TWISTED POINCARÉ DUALITY FOR POISSON HOMOLOGY AND COHOMOLOGY OF AFFINE POISSON ALGEBRAS

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ABSTRACT. This paper investigates the Poisson (co)homology of affine Poisson algebras. It is shown that there is a twisted Poincaré duality between their Poisson homology and cohomology. The relation between the Poisson (co)homology of an affine Poisson algebra and the Hochschild (co)homology of its deformation quantization is also discussed, which is similar to Kassel's result (1988) for homology and is a special case of Kontsevich's theorem (2003) for cohomology.

1. INTRODUCTION

Given a Poisson algebra, deformation quantization seeks a noncommutative algebra whose noncommutativity encodes the Poisson bracket on the previous Poisson algebra. There are deep relations between the Poisson (co)homology of a Poisson algebra with the Hochschild (co)homology of its deformation quantization. Brylinski's spectral sequence [Bry88] and HKR-theorem [HKR62] link the Poisson homology of a Poisson algebra with the Hochschild homology of its deformation quantization, and also provide a way to compute the Hochschild homology of some algebras [VdB94, Mar04, BP14]. For the algebra of smooth functions over $\mathbb{R}^n$, Kontsevich proved that its Poisson cohomology is isomorphic to the Hochschild cohomology of the corresponding deformation as graded algebras [Kon03]. In 1999, Xu obtained a Poincaré duality between Poisson cohomology and Poisson homology for a unimodular Poisson manifold [Xu99]. Van den Bergh duality establishes a twisted Poincaré duality between Hochschild cohomology and Hochschild homology for some Gorenstein algebras [VdB98]. Calabi-Yau algebras are important ones which satisfy Van den Bergh duality [Gin06]. Recently, Dolgushev showed that the deformation quantization algebra, of the algebra of functions on a Poisson variety, is a Calabi-Yau algebra if and only if the corresponding Poisson structure is unimodular [Dol09].

There are also many nonunimodular Poisson algebras and it would be interesting to search Xu's duality in a general sense. Such a duality was constructed in [ELW99, Hue99] in an abstract way. By dualizing the complexes for computing the Poisson (co)homology, Launois and Richard proposed a version of Xu's duality for some quadratic Poisson algebras [LR07]. Using similar methods, in this paper we will study this duality for affine Poisson algebras.

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Let V be a finite dimensional vector space and $S(V)$ be the symmetric algebra of V . A Poisson structure on $S(V)$ is called affine if the Poisson bracket is of degree ≤ 1 where the grading on $S(V)$ is given by setting $\deg x = 1$ for each $x \in V$ (see [LPV13] for details). Let $\mathfrak{g}$ be a Lie algebra of dimension n over the field $\mathbb{k}$, and let $f \in Z^2(\mathfrak{g}, \mathbb{k})$ be an arbitrary 2-cocycle, that is, $f : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{k}$ satisfying that

$$f(x, x) = 0 \quad \text{and} \quad f(x, [z, y]) + f(y, [z, x]) + f(z, [x, y]) = 0$$

for any $x, y, z \in \mathfrak{g}$. Then, there is a Poisson structure on the symmetric algebra $S(\mathfrak{g})$ with the Poisson bracket induced by

$$(1.1) \quad \{x, y\} = [x, y] + f(x, y)$$

for $x, y \in \mathfrak{g}$ ([Kas88]). Here, $[x, y]$ is of degree 1, and $f(x, y)$ of degree 0 in $S(\mathfrak{g})$. Hence, this is an affine Poisson structure. There is an associative algebra $U_f(\mathfrak{g}) = T_{\mathbb{k}}(\mathfrak{g})/I$, where I is the two-sided ideal generated by the elements

$$x \otimes y - y \otimes x - [x, y] - f(x, y).$$

This algebra was introduced by Sridharan [Sri61] and is often called Sridharan enveloping algebra. It is easy to see that Sridharan enveloping algebra is a deformation quantization of the above Poisson algebra (also a PBW deformation [Sri61]). The universal enveloping algebras and Weyl algebras are two important classes of Sridharan enveloping algebra. Moreover, according to [Sri61, Kas88, LPV13], any affine Poisson algebra can be achieved by Sridharan enveloping algebra of a Lie algebra $\mathfrak{g}$ and a 2-cocycle f .

Unless in general settings, we fix an n -dimensional Lie algebra $\mathfrak{g}$ and f is an arbitrary 2-cocycle, $S = S(\mathfrak{g})$ is the Poisson algebra with the Poisson bracket (1.1) and A is the Sridharan enveloping algebra $U_f(\mathfrak{g})$. Denote by ν the Nakayama automorphism of A . The main result of this paper reads

Theorem. *There exists a Poisson module S_ν such that there is a twisted Poincaré duality between the Poisson homology of S with coefficients in S_ν and the cohomology of S ,*

$$\mathrm{HP}_k(S, S_\nu) \cong \mathrm{HP}^{n-k}(S), \forall k \in \mathbb{N}.$$

The relation between the Poisson (co)homology of an affine Poisson algebra and the Hochschild (co)homology of its deformation quantization is discussed.

