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SOME UNUSUAL EPICOMPLETE ARCHIMEDEAN LATTICE-ORDERED GROUPS

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Abstract. An Archimedean l-group is *epicomplete* if it is divisible and σ-complete, both laterally and conditionally. Under various circumstances it has been shown that epicompleteness implies the existence of a compatible reduced f-ring multiplication; the question has arisen whether or not this is always true. We show that a set-theoretic condition weaker than the continuum hypothesis implies "not", and conjecture the converse. The examples also fail decent representation and existence of some other compatible operations.

1. Introduction

Arch is the category of Archimedean l-groups, with l-group homomorphisms, **frA** is the category of f-rings which are reduced (semi-prime) and Archimedean, with l-ring homomorphisms, and $\mathbf{frA} \xrightarrow{F} \mathbf{Arch}$ is the forgetful functor. "$G \in F(\mathbf{frA})$" means $G \in \mathbf{Arch}$ and for some multiplication $*$ compatible with the l-group structure of G, $(G, *) \in \mathbf{frA}$.

In a general category, an object A is called epicomplete (EC) if $\varphi : A \to \bullet$ monic and epic implies φ is an isomorphism.

In **Arch**, monics are one-to-one, and [BH90a] shows: G is EC if and only if G is divisible, laterally and conditionally σ-complete; EC is monoreflective in **Arch**, and thus is the least monoreflective subcategory. (See [HS07] regarding "monoreflective".)

Question 1.1 ([Hol03] and [BHJK09], p. 166). G EC $\Rightarrow G \in F(\mathbf{frA})$?

This has arisen because the implication *does* hold under some extra assumptions (described in 1.6 below). We shall show that the following set-theoretic hypothesis implies "No" (which was conjectured in [BHJK09]).

Hypothesis 1.2. *For a set S of cardinal $|S| = \aleph_1$, there is a family $\mathcal{W}$ of subsets of S with the properties*

(1) $\forall A \in \mathcal{W}, |A| = \aleph_1$,
(2) $\forall A \neq B$ in $\mathcal{W}$, $|A \cap B| \leq \aleph_0$, and
(3) $|\mathcal{W}| = 2^{\aleph_1}$.

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Baumgartner has shown (see [Jec03], pp. 579, 582): In ZFC,

$$2^{\aleph_0} = \aleph_1 \Rightarrow (2^{\aleph_0} < 2^{\aleph_1} \text{ and } 2^{\aleph_0} < \aleph_{\omega_1}) \Rightarrow 1.2 \Rightarrow 2^{\aleph_0} < 2^{\aleph_1}.$$

Thus Hypothesis 1.2 is weaker than the continuum hypothesis (and consistent with ZFC). In the following, $|S| = \aleph_1$ and $\mathbb{R}^S$ (all functions $f : S \to \mathbb{R}$) is given pointwise $+$ and $\leq$; $\mathbb{R}^S \in \mathbf{Arch}$.

Theorem 1.3 (4.1 below). *If Hypothesis 1.2 holds, then there is an EC $G \leq \mathbb{R}^S$ (constructed from $\mathcal{W}$) with $G \notin F(\mathbf{frA})$.*

Conjecture 1.4. *If there is an EC $G \notin F(\boldsymbol{frA})$, then Hypothesis 1.2 holds.*

The rest of this section explains where Question 1.1 comes from, and is technically not required for the proof of Theorem 1.3. Question 1.1 is closed related to issues of representation of Archimedean l-structures in lattices

$$D(Y) = \{f \in C(Y, [-\infty, +\infty]) | f^{-1}\mathbb{R} \text{ dense in } Y\}.$$

Here Y is a Tychonoff space, $[-\infty, +\infty] = \mathbb{R} \cup \{\pm\infty\}$ with the obvious order and topology, and $D(Y)$ is given the pointwise order. Addition is partially defined by $f_1 + f_2 = f_3$ means $f_1(x) + f_2(x) = f_3(x)$ when all three are real, and likewise multiplication. This $+$ (respectively, $\cdot$) is fully defined if and only if each dense cozero set is C^*-embedded in Y [HJ61]. This condition defines quasi-F (QF) spaces; Then $D(Y) \in \mathbf{frA}$ with the constant function 1_Y, the ring identity.

