FULL GROUPS AND SOFICITY

GÁBOR ELEK

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ABSTRACT. First, we answer a question of Giordano and Pestov by proving that the full group of a sofic equivalence relation is a sofic group. Then, we give a short proof of the theorem of Grigorchuk and Medynets that the topological full group of a minimal Cantor homeomorphism is LEF. Finally, we show that for certain non-amenable groups all the generalized lamplighter groups are sofic.

1. INTRODUCTION

1.1. Sofic groups and LEF groups. The notion of sofic groups was introduced by Weiss [12] and Gromov [5] (in a somewhat different form). A group Γ is sofic if for any finite set $F \subset \Gamma$ and $\epsilon > 0$ there exists a finite set A and a mapping $\Theta : \Gamma \rightarrow \text{Map}(A)$ such that ([3])

- If $f, g, fg \in F$, then $d_H(\Theta(fg) - \Theta(f)\Theta(g)) \leq \epsilon$, where

$$d_H(\alpha, \beta) = \frac{|\{x \in A \mid \alpha(x) \neq \beta(x)\}|}{|A|}.$$

- If $1 \neq f \in F$, then $d_H(\Theta(f), 1) > 1 - \epsilon$.
- $\Theta(1) = 1$.

All amenable and residually finite groups are sofic. It is an open question whether non-sofic groups exist. If we add the extra requirement that $\Theta(fg) = \Theta(f)\Theta(g)$, then we get the class of LEF-groups (locally embeddable into finite groups). This class of groups was introduced by Gordon and Vershik [11]. Clearly, all residually finite groups are LEF. However, simple, finitely presented groups are not LEF. Nevertheless, by a recent result of Juschenko and Monod [6] (and Theorem 2), there exist simple, finitely generated LEF-groups.

1.2. Sofic equivalence relations. Let $X = \{0, 1\}^{\mathbb{N}}$ be the standard Borel space with the natural product measure μ . Let $\Phi : \mathbf{F}_{\infty} \curvearrowright X$ be a (not necessarily free) Borel action of the free group of countably infinite generators $\{\gamma_1, \gamma_1^{-1}, \gamma_2, \gamma_2^{-1}, \dots\}$ preserving μ . Note that $\mathbf{F}_{\infty} = \bigcup_{r=1}^{\infty} \mathbf{F}_r$, where $\mathbf{F}_r$ is the free group of rank r . Hence, we also have probability measure preserving (p.m.p.) Borel actions $\Phi_r : \mathbf{F}_r \curvearrowright X$. We say that $x, y \in X$ are equivalent, $x \sim_{\Phi} y$ if there exists $w \in \mathbf{F}_{\infty}$, such that $w(x) = y$. Note that slightly abusing the notation we write $w(x)$ instead of $\Phi(w)(x)$. Thus, the action Φ represents a countable measured equivalence relation E_{Φ} on X . Similarly, each Φ_r represents a countable measured equivalence relation E_{Φ_r} on X ,

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and $E_\Phi = \bigcup_{i=1}^\infty E_{\Phi_r}$. Each equivalence relation E_{Φ_r} defines a graphing [7] G_r on X :

- $V(G_r) = X$.
- $(x, y) \in E(G_r)$ if $\gamma_i x = y$ or $\gamma_i y = x$ for some i (so, there may be loops in G_r).

Observe that each component of G_r is a countable graph of bounded vertex degrees. We label each directed edge (x, y) with all the generators mapping x to y . Thus an edge, even a loop, may have multiple labels.

Now let us consider transitive actions of $\mathbf{F}_r$ on countable sets. If $\alpha : \mathbf{F}_r \curvearrowright Y$ is such an action, then we have a bounded degree graph structure on Y with multiple labels on the edges from the set $\{\gamma_1, \gamma_1^{-1}, \dots, \gamma_r, \gamma_r^{-1}\}$. Let T_r be the set of graphs of all countable $\mathbf{F}_r$ -actions with a distinguished vertex (the root) such that all the vertices are labeled by the elements of $\{0, 1\}^r$. Let $G \in T_r$. We define the k -ball around the root x , $B_k(x)$, as the induced subgraph on vertices of G in the form of $w(x)$, where $w \in \mathbf{F}_r$ is a reduced word of length at most k . That is, $B_k(x)$ is the ball centered at x of radius k with respect to the shortest path metric of G . The ball $B_k(x)$ is a finite rooted graph with edge-colors from the set $\{\gamma_1, \gamma_1^{-1}, \dots, \gamma_r, \gamma_r^{-1}\}$ and vertex labels from the set $\{0, 1\}^r$. We denote the set of all possible k -balls arising from $\mathbf{F}_r$ -actions by U_r^k . We can define a compact metric structure on the set T_r the following way. Let $d_r(G, H) = \frac{1}{2^k}$ if k is the maximal number such that the k -balls around the roots of G , resp. H , are isomorphic as rooted, labeled graphs.