Theorem. *For each $k \in \mathbb{N}$,*

$$\mathrm{HH}_k(A, A^\nu) \cong \mathrm{HP}_k(S, S_\nu),$$

$$\mathrm{HH}^k(A) \cong \mathrm{HP}^k(S).$$

The paper is organized as follows. First, we retrieve the Nakayama automorphism and Van den Bergh duality for Sridharan enveloping algebras by a direct approach. Next, we recall some basic materials on Poisson algebras and their (co)homology, and construct a Poisson module providing a twisted Poincaré duality between Poisson homology and cohomology. In the last section, we study the relation between the Poisson (co)homology of an affine Poisson algebra and the Hochschild (co)homology of its deformation quantization.

Throughout the paper, $\mathbb{k}$ is a base field of characteristic 0 and all algebras are $\mathbb{k}$ -algebras; unadorned $\otimes$ means $\otimes_{\mathbb{k}}$.

2. VAN DEN BERGH DUALITY

Yekutieli [Ye00], and He, Van Oystaeyen and Zhang [HVZ10] compute the rigid dualizing complex for universal and Sridaharan enveloping algebras. Then, it results in a twisted Poincaré duality between the Hochschild cohomology and homology from a theorem of Van den Bergh [VdB98]. In this section we reprove it by computing the Nakayama automorphism. We denote by $\mathrm{HH}_*(A, N)$ the Hochschild homology of an algebra A with coefficients in the A^e -module N , and by $\mathrm{HH}_*(A)$ the Hochschild homology of an algebra A . We use similar notations for the Hochschild cohomologies.

Proposition 2.1 ([Kas88]). *Suppose that A is as in the introduction. Let $L_k(A) := A \otimes \bigwedge^k \mathfrak{g} \otimes A$ and $d_k : L_k(A) \rightarrow L_{k-1}(A)$ be defined by*

$$\begin{aligned} d_k(a \otimes x_1 \wedge \cdots \wedge x_k \otimes b) := & \sum_{1 \leq i \leq k} (-1)^{i+1} ax_i \otimes x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge x_k \otimes b \\ & - \sum_{1 \leq i \leq k} (-1)^{i+1} a \otimes x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge x_k \otimes x_i b \\ & + \sum_{1 \leq i < j \leq k} (-1)^{i+j} a \otimes [x_i, x_j] \wedge x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge \widehat{x_j} \wedge \cdots \wedge x_k \otimes b \end{aligned}$$

for any $a, b \in A$ and $x_1, \dots, x_k \in \mathfrak{g}$. Then $(L_*(A), d_*)$ is a projective resolution of A as an A^e -module.

When $f = 0$, it is just the Chevalley-Eilenberg resolution for a universal enveloping algebra (see [CE56], for example).

Proposition 2.2. *Let A be as in the introduction. Then,*

$$\mathrm{Ext}_{A^e}^i(A, A^e) \cong \begin{cases} 0, & i \neq n, \\ A^\nu, & i = n, \end{cases}$$

as A^e -modules, where ν is an automorphism of A induced by $\nu(x) = x + \mathrm{tr}(\mathrm{ad} x)$ for any $x \in \mathfrak{g}$, and A^ν denotes the A^e -module A with the action

$$(a \otimes b) \rightsquigarrow x := ax\nu(b).$$

Such an automorphism is called a Nakayama automorphism in terms of [BZ08].

Proof. Applying the functor $\mathrm{Hom}_{A^e}(-, A^e)$ to the sequence $(L_*(A), d_*)$ and by using the filtration on this complex which is induced by the filtration of A , we obtain a spectral sequence with $E_1^{p,q} = \mathrm{Ext}_{(\mathrm{gr} A)^e}^{p+q}(\mathrm{gr} A, (\mathrm{gr} A)^e)_p$ the homogeneous part of degree p of the cohomology group $\mathrm{Ext}_{(\mathrm{gr} A)^e}^{p+q}(\mathrm{gr} A, (\mathrm{gr} A)^e)$, and $E_\infty^n = \mathrm{Ext}_{A^e}^{p+q}(A, A^e)$. It is known that $\mathrm{gr} A$ is isomorphic to the symmetric algebra $S(\mathfrak{g})$. Hence, $\mathrm{Ext}_{A^e}^i(A, A^e) = 0$ for $i \neq n$. Next, we compute the cohomology group $\mathrm{Ext}_{A^e}^n(A, A^e)$. First, for each finite dimensional $\mathbb{k}$ -space V , there is an isomorphism

$$\theta : A \otimes V^* \otimes A \rightarrow \mathrm{Hom}_{A^e}(A \otimes V \otimes A, A^e)$$

as A^e -modules, where V^* denotes the dual as a $\mathbb{k}$ -space and $\theta(a \otimes \alpha \otimes b) : c \otimes x \otimes d \mapsto \alpha(x)cb \otimes ad$ for $\alpha \in V^*$, $x \in V$ and $a, b, c, d \in A$. Note that $\bigwedge^{n-1} \mathfrak{g}$ is a $\mathbb{k}$ -space of dimension n and so is $(\bigwedge^{n-1} \mathfrak{g})^*$. We choose $\{x_1, \dots, x_n\}$ as a basis for the $\mathbb{k}$ -algebra $\mathfrak{g}$. Then, $\{x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge x_n, i = 1, \dots, n\}$ is a basis for $\bigwedge^{n-1} \mathfrak{g}$. Denote $\{(\widehat{x_i})^*, i = 1, \dots, n\}$ as the basis for the dual space $(\bigwedge^{n-1} \mathfrak{g})^*$. Similarly,