Y is called extremally (resp. basically) disconnected (ED, resp. BD) if each open (resp. cozero) set has open closure. (See [GJ76]). We have ED $\Rightarrow$ BD $\Rightarrow$ QF, and $D(Y)$ is an l-group which is laterally and conditionally complete (σ-complete) if and only if Y is ED (resp., BD). These extend the Nakano-Stone Theorems for $C(X)$. (See [BHJK09]).

In particular, $D(Y)$ is an **Arch**-object which is EC if and only if Y is BD. (See [BH90a]).

By a *representation* of $G \in \mathbf{Arch}$ (resp. $\mathbf{frA}$), we mean a one-to-one map $\epsilon : G \to D(Y)$ for which $\epsilon(G)$ is closed under the partial operations in $D(Y)$ requisite that $\epsilon(G) \in \mathbf{Arch}$ (resp. $\mathbf{frA}$), and $G \xrightarrow{\epsilon} \epsilon(G)$ is an l-group (resp. l-ring) isomorphism.

A Johnson representation of $G \in \mathbf{Arch}$ is a representation $G \leq D(Y)$ (suppressing the ϵ) satisfying

(J_Y) $y \notin F$ closed in $Y \Rightarrow \exists g \in G$ with $0 < g(y) < +\infty$, and

(J_∞) $\forall g \in G$, the Čech-Stone extension has $\beta g(\beta Y - Y) \subseteq \{0, \pm\infty\}$.

(It follows that Y is locally compact). We may say "J-representation", and indicate such a situation as $G \overset{J}{\leq} D(Y)$. It is so-called because:

Theorem 1.5 ([Joh62], [Joh07]). *For each $(G, *) \in \mathbf{frA}$, there is $G \overset{J}{\leq} D(Y)$ with G closed in $D(Y)$ under the partial $\cdot$.*

Consider Y locally compact. Let $D_0(Y) \equiv \{f \in D(Y) | \beta f(\beta Y - Y) = \{0\}\}$ (the functions that vanish at ∞). If Y is BD, then $D_0(Y) \in \mathbf{Arch}$, is conditionally σ-complete ([BH90a], 3.1), and is laterally σ-complete if and only if either Y is compact (whence $D_0(Y) = D(Y)$) or Y is not compact and in the one-point compactification $Y \cup \{\alpha\}$, α is a P-point ([BH90a], §4,5).

For the nonce, a "good" representation of EC G is an isomorphism $G \approx D_0(Y)$, for Y locally compact and BD (thus, necessarily, α is a P-point).

The following sums up partial results inspiring, and giving four versions of, Question 1.1. Ball's truncations [Bal13] will be discussed in §5.

Theorem 1.6. *Suppose G is EC.*

(1) *These are equivalent.*
   (a) $G \in F(\mathbf{frA})$.
   (b) *G has a Johnson representation.*
   (c) *G has a good representation.*
   (d) *G has a compatible truncation.*

(2) *If G has any weak unit, these four conditions obtain.*

[BH90b] shows (2): For G with weak unit e, there is X compact BD, with $G \approx D(X)$ and $e \mapsto 1_X$.

Regarding (1): Johnson's 1.4 shows (a) $\Rightarrow$ (b), and Ball's 5.1 (below) shows (d) $\Rightarrow$ (b) (neither needing "G EC"). From §5, (c) $\Rightarrow$ (d) is clear. [BH90a] shows (b) $\Rightarrow$ (c), at considerable labor. (c) $\Rightarrow$ (a) is evident.

We mentioned that EC is monoreflective in **Arch**. In that regard, there is more, as follows.

