ASYMPTOTIC VALUES OF SOME CONTINUOUS MAPPINGS

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ABSTRACT. It is shown that the set of asymptotic values of a light continuous mapping defined on $\mathbb{R}^s$ is an analytic set in the sense of Suslin.

A mapping $f : \mathbb{R}^s \rightarrow \mathbb{R}^t$ is *light* if for every $a \in f(\mathbb{R}^s)$ the preimage $f^{-1}(a) := \{x \in \mathbb{R}^s : f(x) = a\}$ is totally disconnected. A particular instance is a *discrete* mapping for which every fiber $f^{-1}(a)$ is a discrete set. For example, non-constant holomorphic functions are discrete. If $f : \mathbb{R}^s \rightarrow \mathbb{R}^t$ is a light continuous mapping, then $s \leq t$ (see [2] and [8]).

A point $a \in \mathbb{R}^t$ is an *asymptotic value* of a continuous mapping $f : \mathbb{R}^s \rightarrow \mathbb{R}^t$ if $f(x) \rightarrow a$ as $|x| \rightarrow \infty$ on some continuous path $\gamma \subset \mathbb{R}^s$. The set of asymptotic values of f will be denoted by $\text{As}(f)$.

A Suslin analytic set in $\mathbb{R}^t$ is a continuous image of a Borel set. There are several equivalent definitions (see, for example, [3, Lemma 11.6]) but in this note there will be used the characterization of analytic sets (in a complete separable metric space) in terms of the Suslin $\mathcal{A}$ -operation. Concretely, $A \subset \mathbb{R}^t$ is analytic if and only if

$$A = \bigcup_{\mathbb{N}^{\mathbb{N}}} \bigcap_{p \geq 1} S_{n_1, \dots, n_p},$$

where the sets $S_{n_1, \dots, n_p} \subset \mathbb{R}^t$ are closed and $\mathbb{N}^{\mathbb{N}}$ is the collection of all infinite sequences of (positive) natural numbers (see [1], [3, Lemma 11.7] or [7, p. 207]). Sierpinski calls the set A the *nucleus* of the defining system $\{S_{n_1, \dots, n_p}\}$.

In [4], S. Mazurkiewicz shows that the set of asymptotic values of a holomorphic (or meromorphic) function, f , defined in $\mathbb{C}$, is an analytic set. In fact, as pointed out to the authors by Alexandre Eremenko, Mazurkiewicz does not use the analyticity of the function f . Indeed, he defines a function $\rho_f : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{R}^+$ as the infimum of the diameters of the f -images of curves connecting two points. When f is light ρ_f is a distance. Let $\mathcal{X}_f^*$ denote the completion of the metric space $\mathcal{X}_f = (\mathbb{C}, \rho_f)$. Mazurkiewicz observes that the set of asymptotic values of f is the image (under the continuous function f) of the Borel set $\mathcal{X}_f^* \setminus \mathcal{X}_f$, and hence, analytic. Consequently, Mazurkiewicz's proof and conclusion can be extended to light mappings defined in $\mathbb{R}^s$ (with $s \geq 3$). An important class of such mappings are non-constant quasiregular

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mappings which, although in general not smooth, are discrete and therefore light (see [6] or [9]).

In this note, we present an alternative approach to Mazurkiewicz's result, where we use the characterization of analytic sets given by the Suslin $\mathcal{A}$ -operation, and work on the range of f rather than on its domain.

Theorem 1. *Let $f : \mathbb{R}^s \rightarrow \mathbb{R}^t$ ($s \leq t$) be a light continuous mapping. Then, the set of asymptotic values of f is an analytic set.*

It will be shown that $\text{As}(f)$ can be written in terms of the Suslin $\mathcal{A}$ -operation. In order to do this, consider a dyadic partition of $\mathbb{R}^t$. Concretely, let $\{X_n\}_{n \geq 1}$ be the family of closed unit cubes with vertices on the integer lattice so that $\mathbb{R}^t = \bigcup_{n \geq 1} X_n$. Each X_n is divided into 2^t congruent closed cubes X_{n_1, n_2} with $n_2 \in \{1, \dots, 2^t\}$ having side length $1/2$. In general, a dyadic cube $X_{n_1, n_2, \dots, n_p}$ of side length 2^{1-p} is divided into 2^t congruent closed cubes of side length 2^{-p} , and each of these cubes is denoted by $X_{n_1, n_2, \dots, n_p, n_{p+1}}$ with $n_{p+1} \in \{1, 2, \dots, 2^t\}$. For $p \geq 1$, let $\mathcal{F}_p$ be the family of all dyadic cubes of generation p , that is, $\mathcal{F}_p = \{X_{n_1, n_2, \dots, n_p} : n_i \in \mathbb{N}, n_i \in \{1, \dots, 2^t\}, 2 \leq i \leq p\}$. For $X_{n_1, n_2, \dots, n_p} \in \mathcal{F}_p$ ($p \geq 1$), consider the set

