROOFS OF THEOREMS AND COROLLARIES

We shall start with the following lemma.

Lemma 2.1. *For all integer $n \geq 1$, we have*

$$(2.1) \quad \sum_{m \geq 0} c(n - (2m + 1)^2) \equiv 0 \pmod{2}.$$

Proof. The well-known Jacobi identity says that

$$(2.2) \quad \prod_{n=1}^{\infty} (1 - q^n)^3 = \sum_{k=0}^{\infty} (-1)^k (2k + 1) q^{k(k+1)/2}.$$

Since $(x + y)^{2^m} \equiv x^{2^m} + y^{2^m} \pmod{2}$, we use (2.2) in (1.2) to write

$$(2.3) \quad \Delta(z) \equiv q \prod_{n=1}^{\infty} (1 - q^{8n})^3 \equiv q \sum_{n=0}^{\infty} q^{8n(n+1)/2} \equiv \sum_{n=0}^{\infty} q^{(2n+1)^2} \pmod{2}.$$

By (1.3), we have $E_4(z) \equiv 1 \pmod{2}$. Therefore, (1.1) becomes

$$j(z)\Delta(z) \equiv 1 \pmod{2}.$$

From (1.4) and (2.3), we have

$$j(z)\Delta(z) \equiv \left(\sum_{n=-1}^{\infty} c(n)q^n \right) \left(\sum_{n=0}^{\infty} q^{(2n+1)^2} \right) \pmod{2}.$$

Therefore, we get

$$1 \equiv \sum_{n=0}^{\infty} \sum_{k \geq 0} c(n - (2k + 1)^2) q^n \pmod{2}.$$

Now by comparing the coefficients of q^n on both sides, we get the required congruence. $\square$

Proof of Theorem 1.1. Let $a \geq 1$ be a given integer. Assume that $c(m)$ is even for every $m \in (a, 4a(a + 1) - 1]$. Put $n = 4a(a + 1)$ in (2.1). We get

$$\sum_{k \geq 0} c(4a(a + 1) - (2k + 1)^2) = \sum_{k \geq 0} c(4a(a + 1) - 4k(k + 1) - 1) \equiv 0 \pmod{2}.$$

In the above congruence, the term corresponding to $k = a$ is $c(-1)$ which is indeed 1 and hence $c(-1) \not\equiv 0 \pmod{2}$. When we put $k = a - j$, we get

$$4a(a + 1) - 4(a - j)(a - j + 1) - 1 = 8ja - 4j^2 + 4j - 1 = 4j(2a - j + 1) - 1.$$

If we vary $j = 1, 2, \dots, a - 1$, then we see that

$$4j(2a - j + 1) - 1 \geq 4(2a - (a - 1) + 1) - 1 = 4(a + 2) - 1 > a$$

for all $a \geq 1$. Therefore, if

$$c(4a(a + 1) - 4k(k + 1) - 1) \text{ are all even for all } k = 1, 2, \dots, a - 1$$

and $k = a$, the above integer is odd. Therefore, their sum cannot be even, which is a contradiction. Hence there is an integer $n \in (a, 4a(a + 1) - 1]$ for which $c(n)$ is an odd integer.

Since

$$j(z) \equiv \frac{1}{q \prod_{n=1}^{\infty} (1 - q^{8n})^3} \equiv \sum_{k=-1}^{\infty} b(k) q^{8k+7} \pmod{2},$$

where $b(k) \equiv 0, 1 \pmod{2}$, by comparing the Fourier coefficients on both sides, we get if $n \not\equiv 7 \pmod{8}$, we have $c(n) \equiv 0 \pmod{2}$ and if $c(n)$ is odd, then $n \equiv 7 \pmod{8}$. Therefore the integer $n \in (a, 4a(a+1) - 1]$ (for which $c(n)$ is odd) must be an odd integer and $n \equiv 7 \pmod{8}$. $\square$