Let $\mathbf{W}$ be the (non-full) subcategory of **Arch** with objects (G, e_G), e_G is a positive weak unit of G, and morphisms $(G, e_G) \xrightarrow{\varphi} (H, e_H)$ the **Arch**-morphisms with $\varphi(e_G) = e_H$. In $\mathbf{W}$, the remarkably canonical Yosida representation has many consequences, which are striking, and food for thought, in comparison with **Arch**. This representation of a (G, e_G) can be described as the essentially unique $G \leq D(YG)$ which is a J-representation, YG is compact, and $e_G \mapsto 1$. See [HR77].

EC in $\mathbf{W}$ has "the same" characterization as in **Arch**, divisible, laterally and conditionally σ-complete. It results that (G, e_G) is $\mathbf{W}$-EC if and only if $(G, e_G) \approx (D(X), 1_X)$, for X compact BD. (The X is the YG above. We alluded to this in Theorem 1.6 (2), above.) Similar to **Arch**, $\mathbf{W}$-EC is the smallest monoreflective subcategory of $\mathbf{W}$. (This information is in [BH90b, BH90a].)

Let $\beta^{\mathbf{Arch}}$ and $\beta^{\mathbf{W}}$ denote their "EC-monoreflection" functors. In [BH90a], we find a concrete calculation/representation/description of $G \leq \beta^{\mathbf{W}}G$ (suppressing units), in terms of $G \leq D(YG)$. In [BHJK05], we found a quite similar description of $G \leq \beta^{\mathbf{Arch}}G$, for $G = C_K(Y)$, or $C_0(Y)$ the functions of compact support, or vanishing at ∞, on locally compact Y, and these $\beta^{\mathbf{Arch}}G$ have the features in Theorem 1.6 (1). Note that $C_K(Y)$ and $C_0(Y)$ are, as presented, in J-representation. One imagines that the descriptions of these $\beta^{\mathbf{Arch}}G$ alluded to above can be carried through for any G in a J-representation, and that if $(G, *) \in \mathbf{frA}$, then there is $*'$ on $\beta^{\mathbf{Arch}}G$ with $(G, *) \leq (\beta^{\mathbf{Arch}}G, *')$ in $\mathbf{frA}$.

2. Some epicomplete l-groups

S is a set of cardinal $|S| = \aleph_1$.

$\mathcal{W}$ is an uncountable family of uncountable subsets of S for which, if $A, B \in \mathcal{W}$ and $A \neq B$, then $|A \cap B| \leq \aleph_0$. (There are no further assumptions on $\mathcal{W}$ in this section.) $\mathbb{R}^S$ carries pointwise $+$ and $\leq$; $\mathbb{R}^S \in \mathbf{Arch}$. $P = \{f \in \mathbb{R}^S | f(x) > 0 \ \forall x \in S\}$. This is the set of positive weak units in $\mathbb{R}^S$.

$\mathcal{W} \xrightarrow{\gamma} P$ is any function.

In this section, we construct epicomplete $G(\mathcal{W}, \gamma) \leq \mathbb{R}^S$. The construction is modeled on the elaboration in [HJ10] of the example in [CM90]. In §4, we will specify the $(\mathcal{W}, \gamma)$ to make $G(\mathcal{W}, \gamma) \notin F(\mathbf{frA})$.

We adopt the following notations.

For $U, V \subseteq S$, $U \overset{*}{\subseteq} V$ means there is countable T with $U \subseteq V \cup T$, and $U \overset{*}{=} V$ means $U \overset{*}{\subseteq} V$ and $V \overset{*}{\subseteq} U$. (For $A \neq B$ in $\mathcal{W}$, $A \cap B \overset{*}{=} \emptyset$.)

For $f, g \in \mathbb{R}^S$, $f \overset{*}{\leq} g$ means there is countable T with $f(x) \leq g(x)$, for $x \in S - T$, and $f \overset{*}{=} g$ means $f \overset{*}{\leq} g$ and $g \overset{*}{\leq} f$.