$$\text{Adj}(X_{n_1, n_2, \dots, n_p}) = \{x \in X : X \in \mathcal{F}_p, X \cap X_{n_1, n_2, \dots, n_p} \neq \emptyset\},$$

consisting of the cube together with its neighbors of generation p . Clearly, $X_{n_1, n_2, \dots, n_p}$ lies in the interior of $\text{Adj}(X_{n_1, n_2, \dots, n_p})$.

Given a light continuous mapping $f : \mathbb{R}^s \rightarrow \mathbb{R}^t$, a dyadic cube $X_{n_1, n_2, \dots, n_p}$ is said to be *admissible* if $f^{-1}(\text{Adj}(X_{n_1, n_2, \dots, n_p}))$ has an unbounded connected component. For any finite sequence of natural numbers, $n_1, \dots, n_p$, define

$$(1) \quad S_{n_1, n_2, \dots, n_p} = \begin{cases} X_{n_1, \dots, n_p}, & \text{if } X_{n_1, \dots, n_p} \text{ is admissible,} \\ \emptyset, & \text{otherwise.} \end{cases}$$

Notice that $S_{n_1, \dots, n_p}$ is closed, with diameter $\text{diam}(S_{n_1, \dots, n_p}) \leq 2^{1-p}\sqrt{t}$ and $S_{n_1, \dots, n_p, n_{p+1}} \subset S_{n_1, \dots, n_p}$. Theorem 1 is a consequence of Proposition 1.

Proposition 1. *Let $f : \mathbb{R}^s \rightarrow \mathbb{R}^t$ ($s \leq t$) be a light continuous mapping and A the analytic set that is the nucleus of the defining system $\{S_{n_1, \dots, n_p}\}$ given by (1). Then,*

$$A = \text{As}(f) \setminus \{\infty\}.$$

Remark 1. If A is analytic, then $A \cup \{\infty\}$ is also analytic.

Proof. First it will be shown that $A \subset \text{As}(f) \setminus \{\infty\}$ or, in words, that any point $a \in A$ is a finite asymptotic value of f . Since $a \in A$ there exists a sequence of natural numbers $\{n_p\}_{p \geq 1}$ such that $a = \bigcap_{p \geq 1} S_{n_1, \dots, n_p}$ with $S_{n_1, \dots, n_p} \neq \emptyset$ for all $p \geq 1$. Thus for every $p \geq 1$, $S_{n_1, \dots, n_p} = X_{n_1, \dots, n_p}$ where $n_2, \dots, n_p \in \{1, \dots, 2^t\}$ and $X_{n_1, \dots, n_p}$ is admissible and therefore $f^{-1}(\text{Adj}(X_{n_1, \dots, n_p}))$ has an unbounded connected component. Given $k \in \mathbb{N}$ there is $p \in \mathbb{N}$ such that $B(a, 1/k) \supset \text{Adj}(X_{n_1, \dots, n_p})$, where $B(a, r)$ denotes the open ball of radius $r > 0$ centered at a . Since $f^{-1}(\text{Adj}(X_{n_1, \dots, n_p}))$ has an unbounded connected component $f^{-1}(B(a, 1/k))$

has an unbounded connected component. Therefore there exists a chain

$$(2) \quad C_1 \supset C_2 \supset \cdots \supset C_k \supset \cdots$$

of nested sets where each C_k is the closure of an unbounded connected component of $f^{-1}(B(a, 1/k))$. The continuity of f implies that each C_k ($k \geq 1$) is path connected.