$\bigwedge^n \mathfrak{g}$ is a $\mathbb{k}$ -space of dimension 1 with the base element $x_1 \wedge \cdots \wedge x_n$ and denote X for the base element of its dual space. Now, there is a commutative diagram

$$\begin{array}{ccc} \cdots \longrightarrow & A \otimes (\bigwedge^{n-1} \mathfrak{g})^* \otimes A & \xrightarrow{\partial} A \otimes (\bigwedge^n \mathfrak{g})^* \otimes A \\ & \theta \downarrow & \theta \downarrow \\ \cdots \longrightarrow & \mathrm{Hom}_{A^e}(A \otimes \bigwedge^{n-1} \mathfrak{g} \otimes A, A^e) & \xrightarrow{d^\vee} \mathrm{Hom}_{A^e}(A \otimes \bigwedge^n \mathfrak{g} \otimes A, A^e), \end{array}$$

where $d^\vee$ represents the morphism $\mathrm{Hom}_{A^e}(d, A^e)$ and ∂ is given by

$$\partial(a \otimes (\widehat{x_k})^* \otimes b) := (-1)^k (ax_k \otimes X \otimes b - a \otimes X \otimes x_k b + \mathrm{tr}(\mathrm{ad} x_k) a \otimes X \otimes b).$$

Define $\rho : A \otimes (\bigwedge^n \mathfrak{g})^* \otimes A \rightarrow A^\nu$ by $a \otimes X \otimes b \mapsto a\nu(b)$. Then, $\rho\partial = 0$. That is, we have a complex of A^e -modules

$$A \otimes (\bigwedge^{n-1} \mathfrak{g})^* \otimes A \xrightarrow{\partial} A \otimes (\bigwedge^n \mathfrak{g})^* \otimes A \xrightarrow{\rho} A^\nu \rightarrow 0.$$

By “filtration-gradation” techniques (see p. 288 Theorem 3 [NVO82], for example) and by the fact that $\mathrm{Ext}_{(\mathrm{gr} A)^e}^n(\mathrm{gr} A, (\mathrm{gr} A)^e) = \mathrm{gr} A$, the above sequence is exact. Therefore, $\mathrm{Ext}_{A^e}^n(A, A^e) = A^\nu$, which finishes the proof. $\square$

By using [VdB98, Theorem 1], we have the following

Proposition 2.3 (Van den Bergh duality). *Let A be as in the introduction. For every A^e -module N and $i \in \mathbb{Z}$, there is a functorial isomorphism*

$$\mathrm{HH}^i(A, N) \cong \mathrm{HH}_{n-i}(A, A^\nu \otimes N).$$

Epecially, there is a twisted Poincaré duality between the Hochschild cohomology and homology

$$\mathrm{HH}^i(A) \cong \mathrm{HH}_{n-i}(A, A^\nu).$$

3. TWISTED POINCARÉ DUALITY FOR POISSON HOMOLOGY AND COHOMOLOGY

In this section, we consider the Poisson (co)homology for affine Poisson algebras. We will construct a Poisson module and prove the main result. First, we recall some preliminaries including Poisson algebra, Poisson module and its (co)homology.

3.1. Poisson algebra and Poisson module. By definition a Poisson algebra is a commutative $\mathbb{k}$ -algebra S equipped with a bilinear map $\{-, -\} : S \otimes S \rightarrow S$ (Poisson bracket) satisfying

- (1) $\{-, -\}$ is a Lie bracket,
- (2) $\{-, -\}$ is a derivation in each variable.

Definition 3.2 ([Oh99]). A right Poisson module over the Poisson algebra S is a $\mathbb{k}$ -vector space M endowed with two bilinear maps $\cdot$ and $\{-, -\}_M : M \otimes S \rightarrow M$ such that

- (1) $(M, \cdot)$ is a right module over the commutative algebra S ,
- (2) $(M, \{-, -\}_M)$ is a right module over the Lie algebra $(S, \{-, -\})$,
- (3) $\{xa, b\}_M = \{x, b\}_M \cdot a + x \cdot \{a, b\}$ for all $a, b \in S$ and $x \in M$,
- (4) $\{x, ab\}_M = \{x, a\}_M \cdot b + \{x, b\}_M \cdot a$ for all $a, b \in S$ and $x \in M$.

The left Poisson module is defined in a similar way. For a Poisson algebra S , the space S has a natural right Poisson module structure. Now, let S be as in the introduction. Let $S_\nu = S(\mathfrak{g})$ as vector space endowed with the following two actions:

- $\cdot$ is just the multiplication of S ,
- $\{-, -\}_{S_\nu} : S_\nu \otimes S \rightarrow S_\nu$ is defined by

$$\{m, x_i\}_{S_\nu} := \{m, x_i\} + \text{tr}(\text{ad } x_i)m$$

for all $m \in S_\nu$ and $i = 1, \dots, n$, where $\{m, x_i\}$ is the Poisson bracket in S . Then, $(S_\nu, \cdot, \{-, -\}_{S_\nu})$ is a right Poisson module over S . As an S -module, S_ν is just S . But as a Poisson module, it differs from the right Poisson module S in general. Here, we can see the Poisson module structure on S_ν is similar to the twisted module A^ν appearing in the Van den Bergh duality.