For $\mathcal{A} \subseteq \mathcal{W}$ with $|\mathcal{A}| \leq \aleph_0$, let $c(\mathcal{A}) = \cup\{A \cap B | A \neq B \text{ in } \mathcal{A}\}$. Observe $|c(\mathcal{A})| \leq \aleph_0$.

For $A \in \mathcal{A}$, let $\dot{A} = A - c(\mathcal{A})$. Let $\dot{\mathcal{A}} = \{\dot{A} | A \in \mathcal{A}\}$. Observe $\dot{A} \cap \dot{B} = \emptyset$ for $A \neq B$ in $\mathcal{A}$.

Again with $\mathcal{A} \subseteq \mathcal{W}$ and $|\mathcal{A}| \leq \aleph_0$, let $r : S \to \mathbb{R}$ have the properties: for $x \notin \bigcup_{\mathcal{A}} \dot{A}$, $r(x) = 0$; for $A \in \mathcal{A}$, $r|\dot{A}$ is constant, denoted r_A.

For $U \subseteq S$, let χ_U be the characteristic function of U.

Definition 2.1. For $(\mathcal{A}, r)$ as above, let

$$u(\mathcal{A}, r) \equiv \sum_{\mathcal{A}} r_A \chi_{\dot{A}} \gamma(A).$$

(Here, the $\sum$ is pointwise in $\mathbb{R}^S$. The supports of the summands are disjoint, so the $\sum_{\mathcal{A}}$ makes sense.)

$$G(\mathcal{W}, \gamma) \equiv \{g \in \mathbb{R}^S | \text{ there is } (\mathcal{A}, r) \text{ with } g \overset{*}{=} u(\mathcal{A}, r)\}.$$

Theorem 2.2. *$G(\mathcal{W}, \gamma)$ is a sub-l-group of $\mathbb{R}^S$, without weak units, and is epicomplete.*

Proof of 2.2. In the following, $\mathcal{A}, \mathcal{B}, \ldots$ always denote countable subsets of $\mathcal{W}$.

The proof has a number of steps. The first may not be completely necessary, but is articulated to avoid confusion.

Lemma 2.3. *The following are equivalent.*

(1) $\mathcal{A} \subseteq \mathcal{B}$.
(2) $\bigcup_{\mathcal{A}} A \subseteq \bigcup_{\mathcal{B}} B$.
(3) $\bigcup_{\mathcal{A}} \dot{A} \subseteq \bigcup_{\mathcal{B}} \dot{B}$.
(2') $\bigcup_{\mathcal{A}} A \overset{*}{\subseteq} \bigcup_{\mathcal{B}} B$.
(3') $\bigcup_{\mathcal{A}} \dot{A} \overset{*}{\subseteq} \bigcup_{\mathcal{B}} \dot{B}$.

Proof. (Note (3) says $\bigcup_{\mathcal{A}}(A - c(\mathcal{A})) \subseteq \bigcup_{\mathcal{B}}(B - c(\mathcal{B}))$, and $\mathcal{A} \subseteq \mathcal{B} \Rightarrow c(\mathcal{A}) \supseteq c(\mathcal{B})$.)

These implications are obvious: (1)⇒(2)⇒(2'); (3)⇒(3'). Also, for any $\mathcal{A}$, $\bigcup_{\mathcal{A}} A = \bigcup_{\mathcal{A}} \dot{A} \cup c(A)$; so (3')⇒ (2').

(2')⇒(1). Suppose (1) fails, with $A \in \mathcal{A} - \mathcal{B}$. Thus, $\forall B \in \mathcal{B}$, $A \cap B$ is countable, and so is $A \cap \bigcup_{\mathcal{B}} B = \bigcup_{\mathcal{B}}(A \cap B)$ (since $\mathcal{B}$ is countable). Then, for any countable T, $A \cap (\bigcup_{\mathcal{B}} B \cup T)$ is countable, so $A \not\subseteq \bigcup_{\mathcal{B}} B \cup T$ (since $|A| \geq \aleph_1$). So (2') fails.