Observe that if $\Theta : \mathbf{F}_\infty \curvearrowright X$ is a p.m.p. action, then for each $r \geq 1$ and $x \in X$ one can associate an element $G(\Theta, x) \in T_r$. Namely, the orbit graph of x , where the vertex labels are given by the X -values, restricted on the first r coordinates. Thus, we have a Borel map $\pi_\Theta : X \rightarrow T_r$. For $\kappa \in U_r^k$, let $\mu_{\Theta_r}^k(\kappa) = (\pi_\Theta)_*(\mu)(L_\kappa)$, where $L_\kappa \subset T_r$ is the set of elements G such that the k -ball around the root of G is isomorphic to κ . In other words, $\mu_{\Theta_r}^k(\kappa)$ is the probability that the k -ball around a μ -random element of X is isomorphic to κ . Now let $\alpha : \mathbf{F}_r \curvearrowright Y$ be an $\mathbf{F}_r$ -action on a finite set. Then for each element y of Y , we can associate an element of T_r ; namely, Y itself with root y . Hence, we can define a probability distribution $\mu_\alpha^{k,r}$ on U_r^k . Following [1] we say that the action $\Theta : \mathbf{F}_\infty \curvearrowright X$ is sofic if for all $r \geq 1$, there exists a sequence of finite $\mathbf{F}_r$ -actions $\{\alpha_n\}_{n=1}^\infty$ such that for each $k \geq 1$ and $\kappa \in U_r^k$,

$$\lim_{n \rightarrow \infty} \mu_{\alpha_n}^{k,r}(\kappa) = \mu_{\Theta_r}^k(\kappa).$$

In [1] the authors proved that

- Soficity is a property of the underlying equivalence relations. That is, if an action Θ_1 is orbit equivalent to a sofic action Θ_2 , then Θ_2 is sofic as well.
- Treeable equivalence relations are sofic.
- Actions associated to Bernoulli shifts of sofic groups are sofic.

1.3. Full groups. Let $E(X, \mu)$ be a countable, measured equivalence relation on a Borel set X with invariant measure μ . The Borel full group of E is the group $[E]_B$ of all Borel bijections $T : X \rightarrow X$ such that for any $x \in X$, $T(x) \sim_E x$. We call two such bijections T_1, T_2 equivalent if

$$\mu(\{x \in X \mid T_1(x) = T_2(x)\}) = 1.$$

The measurable full group $[E]$ is the group formed by the equivalence classes. Obviously, $[E] = [E]_B/N$, where N is the normal subgroup of elements in $[E]_B$ fixing almost all points of X .

Now, let $T : C \rightarrow C$ be a homeomorphism of the Cantor set C . The topological full group $[[T]]$ is the group of homeomorphisms $S : C \rightarrow C$ such that C can be partitioned into finitely many clopen sets $C = \bigcup_{i=1}^n A_i$ such that $S|_{A_i} = T^{n_i}$ for some integer n_i .

1.4. Results. Answering a question of Giordano and Pestov,¹ we prove the following theorem.

Theorem 1. *The measurable full group of a sofic equivalence relation is sofic.*

Then, we give a very short proof of a result of Grigorchuk and Medynets [4].

Theorem 2. *The topological full group of a minimal Cantor homeomorphism is LEF.*

Let X be a countably infinite set and Γ be a countable group acting faithfully and transitively on X . Then Γ can be represented by automorphisms on the Abelian group $\bigoplus_{x \in X} \{0, 1\}$. The groups $\bigoplus_{x \in X} \{0, 1\} \rtimes \Gamma$ are called the lamplighter group of the Γ -action. If the action is the natural translation action on Γ , then we get the classical lamplighter group of Γ . Paunescu [10] proved that if Γ is sofic, then the classical lamplighter group $\bigoplus_{\gamma \in \Gamma} \{0, 1\} \rtimes \Gamma$ is sofic. If Γ is amenable, then all its generalized lamplighter groups are amenable, hence sofic. Nevertheless, we show that there exist non-amenable groups for which all the generalized lamplighter groups are sofic.