3.3. Poisson homology and cohomology. Let M be a right Poisson module over a Poisson algebra S . Then there is a chain complex on the S -module $M \otimes_S \Omega^k(S)$ (see [Mas06]), where $\Omega^k(S)$ denotes the Kähler differential k -forms. The boundary operator $\partial_k : M \otimes_S \Omega^k(S) \rightarrow M \otimes_S \Omega^{k-1}(S)$ is defined by

$$\begin{aligned} \partial_k(m \otimes da_1 \wedge \dots \wedge da_k) &= \sum_{1 \leq i \leq k} (-1)^{i+1} \{m, a_i\}_M \otimes da_1 \wedge \dots \wedge \widehat{da_i} \wedge \dots \wedge da_k \\ &+ \sum_{1 \leq i < j \leq k} (-1)^{i+j} m \otimes d\{a_i, a_j\} \wedge da_1 \wedge \dots \wedge \widehat{da_i} \wedge \dots \wedge \widehat{da_j} \wedge \dots \wedge da_k, \end{aligned}$$

where d denotes the exterior differential. It is easy to see that ∂_k is well defined and that $\partial_{k-1}\partial_k = 0$. The homology of this complex is denoted by $\text{HP}_*(S, M)$ and is called the Poisson homology of the Poisson algebra S with coefficients in the Poisson module M .

Denote by $\chi^k(S)$ the space of all skew-symmetric $\mathbb{k}$ -linear maps $S^{\otimes k} \rightarrow S$ that are derivations in each argument. Here, a map $f : S^k \rightarrow S$ is called skew-symmetric if $f(a_1, \dots, a_k) = \varepsilon(\sigma)f(a_{\sigma_1}, \dots, a_{\sigma_k})$ for any permutation $\sigma \in S_k$, where $\varepsilon(\sigma)$ denotes its sign and σ_i means $\sigma(i)$. Then there is a cochain complex $(\chi^*(S), \delta_*)$ ([Lic77, Hue90]), where $\delta_k : \chi^k(S) \rightarrow \chi^{k+1}(S)$ is defined by

$$\begin{aligned} \delta_k(P)(y_0, y_1, \dots, y_k) &:= \sum_{0 \leq i \leq k} (-1)^i \{y_i, P(y_0, y_1, \dots, \widehat{y_i}, \dots, y_k)\} \\ &+ \sum_{0 \leq i < j \leq k} (-1)^{i+j} P(\{y_i, y_j\}, y_0, y_1, \dots, \widehat{y_i}, \dots, \widehat{y_j}, \dots, y_k) \end{aligned}$$

for all $P \in \chi^k(S)$. It is not hard to check that $\delta_k(P)$ belongs to $\chi^{k+1}(S)$ and that $\delta_{k+1}\delta_k = 0$. The cohomology of this complex is denoted by $\text{HP}^*(S)$ and is called the Poisson cohomology of the Poisson algebra S .

3.4. Twisted Poincaré duality. Let S be as in the introduction. It is known that the set $\chi^k(S)$ is isomorphic as a vector space to $\Omega^{n-k}(S)$ via an isomorphism $\dagger$ defined as follows. Denote by $S_{k, n-k}$ the set of $(k, n-k)$ -shuffle, i.e., those permutations $\sigma \in S_n$ such that $\sigma_1 < \sigma_2 < \dots < \sigma_k$ and $\sigma_{k+1} < \sigma_{k+2} < \dots < \sigma_n$. Then, for any $P \in \chi^k(S)$ let $\dagger(P)$ be the unique element of $\Omega^{n-k}(S)$ defined by

$$\dagger(P) = \sum_{\sigma \in S_{k, n-k}} \varepsilon(\sigma) P(x_{\sigma_1}, x_{\sigma_2}, \dots, x_{\sigma_k}) dx_{\sigma_{k+1}} \wedge \dots \wedge dx_{\sigma_n}.$$

Since $S_\nu = S$ as a vector space the map $\dagger$ will induce an isomorphism between the vector spaces $\chi^k(S)$ and $S_\nu \otimes_S \Omega^{n-k}(S)$. Now, we have the following diagram:

$$\begin{array}{ccc} \chi^k(S) & \xrightarrow{\dagger} & S_\nu \otimes_S \Omega^{n-k}(S) \\ \downarrow \delta_k & & \downarrow \partial'_{n-k} \\ \chi^{k+1}(S) & \xrightarrow{\dagger} & S_\nu \otimes_S \Omega^{n-k-1}(S) \end{array}$$

where the map ∂' stands for $\text{id} \otimes \partial$. First, we prove the following.