(1)⇒(3). Suppose $A \in \mathcal{A}$. Then, $A - c(\mathcal{A}) \overset{*}{=} A - c(\mathcal{B})$, so $\bigcup_{\mathcal{A}}(A - c(\mathcal{A})) \overset{*}{=} \bigcup_{\mathcal{A}}(A - c(\mathcal{B})) \subseteq \bigcup_{\mathcal{B}}(B - c(\mathcal{B}))$ (the last $\subseteq$ because $\mathcal{A} \subseteq \mathcal{B}$).

□

In the following, $G(\mathcal{W}, \gamma)$ is denoted G. We shall prove below that $G \leq \mathbb{R}^S$ (i.e., is a sub-l-group,) and G is EC.

Granted $G \leq \mathbb{R}^S$, G has no weak units:

If $g \in G^+$ has $g^\perp = \{0\}$, then $\forall x \in S$, $g \wedge \chi_x > 0$, which means $g(x) > 0$, so $\operatorname{coz} g = S$. But $g \overset{*}{=} u(\mathcal{A}, r)$, so $S = \operatorname{coz} g \overset{*}{=} \operatorname{coz} u(\mathcal{A}, r) \subseteq \bigcup_{\mathcal{A}} \dot{A}$. But $|\mathcal{A}| \leq \aleph_0$

and $\mathcal{W}$ is uncountable, so there is $B \in \mathcal{W}$ with $B \notin \mathcal{A}$, thus $|B - \bigcup_{\mathcal{A}} \dot{A}| \geq \aleph_1$. Contradiction.

Lemma 2.4. *Suppose $\mathcal{A} \subseteq \mathcal{B}$. Given $u(\mathcal{A}, r)$, there is $\overline{r}$ with $u(\mathcal{B}, \overline{r}) \overset{*}{=} u(\mathcal{A}, r)$ as follows.*

$\overline{r}(x) = 0$ if $x \notin \bigcup_{\mathcal{B}} \dot{B}$ or if $x \in \dot{B}$ for $B \in \mathcal{B} - \mathcal{A}$.

If $A \in \mathcal{A}$, $\overline{r}|(A - c(\mathcal{B}))| = r_A$ (the constant value of r on $A - c(\mathcal{A})$; i.e. $\overline{r}_A = r_A$).

Proof. $\bigcup_{\mathcal{A}}(A - c(\mathcal{B})) \overset{*}{=} \bigcup_{\mathcal{A}}(A - c(\mathcal{A}))$, and $\overline{r} = r$ on the left set. □

Corollary 2.5. *Suppose given $u(\mathcal{A}_n, r_n)$ for $n = 1, 2$ or $n = 1, 2, \ldots$. Let $\mathcal{B} = \bigcup_n \mathcal{A}_n$. Apply Lemma 2.4 to obtain $u(\mathcal{B}, \overline{r}_n) \overset{*}{=} u(\mathcal{A}_n, r_n)$ for each n.*

Corollary 2.6. *$G \subseteq \mathbb{R}^S$ is closed under the inherited $+, -, \vee, \wedge$. I.e., $G \leq \mathbb{R}^S$.*

Proof. Let $\otimes \in \{+, -, \vee, \wedge\}$. Let $g_1, g_2 \in G$, as $g_{\dot{n}} \overset{*}{=} u(\mathcal{A}_{\dot{n}}, r_{\dot{n}})$. Take $\mathcal{B} = \mathcal{A}_1 \cup \mathcal{A}_2$ and apply Corollary 2.5 to get $u(\mathcal{B}, \overline{r}_{\dot{n}}) \overset{*}{=} u(\mathcal{A}_n, r_n)$. Let $r = \overline{r}_1 \otimes \overline{r}_2$. Observe that $\mathcal{B} = (\mathcal{A}_1 \cap \mathcal{A}_2) \cup (\mathcal{A}_1 - \mathcal{A}_2) \cup (\mathcal{A}_2 - \mathcal{A}_1)$; it follows that $\overline{r}|\dot{B}$ is constant $\forall B \in \mathcal{B}$. □

For countable suprema, we proceed as follows.