Theorem 3. *Let Γ^k be the k -fold free product of the cyclic group of two elements. Then, for any transitive, faithful action of Γ^k on a countable set the associated lamplighter group is LEF.*

2. COMPRESSED SOFIC REPRESENTATIONS

Let Γ be a countable sofic group with elements $\{\gamma_1, \gamma_2, \dots\}$. A compressed sofic representation of Γ is defined in the following way. For any $i \geq 1$ we have a constant $\epsilon_i > 0$, and for any $n \geq 1$ we have mappings $\Theta_n : \Gamma \rightarrow \text{Map}(A_n)$ such that $|A_n| < \infty$ satisfying the following condition: For all $r > 0$ and $\epsilon > 0$ there exists $K_{r,\epsilon} > 0$ such that if $n > K_{r,\epsilon}$, then

- $d_H(\Theta_n(\gamma_i \gamma_j) \Theta_n(\gamma_i) \Theta_n(\gamma_j)) < \epsilon$ if $1 \leq i, j \leq r$.
- $d_H(\Theta_n(\gamma_i), Id) > \epsilon_i$ if $1 \leq i \leq r$.

Thus, in a compressed sofic representation we allow a large amount of fixed points for each $\gamma \in \Gamma$.

Lemma 2.1. *If Γ has a compressed sofic representation, then Γ is sofic.*

Proof. Let $\tilde{\Theta}_n^k : \Gamma \rightarrow \text{Map}(A_n^k)$ be defined by

$$\tilde{\Theta}_n^k(\gamma)(x_1, x_2, \dots, x_k) = (\Theta_n(\gamma)(x_1), \Theta_n(\gamma)(x_2), \dots).$$

Observe that if $\gamma, \delta \in \Gamma$, then

- $d_H(\tilde{\Theta}_n^k(\gamma\delta), \tilde{\Theta}_n^k(\gamma)\tilde{\Theta}_n^k(\delta)) \leq (1 - d_H(\Theta_n(\gamma\delta), \Theta_n(\gamma)\Theta_n(\delta)))^k$.
- $d_H(\tilde{\Theta}_n^k(\gamma), Id) > 1 - (1 - d_H(\Theta_n(\gamma), Id))^k$.

¹MR2566316-MathSciNet Review.

Hence, we can choose ϵ , n and k appropriately to obtain for any $F \subset \Gamma$ and $\epsilon' > 0$ a map Θ as in the Introduction, proving the soficity of Γ . $\square$

3. THE PROOF OF THEOREM 1

Let $\Phi : \mathbf{F}_\infty \curvearrowright \{0, 1\}^\mathbb{N}$ be a sofic action preserving the product measure μ . Let $\Gamma \subset [E]$ be a finitely generated group, where $[E]$ is the equivalence relation defined by Φ . So, we have an action $\Phi_\Gamma : \Gamma \curvearrowright \{0, 1\}^\mathbb{N}$. Our goal is to construct a compressed sofic representation of Γ . Let $\{\gamma_n\}_{n=1}^\infty$ be an enumeration of the elements of Γ . Let $\epsilon_n = \mu(\text{Fix}(\Phi_\Gamma(\gamma_n)))/2$. Since Γ is in the full group, $\epsilon_n > 0$. Now, fix a subset $F \subseteq \Gamma$ and $\epsilon > 0$. We need to construct a map $\Theta : F \rightarrow \text{Map}(A)$ for some finite set A such that if $\gamma_i, \gamma_j, \gamma_i\gamma_j \in F$, then

$$(1) \quad d_H(\Theta(\gamma_i\gamma_j)\Theta(\gamma_i)\Theta(\gamma_j)) < \epsilon,$$