Proof Corollary 1.2. We want to count $n \leq x$ for which $c(n)$ is odd. For that we define $a_0 = 1, a_1 = 7$, for every $k \geq 2$

$$a_k = 4a_{k-1}(a_{k-1} + 1) - 1 = 4a_{k-1}^2 + 4a_{k-1} - 1.$$

Then, we partition the interval

$$[1, x] = [1, 7] \cup (7, a_2) \cup [a_2, a_3) \cup \cdots \cup [a_{\ell-1}, a_{\ell}) \cup [a_{\ell}, x]$$

where ℓ is the largest integer k such that $a_k \leq x$. By Theorem 1.1, we know each interval $[a_{k-1}, a_k]$ contains at least one integer $n \equiv 7 \pmod{8}$ for which $c(n)$ is odd. Hence, the number of $n \leq x$ with $n \equiv 7 \pmod{8}$ for which $c(n)$ is odd is at least ℓ and it remains to find the value of ℓ as a function of x . Since

$$a_k = 4a_{k-1}^2 + 4a_{k-1} - 1 < 8a_{k-1}^2 \text{ for all } k \geq 0,$$

we get

$$a_k \leq 8^k a_1^{2^{k-1}} \leq 8^{2^k} \text{ for all } k \geq 0.$$

Since $a_{\ell} \leq x$, we see that $\ell \geq c_0 \log x$ which proves the corollary. $\square$

Proof of Theorem 1.3. For every $a \geq 1$, we denote the interval

$$I_a := [16a - 1, (4a + 1)^2 - 1].$$

We need to prove that I_a contains an integer $n \equiv 7 \pmod{8}$ for which $c(n)$ is even.

Suppose we assume that $c(n)$ is odd for every integer $n \equiv 7 \pmod{8}$ and n lies in the interval I_a . Put $n = (4a + 1)^2 - 1$ in (2.1) and we get

$$\sum_{k \geq 0} c((4a + 1)^2 - 1 - (2k + 1)^2) \equiv 0 \pmod{2}.$$

Note that the argument of c in the summands is $(4a + 1)^2 - 1 - (2k + 1)^2 \equiv -1 \pmod{8}$ and $(4a + 1)^2 - 1 - (2k + 1)^2 \in I_a$ for all $k = 0, 1, \dots, 2a - 1$. When we put $j = 2a$, we get $c(-1)$ which is an odd integer. By assumption, we get $2a$ number of 1's and $c(-1)$ add up to $0 \pmod{2}$, which is a contradiction as $c(-1)$ is odd. Thus, there exists $n \in I_a$ with $n \equiv 7 \pmod{8}$ such that $c(n)$ is an even integer. $\square$

Proof Corollary 1.4. We want to count $n \leq x$ with $n \equiv 7 \pmod{8}$ for which $c(n)$ is even. Since we know $c(15)$ and $c(23)$ are even integers, we define $a_0 = 1, a_1 = 15$, for every $k \geq 2$ as

$$a_k = (4a_{k-1} + 1)^2 - 1.$$

Then, we see that the disjoint union of the following intervals

$$[1, 15] \cup (15, 25) \cup [a_1, a_2) \cup \cdots \cup [a_{\ell-1}, a_{\ell}) \cup [a_{\ell}, x] \subset [1, x]$$

where ℓ is the largest integer k such that $a_k \leq x$. By Theorem 1.3, we know each interval $[a_{k-1}, a_k]$ contains at least one integer $n \equiv 7 \pmod{8}$ for which $c(n)$ is

even. Hence, the number of $n \leq x$ and $n \equiv 7 \pmod{8}$ for which $c(n)$ is even is at least ℓ . Since $a_k \leq 32a_{k-1}^2$ for all $k \geq 0$, we get,

$$a_k \leq 32^k a_1^{2^{k-1}} \leq 32^{2^k} \text{ for all } k \geq 0$$

and hence we get the result. $\square$

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REFERENCES

- [1] Claudia Alfes, *Parity of the coefficients of Klein's j -function*, Proc. Amer. Math. Soc. **141** (2013), no. 1, 123–130, DOI 10.1090/S0002-9939-2012-11502-3. MR2988716
- [2] O. Kolberg, *Note on the parity of the partition function*, Math. Scand. **7** (1959), 377–378. MR0117213 (22 #7995)
- [3] Ken Ono and Nick Ramsey, *A mod ℓ Atkin-Lehner theorem and applications*, Arch. Math. (Basel) **98** (2012), no. 1, 25–36, DOI 10.1007/s00013-011-0347-x. MR2885530
- [4] Ken Ono and Yuichiro Taguchi, *2-adic properties of certain modular forms and their applications to arithmetic functions*, Int. J. Number Theory **1** (2005), no. 1, 75–101, DOI 10.1142/S1793042105000066. MR2172333 (2006e:11057)

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