Lemma 2.7. *Suppose $\{g_n\}_{\mathbb{N}} \subseteq G^+$ and in $\mathbb{R}^S$, $\vee g_n$ exists. (This $\vee$ is pointwise, and saying it exists just means $\{g_n\}$ is bounded above in $\mathbb{R}^S$, i.e., $\forall x$, $\{g_n(x)\}_{\mathbb{N}}$ is bounded above in $\mathbb{R}$.) Call this $\vee g_n$ as g.*

Then, $g \in G$ and therefore $g = \vee^G g_n$.

Proof. Express each g_n as $g_n \overset{*}{=} u(\mathcal{A}_n, r_n)$; note $r_n \geq 0$. Let $\mathcal{B} = \bigcup_n \mathcal{A}_n$, and apply 2.5 $\forall n$, getting $u(\mathcal{B}, \overline{r}_n) \overset{*}{=} u(\mathcal{A}_n, r_n)$ (with $\overline{r}_n = 0$ on $D - \bigcup_{\mathcal{B}} \dot{B}$, and $\overline{r}_n \geq 0$).

Since $g_n \leq g$ and $g_n = u(\mathcal{B}, \overline{r}_n)$ except on a countable T_n, $u(\mathcal{B}, \overline{r}_n) \leq g$ except on T_n. We have successively: $\forall n$ $u(\mathcal{B}, \overline{r}_n) \leq g$ except on the countable $T = \bigcup_n T_n$; $\forall n$ $\overline{r}_n \chi_{\dot{B}} \gamma(B) \leq u(\mathcal{B}, \overline{r}_n) \leq g$ except on T $\forall B \in \mathcal{B}$; $\forall n$ $\forall B$, on $\dot{B}$ except on T, $\overline{r}_n \gamma(B) \leq g$, or $\overline{r}_n \leq \frac{g}{\gamma(B)}$.

Now define $r : S \to \mathbb{R} \equiv 0$ on $S - \bigcup_{\mathcal{B}} \dot{B}$, and for $B \in \mathcal{B}$, on $\dot{B}$ $r \equiv \vee_n \overline{r}_n$. (This is pointwise $\vee$ of the $\overline{r}_n$ which are constant on $\dot{B}$, so it is the $\vee$ in $\mathbb{R}$ of those constant values "$(\overline{r}_n)_B$".)

Then, $g \overset{*}{=} u(\mathcal{B}, r)$:

Clearly, each is 0 on $S - \bigcup_{\mathcal{B}} \dot{B}$ except for T.

For $B \in \mathcal{B}$ and $x \in \dot{B} - T$, we have $u(\mathcal{B}, \overline{r}_n(x)) = \overline{r}_n \gamma(B)(x) = g_n(x)$, so $r(x)\gamma(\mathcal{B})(x) = (\vee \overline{r}_n(x))\gamma(B)(x) = \vee(\overline{r}_n \gamma(B))(x) = \vee g_n(x) = g(x)$. □

Note, Corollary 2.6 says $\{g_n\} \subseteq G^+$ is bounded above in G if and only if in $\mathbb{R}^S$.

Corollary 2.8. *G is EC, i.e., divisible, laterally σ-complete, and conditionally σ-complete.*

Proof. Divisibility is obvious. If $\{g_n\}$ is disjoint (resp., bounded above in G), then in $\mathbb{R}^S$, $\vee g$ exists because $\mathbb{R}^S$ is laterally σ-complete (resp., conditionally σ-complete). Apply Corollary 2.6. □

Theorem 2.2 is proved. □

3. Essential completions and compatible multiplications

We explain essential completions, and note that the $G(\mathcal{W},\gamma) \leq \mathbb{R}^S$ (in Theorem 2.2) is one (Corollary 3.3). The point is that, in general, "$G \in F(\mathbf{frA})$" can be recognized in terms of any particular essential completion of G (Corollaries 3.5 and 3.6).