Proposition 3.5. *For any $P \in \chi^k(S)$, the following equality holds:*

$$\dagger \delta(P) = (-1)^{k+1} \partial' \dagger(P).$$

Proof. First, it follows from the definition that $\dagger \delta(P) = U_1 + V_1$, where

$$U_1 := \sum_{\substack{\sigma \in S_{k+1, n-k-1} \\ 1 \leq i \leq k+1}} \varepsilon(\sigma) (-1)^{i+1} \{x_{\sigma_i}, P(x_{\sigma_1}, \dots, \widehat{x_{\sigma_i}}, \dots, x_{\sigma_{k+1}})\} dx_{\sigma_{k+2}} \wedge \dots \wedge dx_{\sigma_n}$$

and

$$\begin{aligned} V_1 := & \sum_{\substack{\sigma \in S_{k+1, n-k-1} \\ 1 \leq i < j \leq k+1}} \varepsilon(\sigma) (-1)^{i+j} P(\{x_{\sigma_i}, x_{\sigma_j}\}, x_{\sigma_1}, \dots, \widehat{x_{\sigma_i}}, \dots, \widehat{x_{\sigma_j}}, \dots, x_{\sigma_{k+1}}) \\ & \otimes dx_{\sigma_{k+2}} \wedge \dots \wedge dx_{\sigma_n}. \end{aligned}$$

Since S_ν is a Poisson module, we have

$$\begin{aligned} \partial' \dagger(P) = & \sum_{\sigma \in S_{k, n-k}} \varepsilon(\sigma) \left(\sum_{i=1}^{n-k} (-1)^{i+1} \{P(x_{\sigma_1}, \dots, x_{\sigma_k}), x_{\sigma_{k+i}}\}_{S_\nu} dx_{\sigma_{k+1}} \wedge \dots \wedge \widehat{dx_{\sigma_{k+i}}} \dots \wedge dx_{\sigma_n} \right. \\ & + \sum_{1 \leq i < j \leq n-k} (-1)^{i+j} P(x_{\sigma_1}, \dots, x_{\sigma_k}) d\{x_{\sigma_{k+i}}, x_{\sigma_{k+j}}\} \\ & \left. \wedge dx_{\sigma_{k+1}} \wedge \dots \wedge \widehat{dx_{\sigma_{k+i}}} \dots \wedge \widehat{dx_{\sigma_{k+j}}} \dots \wedge dx_{\sigma_n} \right) \\ = & \sum_{\sigma \in S_{k, n-k}} \varepsilon(\sigma) \left(\sum_{i=1}^{n-k} (-1)^{i+1} \{P(x_{\sigma_1}, \dots, x_{\sigma_k}), x_{\sigma_{k+i}}\} dx_{\sigma_{k+1}} \wedge \dots \wedge \widehat{dx_{\sigma_{k+i}}} \wedge \dots \wedge dx_{\sigma_n} \right. \\ & + \sum_{i=1}^{n-k} (-1)^{i+1} \text{tr}(\text{ad } x_{\sigma_{i+k}}) P(x_{\sigma_1}, \dots, x_{\sigma_k}) dx_{\sigma_{k+1}} \wedge \dots \wedge \widehat{dx_{\sigma_{k+i}}} \wedge \dots \wedge dx_{\sigma_n} \\ & \left. + \sum_{1 \leq i < j \leq n-k} (-1)^{i+j} P(x_{\sigma_1}, \dots, x_{\sigma_k}) d\{x_{\sigma_{k+i}}, x_{\sigma_{k+j}}\} \wedge dx_{\sigma_{k+1}} \wedge \dots \wedge \widehat{dx_{\sigma_{k+i}}} \dots \wedge \widehat{dx_{\sigma_{k+j}}} \dots \wedge dx_{\sigma_n} \right). \end{aligned}$$

Denote the first term by U_2 , and let V_2 be the sum of the last two terms on the right-hand side in the above equality.

We will present this in four steps.

Step 1. We rewrite U_1 . For r distinct integers $i_1, \dots, i_r$, let $(i_1, \dots, i_r)$ be the cyclic permutation sending i_1 to i_2 , $\dots$, i_{r-1} to i_r and i_r to i_1 . Let $S_{k,1,n-k-1}$ be the set of those permutations $\tau \in S_n$ with $\tau_1 < \tau_2 < \dots < \tau_k$ and $\tau_{k+2} < \tau_{k+3} < \dots < \tau_n$. Then, the map $(\sigma, i) \in S_{k+1, n-k-1} \times \{1, 2, \dots, k+1\} \mapsto \tau := \sigma \circ (i, i+1, \dots, k+1) \in S_{k,1,n-k-1}$ is well defined and is a bijection. Moreover,

one has $\varepsilon(\tau) = (-1)^{k+1-i}\varepsilon(\sigma)$. Hence, by the changing of variables induced by this bijection, we have

$$U_1 = (-1)^k \sum_{\tau \in S_{k,1,n-k-1}} \varepsilon(\tau) \{x_{\tau_{k+1}}, P(x_{\tau_1}, \dots, x_{\tau_k})\} dx_{\tau_{k+2}} \wedge \dots \wedge dx_{\tau_n}.$$