$$(2) \quad d_H(\Theta(\gamma_i), 1) > \epsilon_i.$$

Let $\{s_1, s_1^{-1}, s_2, s_2^{-1}, \dots, s_m, s_m^{-1}\}$ be a symmetric generating set for Γ . Observe that we have an action $\Sigma_\Gamma : \mathbf{F}_m \curvearrowright \{0, 1\}^\mathbb{N}$ preserving μ such that $\Sigma_\Gamma(\delta) = \Phi_\Gamma(\tau(\delta))$, where $\tau : \mathbf{F}_m \rightarrow \Gamma$ is the natural quotient map. A dyadic E -map of depth k is a Borel map $Q : X \rightarrow X$ defined in the following way. For each $\rho \in \{0, 1\}^k$ we pick $w_Q(\rho) \in \mathbf{F}_k \subset \mathbf{F}_\infty$ and define $Q(x) = \Phi(w_Q(\rho))(x)$ if the first k -coordinate of x is ρ .

A dyadic approximation of Γ is a sequence of families $\{Q_k(s_i)\}_{i=1}^m, \{Q_k(s_i^{-1})\}_{i=1}^m$, where for any $1 \leq i \leq m$:

- $Q_k(s_i) : X \rightarrow X, Q_k(s_i^{-1}) : X \rightarrow X$ are dyadic E -maps of depth k .
- $\lim_{k \rightarrow \infty} \mu(\{x \in X \mid Q_k(s_i)(x) \neq \Sigma_\Gamma(s_i)(x)\}) = 0$.
- $\lim_{k \rightarrow \infty} \mu(\{x \in X \mid Q_k(s_i^{-1})(x) \neq \Sigma_\Gamma(s_i^{-1})(x)\}) = 0$.

We do not require Q_k to be a bijection. Nevertheless, Q_k can be extended to a homomorphism from $\mathbf{F}_m$ to $\text{Map}(X)$. Note that for simplicity we identified the generating set of $\mathbf{F}_m$ by the set $\{s_1, s_1^{-1}, s_2, s_2^{-1}, \dots, s_m, s_m^{-1}\}$.

Since all the $\Sigma_\Gamma(s_i)$'s are Borel bijections, such dyadic approximations clearly exist. The following lemma is an immediate consequence of the definition of the dyadic approximation.

Lemma 3.1. *For any $\delta \in \mathbf{F}_m$*

$$\lim_{k \rightarrow \infty} \mu(\text{Fix}(Q_k(\delta))) = \mu(\text{Fix}(\Sigma_\Gamma(\delta))).$$

Proposition 3.1. *There exists a sequence of mappings $\hat{\Theta}_k : \mathbf{F}_m \rightarrow \text{Map}(B_k)$, where $|B_k| < \infty$ such that for any $\delta \in \mathbf{F}_m$,*

$$\lim_{k \rightarrow \infty} \left(\mu(\text{Fix}(Q_k(\delta))) - \frac{|\text{Fix}(\hat{\Theta}_k(\delta))|}{|B_k|} \right) = 0.$$

That is,

$$\lim_{k \rightarrow \infty} \frac{|\text{Fix}(\hat{\Theta}_k(\delta))|}{|B_k|} = \mu(\text{Fix}(\Sigma_\Gamma(\delta))).$$

Proof. Let $\Phi_k : \mathbf{F}_k \curvearrowright \{0, 1\}^\mathbb{N}$ be the restriction of Φ . Since Φ is sofic, there exists a sequence of mappings $\{\iota_k^n : \mathbf{F}_k \curvearrowright \text{Perm}(C_{k,n})\}_{n=1}^\infty$, where $C_{k,n}$ is a finite $\{0, 1\}^k$ -vertex labeled graph such that for any $t \geq 1$ and $\kappa \in U_k^t$,

$$\lim_{n \rightarrow \infty} \mu_{\iota_k^n}^{t,k}(\kappa) = \mu_{\Phi_k}^t(\kappa).$$

Recall that Q_k is not necessarily an action, only a homomorphism from $\mathbf{F}_m$ to $\text{Map}(X)$. Hence, the local statistics of Q_k cannot be described using the elements of U_k^t as in the case of honest $\mathbf{F}_m$ -actions. So, let W_k^t be the set of isomorphism classes of rooted t -balls of vertex degrees at most $2m$, where the vertices are labeled by elements of the set $\{0, 1\}^k$ and the edges (possibly loops) are labeled by subsets of $\{s_1, s_1^{-1}, s_2, s_2^{-1}, \dots, s_m, s_m^{-1}\}$. Note that $U_k^t \subset W_k^t$. Let $x, y \in X$ be points such that $B_{k^2}^{\Phi_k}(x)$ and $B_{k^2}^{\Phi_k}(y)$ represent the same element in $U_k^{k^2}$. Here $B_{k^2}^{\Phi_k}(x)$ denotes the k -ball with respect to the graphing associated to Φ_k . Then, by the definition of the dyadic approximations $B_k^{Q_k}(x)$ and $B_k^{Q_k}(y)$ represent the same elements in W_k^k . Now we construct a sequence of maps $\hat{\Theta}_k^n : \mathbf{F}_m \curvearrowright \text{Map}(C_{k,n})$ the following way:

$$\hat{\Theta}_k^n(s_i)(x) = \iota_k^n(w_{Q_k(s_i)}(\rho(x)))(x),$$

where $\rho(x)$ is the $\{0, 1\}^k$ -label of x . By the previous observation, for any $\delta \in \mathbf{F}_m$

$$\lim_{n \rightarrow \infty} \frac{|Fix(\hat{\Theta}_k^n(\delta))|}{|C_{k,n}|} = \mu(Fix(Q_k(\delta))).$$

This finishes the proof of the proposition. $\square$

Pick a section $\sigma : \Gamma \rightarrow \mathbf{F}_m$, that is, a map such that $\tau\sigma = Id$. Let $\hat{\Theta}_k$ be as in Proposition 3.1. Define $\Theta_k : \Gamma \rightarrow \text{Map}(B_k)$ by

$$\Theta_k(\gamma) = \hat{\Theta}_k(\sigma(\gamma)).$$

Then $\{\Theta_k\}_{k=1}^\infty$ is a compressed sofic representation of Γ . $\square$

4. THE PROOF OF THEOREM 2

Let $T : C \rightarrow C$ be a minimal homeomorphism and $\Gamma \subset [[T]]$ be a finitely generated subgroup of the topological full group of T with symmetric generating set $S = \{a_1, a_2, \dots, a_k\}$. It is enough to prove that Γ is LEF. Let $x \in C$ and consider the T -orbit $\{T^n(x)\}_{n=-\infty}^\infty$. We define the map $\phi : \Gamma \rightarrow \text{Perm}(\mathbf{Z})$ of Γ into the permutation group of the integers in the following way. Let $\phi(\gamma)(n) = m$, if $\gamma(T^n(x)) = T^m(x)$. Since T acts freely on C , ϕ is well-defined.

Lemma 4.1. *ϕ is an injective homomorphism.*

Proof. If $\phi(\gamma) = Id$, then γ fixes all the elements of the orbit of x . Since all the orbits are dense, this implies that $\gamma = 1$. The fact that ϕ is a homomorphism follows immediately, since ϕ is the restriction of the Γ -action onto the orbit of x . $\square$

Let $a = \max |n|$, where for some $p \in C$ and $a_i \in S$, $a_i(p) = T^n(p)$. We define a sequence

$$l : \mathbf{Z} \rightarrow \{-a, -a+1, \dots, 0, 1, \dots, a-1, a\}^S$$

in the following way. Let $l(n) := (t_{a_1}, t_{a_2}, \dots, t_{a_k})$, where $a_i(T^n(x)) = T^{n+t_{a_i}}(x)$. The following lemma is well-known; we prove it for the sake of completeness.

Lemma 4.2. *l is a repetitive sequence; that is, if we find a substring σ in l , then there exists $m \geq 1$ such that for any interval of length m we can find σ .*

Proof. For a point $p \in C$, we can define its n -pattern

$$q_n(p) := \{-n, -n+1, \dots, 0, 1, \dots, n-1, n\} \rightarrow \{-a, -a+1, \dots, a-1, a\}$$

by $q_n(p)(j) := (t_{a_1}, t_{a_2}, \dots, t_{a_k})$, where $a_i(T^j(x)) = T^{j+t_{a_i}}(x)$. Observe that the set of points with a given n pattern is closed. Now, let us suppose that for a sequence $\{k_r\}_{r=1}^\infty \subset \mathbf{Z}$ the intervals $(k_r - r, k_r + r)$ do not contain σ as a substring. Then, if z is a limit point of $\{T^{k_r}(x)\}_{r=1}^\infty$, no translates of z have σ as a part of their n -patterns. Therefore the orbit closure of z does not contain x , in contradiction with the minimality of T . $\square$