The basic framework is due to Bernau [Ber65] and Conrad [Con71, Con74]. Embellishments are added for clarity, and to the present purpose.

Let X be a compact extremally disconnected (ED) space; so that $D(X) \in \mathbf{Arch}$ and $(D(X), \cdot) \in \mathbf{frA}$ (see §1).

Suppose $X \overset{\tau}{\leftarrow} X$ is a homeomorphism. Define $D(X) \xrightarrow{\overline{\tau}} D(X)$ as $\overline{\tau}(f) = f \circ \tau$. Suppose $d \in P \equiv$ the set of positive weak units of $D(X)$ ($d \in P \equiv d^{\perp} = \{0\}$ if and only if $\operatorname{coz} d \equiv \{x | d(x) \neq 0\}$ is dense in X). Define $D(X) \xrightarrow{\overline{d}} D(X)$ as $d(f) = d \cdot f$ (pointwise multiplication in $D(X)$). Note that $d^{-1} \in P$. (d^{-1} is the Čech-Stone extension of $\frac{1}{d}|\operatorname{coz} d$.) Define $\otimes_d$ on $D(X)$ as $f_1 \otimes_d f_2 \equiv d^{-1} f_1 f_2$.

In the rest of this section, "d" denotes a $d \in P$.

Theorem 3.1.

(1) The $\mathbf{frA}$*-isomorphisms of* $(D(X), \cdot)$ *are exactly the* $\overline{\tau}$.

(2) Each $\overline{d}$ *is an l-group isomorphism of* $D(X)$.

(3) The l-group isomorphisms of $D(X)$ *are exactly the* $\overline{d}\overline{\tau}$.

(4) $(D(X), \otimes_d) \in \mathbf{frA}$ *with identity* d *and* $(D(X), \cdot) \xrightarrow{\overline{d}} (D(X), \otimes_d)$ *is a* $\mathbf{frA}$*-isomorphism.*

All of Theorem 3.1 was certainly known to Bernau and Conrad, but we note: In (1), that $\overline{\tau}$ is a $\mathbf{frA}$-isomorphism is easy, and the converse is a special case of [HR77]. (2) is easy. In (3), that $\overline{d}\overline{\tau}$ is an l-group isomorphism follows from (1) and (2), and the converse is a special case of [BH09]. (4) is (now easy and) completely recognized in [Con74], §2.

In a general category: An essential monic is a monic φ for which $\psi\varphi$ monic implies ψ monic. E is *essentially complete* if $E \xrightarrow{\varphi} \bullet$ essential monic implies φ an isomorphism. An essential completion (of G) is a $G \xrightarrow{\varphi} E$ with φ essential monic and E essentially complete. (If the category lacks injective hulls, this notion substitutes.)

In $\mathbf{Arch}$, monic means one-to-one, and (suppressing label) $G \leq H$ is essential if and only if G is large in H, meaning: if $0 < h \in H$, there are $0 < g \in G$ and $n \in \mathbb{N}$ with $g \leq nh$.

The following is all in $\mathbf{Arch}$.

Theorem 3.2.

(1) E *is essentially complete if and only if* $E \approx$ some $D(X)$ *for* X *compact ED if and only if* E *is divisible, laterally complete, and conditionally complete.*

(2) For any G*, there is an essential completion* $G \overset{\varphi}{\leq} D(X)$.

(3) For every two essential completions, $G \overset{\varphi_i}{\leq} D(X)$*, there is a unique l-group isomorphism* μ *with* $\mu\varphi_1 = \varphi_2$*. (Note* $\mu = \overline{d}\overline{\tau}$ *as in Theorem* 3.1.*)*

In Theorem 3.2: (1) is due to Conrad [Con71]. (2) and (3) combine (1) with facts about Bernau's representation [Ber65].