Step 2. We rewrite V_1 . First, note that $P(y_1, \dots, y_k) = 0$ if there exists $i \neq j$ such that $y_i = y_j$ or if some $y_i \in k$ by definition. Suppose that $[x_i, x_j] := \sum_{1 \leq l \leq n} a_l^{i,j} x_l$; then $\{x_i, x_j\} = \sum_{1 \leq l \leq n} a_l^{i,j} x_l + f(x_i, x_j)$. Obviously, $a_l^{i,j} = -a_l^{j,i}$ by the antisymmetry of the Poisson bracket. Then,

$$\begin{aligned} V_1 &:= \sum_{\substack{\sigma \in S_{k+1,n-k-1} \\ 1 \leq i < j \leq k+1}} \varepsilon(\sigma) (-1)^{i+j} P\left(\sum_{1 \leq l \leq n} a_l^{\sigma_i, \sigma_j} x_l, x_{\sigma_1}, \dots, \widehat{x_{\sigma_i}}, \dots, \widehat{x_{\sigma_j}}, \dots, x_{\sigma_{k+1}}\right) \\ &\quad \otimes dx_{\sigma_{k+2}} \wedge \dots \wedge dx_{\sigma_n} \\ &= \sum_{\substack{\sigma \in S_{k+1,n-k-1} \\ 1 \leq i < j \leq k+1}} \varepsilon(\sigma) (-1)^{i+j} P(a_{\sigma_i}^{\sigma_i, \sigma_j} x_{\sigma_i}, x_{\sigma_1}, \dots, \widehat{x_{\sigma_i}}, \dots, \widehat{x_{\sigma_j}}, \dots, x_{\sigma_{k+1}}) \\ &\quad \otimes dx_{\sigma_{k+2}} \wedge \dots \wedge dx_{\sigma_n} \\ &\quad + \sum_{\substack{\sigma \in S_{k+1,n-k-1} \\ 1 \leq i < j \leq k+1}} \varepsilon(\sigma) (-1)^{i+j} P(a_{\sigma_j}^{\sigma_i, \sigma_j} x_{\sigma_j}, x_{\sigma_1}, \dots, \widehat{x_{\sigma_i}}, \dots, \widehat{x_{\sigma_j}}, \dots, x_{\sigma_{k+1}}) \\ &\quad \otimes dx_{\sigma_{k+2}} \wedge \dots \wedge dx_{\sigma_n} \\ &= \sum_{\sigma \in S_{k+1,n-k-1}} \varepsilon(\sigma) \sum_{i=1}^k (-1)^i \sum_{j=i+1}^{k+1} a_{\sigma_j}^{\sigma_i, \sigma_j} P(x_{\sigma_1}, \dots, \widehat{x_{\sigma_i}}, \dots, x_{\sigma_{k+1}}) \\ &\quad \otimes dx_{\sigma_{k+2}} \wedge \dots \wedge dx_{\sigma_n} \\ &\quad + \sum_{\sigma \in S_{k+1,n-k-1}} \varepsilon(\sigma) \sum_{j=2}^{k+1} (-1)^{j+1} \sum_{i=1}^{j-1} a_{\sigma_i}^{\sigma_i, \sigma_j} P(x_{\sigma_1}, \dots, \widehat{x_{\sigma_j}}, \dots, x_{\sigma_{k+1}}) \\ &\quad \otimes dx_{\sigma_{k+2}} \wedge \dots \wedge dx_{\sigma_n} \\ &= \sum_{\sigma \in S_{k+1,n-k-1}} \varepsilon(\sigma) \sum_{i=1}^k (-1)^i \sum_{j=i+1}^{k+1} a_{\sigma_j}^{\sigma_i, \sigma_j} P(x_{\sigma_1}, \dots, \widehat{x_{\sigma_i}}, \dots, x_{\sigma_{k+1}}) \\ &\quad \otimes dx_{\sigma_{k+2}} \wedge \dots \wedge dx_{\sigma_n} \\ &\quad + \sum_{\sigma \in S_{k+1,n-k-1}} \varepsilon(\sigma) \sum_{i=2}^{k+1} (-1)^{i+1} \sum_{j=1}^{i-1} a_{\sigma_j}^{\sigma_j, \sigma_i} P(x_{\sigma_1}, \dots, \widehat{x_{\sigma_i}}, \dots, x_{\sigma_{k+1}}) \\ &\quad \otimes dx_{\sigma_{k+2}} \wedge \dots \wedge dx_{\sigma_n} \\ &= \sum_{\sigma \in S_{k+1,n-k-1}} \varepsilon(\sigma) \sum_{i=1}^{k+1} (-1)^i \sum_{j=1}^{k+1} a_{\sigma_j}^{\sigma_i, \sigma_j} P(x_{\sigma_1}, \dots, \widehat{x_{\sigma_i}}, \dots, x_{\sigma_{k+1}}) \otimes \\ &\quad \otimes dx_{\sigma_{k+2}} \wedge \dots \wedge dx_{\sigma_n}. \end{aligned}$$

Then, using the same technique as in the previous step, we get

$$V_1 = \sum_{\tau \in S_{k,1,n-k-1}} \varepsilon(\tau) (-1)^{k+1} \sum_{j=1}^k a_{\tau_j}^{\tau_{k+1}, \tau_j} P(x_{\tau_1}, \dots, x_{\tau_k}) \otimes dx_{\tau_{k+2}} \wedge \dots \wedge dx_{\tau_n}.$$

Step 3. By a similar progress as in Step 1, one can easily get

$$U_2 = \sum_{\tau \in S_{k,1,n-k-1}} \varepsilon(\tau) \{P(x_{\tau_1}, \dots, x_{\tau_k}), x_{\tau_{k+1}}\} dx_{\tau_{k+2}} \wedge \dots \wedge dx_{\tau_n}.$$

Hence, $U_1 = (-1)^{k+1} U_2$.