Now let $r \geq 1$ and consider the string $\sigma_r = l_{|\{-ar, -ar+1, \dots, ar-1, ar\}}$, where a is the constant defined above. Note that if $\gamma \in \Gamma$ is the product of at most r generators, then $|\phi(\gamma)(i) - i| \leq ar$. Pick $n > 10a^r$ such that

- $l_{|\{-ar+n, -ar+1+n, \dots, ar-1+n, ar+n\}} = \sigma_r$,
- for any $\gamma \in \Gamma$ that is the product of at most r generators, there is $0 < j < n$ such that $\gamma(j) \neq j$.

Now we define $\phi_r : W^r \rightarrow \text{Perm}(\mathbf{Z}_n)$, where W^r is the set of elements in Γ that are products of at most r generators by $\phi_r(i) = \phi(i) \pmod n$. Clearly, ϕ_r is injective, and if $x, y, xy \in W^r$, then $\phi_r(x)\phi_r(y) = \phi_r(xy)$. This implies that Γ is LEF. $\square$

5. THE PROOF OF THEOREM 3

Let $\alpha : \Gamma^k \rightarrow X$ be a transitive and faithful action of the free product group. Consider the Schreier graph G_α of the action with respect to the generators of the k cyclic groups $\{a_1, a_2, \dots, a_k\}$. Recall that $V(G_\alpha)$ is X and $(x, y) \in E(G)$ if $y = a_i x$ for some $i \geq 1$. Hence G_α is a connected graph of vertex degree bound k .

Proposition 5.1. *Let α be as above. Then for any $1 \neq w \in \Gamma^k$, there exist infinitely many $y \in X$ such that $\alpha(w)(y) \neq y$.*

Proof. We will need the following lemma.

Lemma 5.1. *For any finite set $S \subseteq X$, there exists $g \in \Gamma^k$ such that $gS \cap S = \emptyset$.*

Proof. We define a lazy random walk on X in the following way. For $y \in X$ the transition probability $p(x, y) = l/k$, where l is the number of generators a_i such that $a_i x = y$. It is well-known (see e.g. [9], [8]) that the probabilities $p_n(x, y)$ tend to zero for each pair $x, y \in X$. Now consider the standard random walk on the Cayley graph of Γ^k , the k -regular tree. Let $P_n(g)$ be the probability being at g after taking n steps starting from the identity. Then,

$$p_n(x, y) = \sum_{g \in \Gamma, gx=y} P_n(g).$$

By the previous observation, if n is large enough, then

$$\sum P_n(g) < 1,$$

where the summation is taken for all $g \in \Gamma^k$ such that $gx \in S$, for some $x \in S$. Hence, there exists $g \in \Gamma^k$ such that $gS \cap S = \emptyset$. $\square$

Now let us suppose that $w \in \Gamma^k$ fixes all points of X outside a finite set S . That is, $\alpha(w)(S) = S$. Let $gS \cap S = \emptyset$. Then gwg^{-1} fixes all the points of X outside gS . Therefore the commutator $[w, gwg^{-1}]$ fixes all elements of X , in contradiction with the assumption that the action is faithful. $\square$

Now fix a vertex $x \in X$ and consider the ball of radius n , $B_n(x)$ around x . We define an action $\alpha_n : \Gamma^k \curvearrowright B_n(x)$ in the following way. Let $\partial B_n(x)$ be the boundary of the ball $B_n(x)$, that is, the set of all $y \in B_n(x)$ such that there exists a_i for which $\alpha(a_i)y \notin B_n(x)$. If $y \notin \partial B_n(x)$, then let $\alpha_n(a_i)y = \alpha(a_i)y$. If $y \in \partial B_n(x)$ and $\alpha(a_i)y \notin B_n(x)$, then let $\alpha_n(a_i)(y) = y$. Finally, if $y \in \partial B_n(x)$ and $\alpha(a_i)y \in B_n(x)$, then let $\alpha_n(a_i)(y) = \alpha(a_i)(y)$. Now let $L_k^n = \{0, 1\}^{B_n(x)} \rtimes_{\alpha_n} \alpha_n(\Gamma^k)$ be the associated finite lamplighter group and $L^k = \bigoplus_{x \in X} \{0, 1\} \rtimes_{\alpha} \Gamma^k$. Our goal is to embed L^k into L_k^n locally. That is, for any finite set $F \subset L^k$ we construct an injective map $\Theta : F \rightarrow L_k^n$ such that $\Theta(fg) = \Theta(f)\Theta(g)$. Recall that each element of L^k can be uniquely written in the form $a \cdot w$, where $a \in \bigoplus_{x \in X} \{0, 1\}$ and $w \in \Gamma^k$. We regard the elements of the lamplighter group as permutations of the set $\bigoplus_{x \in X} \{0, 1\}$. If $\kappa \in \bigoplus_{x \in X} \{0, 1\}$ and $p \in X$, then