Since f is light there can be found an increasing subsequence of natural numbers, $\{k_j\}_{j \geq 1}$, and a strictly increasing sequence of positive real numbers, $\{r_j\}_{j \geq 0}$, with $r_j \nearrow \infty$ ($j \rightarrow \infty$) such that for any $j \in \mathbb{N}$,

- a) $C_{k_j} \cap \overline{B(r_{j-1})} = \emptyset$, and,
- b) $C_{k_j} \cap S(r_j) \neq \emptyset$,

(where $B(r) = B(0, r)$, $S(r) = \partial B(r)$ and $\overline{A}$ is the closure of the set A). Otherwise, there exists $\rho > 0$ such that $C_k \cap B(\rho) \neq \emptyset$ for all $k \geq 1$. Consider $\Omega = \bigcap_{k \geq 1} C_k$ a subset of $f^{-1}(a)$. Since $\{C_k\}_{k \geq 1}$ is a decreasing sequence of unbounded closed sets that intersect $S(\rho)$, $\Omega \neq \emptyset$, and moreover, $\Omega \cap S(r) \neq \emptyset$ for any $r \geq \rho$. Let $\widehat{\mathbb{R}}^s = \mathbb{R}^s \cup \{\infty\}$ be the Alexandroff one-point compactification of $\mathbb{R}^s$. Since every C_k , $k \geq 1$, is an unbounded connected closed set, $\widehat{C_k} = C_k \cup \{\infty\}$ is a connected compact set in $\widehat{\mathbb{R}}^s$ and $\{\widehat{C_k}\}_{k \geq 1}$ decreases to $\widehat{\Omega} = \Omega \cup \{\infty\}$ where $\widehat{\Omega}$ is compact and connected with more than one point. Hence by [5, Corollaire XIII], $\widehat{\Omega}$ can be decomposed as $\widehat{\Omega} = U \cup V$ with U and V disjoint connected sets, each containing more than one point. Let U be the set that contains ∞ ; then $V \subset \mathbb{R}^s$ is a connected subset of $f^{-1}(a)$ with more than one point, which contradicts the assumption that f is light.

Hence we can use a) and b) above to find an asymptotic path for a . Pick $x_1 \in C_{k_1} \cap S(r_1)$ and let γ_1 be a continuous path in C_{k_1} that joins x_1 with $C_{k_2} \cap S(r_2)$. Let $x_2 \in C_{k_2} \cap S(r_2)$ be the other end of γ_1 . In general for $j \geq 2$, if $x_j \in C_{k_j} \cap S(r_j)$ let γ_j be a continuous path in C_{k_j} that joins x_j with $C_{k_{j+1}} \cap S(r_{j+1})$ and define x_{j+1} to be the other end of γ_j . Let $\gamma = \bigcup_{j \geq 1} \gamma_j$. By construction, γ is a continuous curve and $\gamma \rightarrow \infty$. Indeed, for any $r > 0$ there exists $j \in \mathbb{N}$ such that $r_j > r$ and by condition a) above

$$\bigcup_{i \geq j+1} \gamma_i \subset C_{k_{j+1}} \subset \{|x| > r_j\} \subset \{|x| > r\}.$$

Finally it remains to show that γ is an asymptotic curve with a as an asymptotic value. Let $\varepsilon > 0$ and choose $j \in \mathbb{N}$ so that $1/k_j < \varepsilon$. Take $x \in \bigcup_{i \geq j+1} \gamma_i \subset \gamma$. Then $x \in C_{k_{j+1}} \subset \overline{f^{-1}(B(a, 1/k_{j+1}))} \subset f^{-1}(\overline{B(a, 1/k_j)})$ and therefore $f(x) \in \overline{B(a, 1/k_j)}$, that is, $|a - f(x)| \leq 1/k_j < \varepsilon$, as desired.

Next there will be shown the other inclusion, that is, that any finite asymptotic value of f belongs to the set A . Let $b \in \text{As}(f) \setminus \{\infty\}$ and consider the family of dyadic cubes such that $b = \bigcap_{p \geq 1} X_{n_1, \dots, n_p}$. Then, for any $p \geq 1$, there is an $\varepsilon > 0$ so that $b \in B(b, \varepsilon) \subset \text{Adj}(X_{n_1, \dots, n_p})$ since $b \in \text{int}(\text{Adj}(X_{n_1, \dots, n_p}))$. Since b is a finite asymptotic value of f , $f^{-1}(B(b, \varepsilon))$ has an unbounded connected component and therefore the set $f^{-1}(\text{Adj}(X_{n_1, \dots, n_p}))$ also has an unbounded connected component. Thus $X_{n_1, \dots, n_p}$ is admissible and therefore $S_{n_1, \dots, n_p} = X_{n_1, \dots, n_p}$ for every $p \geq 1$ which implies $b = \bigcap_{p \geq 1} S_{n_1, \dots, n_p}$; that is, $b \in A$. $\square$

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