Step 4. We rewrite V_2 and conclude. Note that $\text{tr}(\text{ad } x_i) = \sum_{1 \leq l \leq n} a_l^{i,l}$. To simplify the notation, write Q for $P(x_{\sigma_1}, \dots, x_{\sigma_k})$. We obtain

$$\begin{aligned} V_2 &= \sum_{\sigma \in S_{k,n-k}} \varepsilon(\sigma) \left(\sum_{i=1}^{n-k} (-1)^{i+1} \text{tr}(\text{ad } x_{\sigma_{i+k}}) Q dx_{\sigma_{k+1}} \wedge \dots \wedge \widehat{dx_{\sigma_{k+i}}} \wedge \dots \wedge dx_{\sigma_n} \right. \\ &\quad + \sum_{1 \leq i < j \leq n-k} (-1)^{i+j} Q d \left(\sum_{1 \leq l \leq n} a_l^{\sigma_{k+i}, \sigma_{k+j}} x_l + f(x_i, x_j) \right) \\ &\quad \left. \wedge dx_{\sigma_{k+1}} \wedge \dots \wedge \widehat{dx_{\sigma_{k+i}}} \dots \widehat{dx_{\sigma_{k+j}}} \dots \wedge dx_{\sigma_n} \right) \\ &= \sum_{\sigma \in S_{k,n-k}} \varepsilon(\sigma) \left(\sum_{i=1}^{n-k} (-1)^{i+1} \text{tr}(\text{ad } x_{\sigma_{i+k}}) Q dx_{\sigma_{k+1}} \wedge \dots \wedge \widehat{dx_{\sigma_{k+i}}} \wedge \dots \wedge dx_{\sigma_n} \right. \\ &\quad + \sum_{i=1}^{n-k-1} \sum_{j=i+1}^{n-k} (-1)^{j+1} a_{\sigma_{i+k}}^{\sigma_{k+i}, \sigma_{k+j}} Q dx_{\sigma_{k+1}} \wedge \dots \wedge \widehat{dx_{\sigma_{k+j}}} \wedge \dots \wedge dx_{\sigma_n} \\ &\quad + \sum_{j=2}^{n-k} \sum_{i=1}^{j-1} (-1)^i a_{\sigma_{j+k}}^{\sigma_{k+i}, \sigma_{k+j}} Q dx_{\sigma_{k+1}} \wedge \dots \wedge \widehat{dx_{\sigma_{k+i}}} \wedge \dots \wedge dx_{\sigma_n} \left. \right) \\ &= \sum_{\sigma \in S_{k,n-k}} \varepsilon(\sigma) \left(\sum_{i=1}^{n-k} (-1)^{i+1} \sum_{l=1}^n a_l^{\sigma_{k+i}, l} Q dx_{\sigma_{k+1}} \wedge \dots \wedge \widehat{dx_{\sigma_{k+i}}} \wedge \dots \wedge dx_{\sigma_n} \right. \\ &\quad + \sum_{i=1}^{n-k} \sum_{j=1}^{n-k} (-1)^i a_{\sigma_{j+k}}^{\sigma_{k+i}, \sigma_{k+j}} Q dx_{\sigma_{k+1}} \wedge \dots \wedge \widehat{dx_{\sigma_{k+i}}} \wedge \dots \wedge dx_{\sigma_n} \left. \right) \\ &= \sum_{\sigma \in S_{k,n-k}} \varepsilon(\sigma) \sum_{i=1}^{n-k} (-1)^{i+1} \sum_{l=1}^k a_{\sigma_l}^{\sigma_{k+i}, \sigma_l} Q dx_{\sigma_{k+1}} \wedge \dots \wedge \widehat{dx_{\sigma_{k+i}}} \wedge \dots \wedge dx_{\sigma_n} \\ &= \sum_{\tau \in S_{k,1,n-k-1}} \varepsilon(\tau) \sum_{l=1}^k a_{\tau_l}^{\tau_{k+1}, \tau_l} P(x_{\tau_1}, \dots, x_{\tau_k}) dx_{\tau_{k+1}} \wedge \dots \wedge \widehat{dx_{\tau_{k+i}}} \wedge \dots \wedge dx_{\tau_n}. \end{aligned}$$

Thus, $V_1 = (-1)^{k+1} V_2$. We must point out that in Step 3 and Step 4 we use the bijection map $(\sigma, i) \in S_{k,n-k} \times \{1, 2, \dots, n-k\} \mapsto \tau := \sigma \circ (k+i, k+i-1, \dots, k+1) \in S_{k,1,n-k}$ and $\varepsilon(\tau) = (-1)^{i-1} \varepsilon(\sigma)$.

Therefore, we obtain that $\dagger \delta(P) = (-1)^{k+1} \partial' \dagger(P)$. $\square$

Now, the diagram considered is commutative up to sign. As a consequence, the following theorem is proved.