$$(a \cdot w)(\kappa)|_p = a(p) + \kappa(\alpha(w^{-1})(p)).$$

We will also use the product formula

$$(a_2 \cdot w_2)(a_1 \cdot w_1) = (a_2 + \alpha(w_2)(a_1), w_2 w_1),$$

where $\alpha(w_2)(a_1)(q) = a_1(\alpha(w_2^{-1})(q))$. For $l \geq 1$, let H_l be the set of elements of L^k in the form of $a \cdot w$, where w is a word of length at most l and the support of a is contained in $B_l(x)$. For $n \geq l$ we define the map $\tau_l^n : H_l \rightarrow L_k^n$ by $\tau_l^n(a \cdot w) := a \cdot \alpha_n(w)$.

Lemma 5.2. *If n is large enough, then τ_l^n is injective.*

Proof. If n is large enough, then $B_n(x)$ contains a point y such that

- $\alpha(w)(y) \neq y$,
- $d(y, \partial B_n(x)) > l$,
- $d(y, B_l(x)) > l$,

where d is the shortest path distance on the Schreier graph G_α . Let $\kappa \in \bigoplus_{x \in X} \{0, 1\}$ be the element which is 1 at y and zero otherwise. Then

$$\tau_l^n(a \cdot w)(\kappa)|_{\alpha_n(w)(y)} = 1,$$

hence $\tau_l^n(a \cdot w)$ is not trivial. □

The following lemma finishes the proof of Theorem 3.

Lemma 5.3. *Suppose that $(a_1 \cdot w_1), (a_2 \cdot w_2)$ and $(a_2 \cdot w_2)(a_1 \cdot w_1) \in H_l$ and n is large enough. Then*

$$\tau_l^n((a_2 \cdot w_2))\tau_l^n((a_1 \cdot w_1)) = \tau_l^n((a_2 \cdot w_2)(a_1 \cdot w_1)).$$

Proof. We need to prove that

$$(a_2 \cdot \alpha_n(w_2))(a_1 \cdot \alpha_n(w_1)) = (a_2 + \alpha(w_2)(a_1)) \cdot \alpha_n(w_2 w_1)$$

holds in L_k^n . Fix an element $\kappa \in \{0, 1\}^{B_n(x)}$. Let $n > 10l$ and $d(p, \partial B_n(x)) > 5l$. Then

$$(a_2 \cdot \alpha_n(w_2))(a_1 \cdot \alpha_n(w_1))(\kappa)|_p = (a_2 \cdot w_2)(a_1 \cdot w_1)(\bar{\kappa})|_p$$

and

$$(a_2 + \alpha(w_2)(a_1) \cdot \alpha_n(w_2 w_1))(\kappa)|_p = (a_2 + \alpha(w_2)(a_1) \cdot (w_2 w_1))(\bar{\kappa})|_p,$$

where $\bar{\kappa}$ is an extension of κ onto X . On the other hand, if $d(p, \partial B_n(x)) \leq 5l$, then

$$\begin{aligned} (a_2 \cdot \alpha_n(w_2))(a_1 \cdot \alpha_n(w_1))(\kappa)|_p &= \alpha_n(w_2)\alpha_n(w_1)(\kappa)|_p \\ &= \alpha_n(w_2w_1)(\kappa)|_p = (a_2 + \alpha(w_2)(a_1)) \cdot \alpha_n(w_2w_1)(\kappa)|_p. \end{aligned} \quad \square$$

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DEPARTMENT OF MATHEMATICS, LANCASTER UNIVERSITY, BAILRIGG, LANCASTER, LA1 4YW, UNITED KINGDOM

E-mail address: g.elek@lancaster.ac.uk