Theorem 3.6. *For all $k \in \mathbb{N}$, we have $\mathrm{HP}_k(S, S_\nu) \cong \mathrm{HP}^{n-k}(S)$.*

4. APPLICATION

Suppose that an algebra A has a filtration such that the associated graded algebra $\mathrm{gr} A$ is commutative and smooth. Brylinski introduced a spectral sequence $E^1 := \mathrm{HH}_*(\mathrm{gr} A) \Rightarrow \mathrm{HH}_*(A)$ in [Bry88]. Moreover, by HKR-theorem [HKR62], the E^2 page of this spectral sequence is equal to the Poisson homology. Kassel showed that when the associated graded algebra is a symmetric algebra, this spectral sequence degenerates. Therefore, $\mathrm{HH}_*(A) \cong \mathrm{HP}_*(S)$ ([Kas88]). The following is similar to Kassel's result.

Proposition 4.1. *Let A and S be as in the introduction. Then, for all $k \in \mathbb{N}$, we have*

$$\mathrm{HH}_k(A, A^\nu) \cong \mathrm{HP}_k(S, S_\nu).$$

Proof. Applying the functor $A^\nu \otimes_{A^e} -$ to the resolution in Proposition 2.1 and by the canonical isomorphism $A^\nu \otimes_{A^e} (A \otimes V \otimes A) \cong A^\nu \otimes V$, the Hochschild homology $\mathrm{HH}_*(A, A^\nu)$ will be equal to the homology group of the following chain complex:

$$\cdots \rightarrow A^\nu \otimes \bigwedge^k \mathfrak{g} \xrightarrow{b'} A^\nu \otimes \bigwedge^{k-1} \mathfrak{g} \rightarrow \cdots \rightarrow A^\nu,$$

where the boundary b' is

$$\begin{aligned} b'(a \otimes x_1 \wedge \cdots \wedge x_k) &:= \sum_{1 \leq i \leq k} (-1)^{i+1} (a\nu(x_i) - x_i a) \otimes x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge x_k \\ &+ \sum_{1 \leq i < j \leq k} (-1)^{i+j} a \otimes [x_i, x_j] \wedge x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge \widehat{x_j} \cdots \wedge x_k. \end{aligned}$$

Recall that ν is an automorphism of A induced by $\nu(x) = x + \mathrm{tr}(\mathrm{ad} x)$ for any $x \in \mathfrak{g}$. Hence, $a\nu(x_i) - x_i a = ax_i - x_i a + \mathrm{tr}(\mathrm{ad} x_i)a$.

Obviously, the S -module of the Kähler differential k -forms $\Omega^k(S)$ is isomorphic to $S \otimes \bigwedge^k \mathfrak{g}$ as S -modules. Hence, the Poisson homology group $\mathrm{HP}_*(S, S_\nu)$ will be equal to the homology group of the following chain complex:

$$\cdots \rightarrow S_\nu \otimes \bigwedge^k \mathfrak{g} \xrightarrow{\delta'} S_\nu \otimes \bigwedge^{k-1} \mathfrak{g} \rightarrow \cdots \rightarrow S_\nu,$$

where the boundary δ' is

$$\begin{aligned} \delta'(a \otimes x_1 \wedge \cdots \wedge x_k) &:= \sum_{1 \leq i \leq k} (-1)^{i+1} \{a, x_i\}_{S_\nu} \otimes x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge x_k \\ &+ \sum_{1 \leq i < j \leq k} (-1)^{i+j} a \otimes [x_i, x_j] \wedge x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge \widehat{x_j} \cdots \wedge x_k. \end{aligned}$$

Next, we claim that the following diagram is commutative:

$$\begin{array}{ccc} S_\nu \otimes \bigwedge^k \mathfrak{g} & \xrightarrow{\epsilon \otimes \mathrm{id}} & A^\nu \otimes \bigwedge^k \mathfrak{g} \\ \downarrow \delta'_k & & \downarrow b'_k \\ S_\nu \otimes \bigwedge^{k-1} \mathfrak{g} & \xrightarrow{\epsilon \otimes \mathrm{id}} & A^\nu \otimes \bigwedge^{k-1} \mathfrak{g} \end{array}$$

where $\epsilon : S \rightarrow A$ is an isomorphism of $\mathbb{k}$ -vector space by the Poincaré-Birkhoff-Witt theorem and is defined by

$$\epsilon(x_1 x_2 \cdots x_m) := \frac{1}{m!} \sum_{\sigma \in S_m} x_{\sigma_1} x_{\sigma_2} \cdots x_{\sigma_m}.$$

According to [Kas88, Lemma 6], the equation

$$\epsilon(\{g, x_i\}) = \epsilon(g)x_i - x_i\epsilon(g)$$

holds in A for all $g \in S$ and $x_i \in \mathfrak{g}$. Then, the diagram is commutative and we get the desired conclusion. $\square$

In [Kon03], Kontsevich proved that the Poisson cohomology of the algebra of smooth functions over $\mathbb{R}^n$ is isomorphic to the Hochschild cohomology of the corresponding deformation quantization. Here, we get the following which is a special case of Kontsevich's result.

Corollary 4.2. *Let A and S be as above. Then, for each $k \in \mathbb{N}$ we have*

$$\mathrm{HH}^k(A) \cong \mathrm{HP}^k(S).$$

Proof. This is a direct consequence of Propositions 2.3 and 4.1 and Theorem 3.6. $\square$

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