Corollary 3.3. *In 2.2, $G(\mathcal{W},\gamma) \leq \mathbb{R}^S$ is an essential completion. (The X in Theorem 3.2 is βS; $\mathbb{R}^S \approx D(\beta S)$.)*

Proof. $\mathbb{R}^S$ is visibly divisible, laterally and conditionally complete (or, referring to Theorem 3.2, βS is ED). $G(\mathcal{W},\gamma)$ is large in $\mathbb{R}^S$: If $f > 0$, there is x with $f(x) > 0$, so there is n with $nf(x) > 1$, so $0 < \chi_x \leq nf$, and $\chi_x \in G(\mathcal{W},\gamma)$. □

Theorem 3.4. *If $(G,*) \in \mathbf{frA}$, then there is $(G,*) \overset{\psi}{\leq} (D(X),\cdot)$ in $\mathbf{frA}$ with $G \overset{F\psi}{\leq} D(X)$ an essential completion.*

Bernau shows Theorem 3.4 for his representation [Ber65], which Conrad identified as an essential completion [Con71]. (Another proof of Theorem 3.4 is indicated after Remark 5.3.)

Corollary 3.5. *Suppose $G \overset{\eta}{\leq} D(X)$ is any essential completion.*

(1) *If there is d such that $\eta(G)$ is closed under $\otimes_d$ in $D(X)$, then $(\eta(G),\otimes_d) \in \mathbf{frA}$ and $G \in F(\mathbf{frA})$.*
(2) *If there is $*$ with $(G,*) \in \mathbf{frA}$ (i.e., $G \in F(\mathbf{frA})$), then there is d with $(G,*) \approx (\eta(G),\otimes_d)$.*

Proof. (1) is obvious.

(2). We use Theorems 3.1, 3.2, and 3.4. Consider

$$\begin{tikzcd}
& D(X) & \\
G \arrow[ur, "\eta"] \arrow[dr, "(F\psi =)\ \psi"'] & & D(X) \arrow[ul, dashed, "\overline{d}"'] \\
& D(X) \arrow[uu, dashed, "\mu"] \arrow[ur, dashed, "\overline{\tau}"'] &
\end{tikzcd}$$

where η is the given, $(G,*) \overset{\psi}{\leq} (D(X),\cdot)$ is from Theorem 3.4, and μ with $\mu\psi = \eta$ comes from Theorem 3.2 (3), and $\mu = \overline{d}\overline{\tau}$ by Theorem 3.1.

The claim is that $\eta(G)$ is closed under $\otimes_d$, as $d^{-1}\eta(g_1)\eta(g_2) = \eta(g_1 * g_2)$:

$$\begin{aligned}
\eta(g_1 * g_2) =& \mu\psi(g_1 * g_2) = \mu(\psi(g_1)\cdot\psi(g_2)) \\
=& \overline{d}\overline{\tau}(\psi(g_1)\cdot\psi(g_2)) = \overline{d}(\overline{\tau}\psi(g_1)\cdot\overline{\tau}\psi(g_2)) \\
=& d\cdot\overline{\tau}\psi(g_1)\cdot\overline{\tau}\psi(g_2) = d\cdot\overline{\tau}\psi(g_1)\cdot(d^{-1}d)\cdot\overline{\tau}\psi(g_2) \\
=& d^{-1}\cdot d\overline{\tau}\psi(g_1)\cdot d\overline{\tau}\psi(g_2) \\
=& d^{-1}\cdot\overline{d}\overline{\tau}\psi(g_1)\cdot\overline{d}\overline{\tau}\psi(g_2) = d^{-1}\cdot\mu\psi(g_1)\cdot\mu\psi(g_2) \\
=& d^{-1}\cdot\eta(g_1)\cdot\eta(g_2).
\end{aligned}$$

□

Corollary 3.5 is, more-or-less, in [Con74], §2.

Corollary 3.6. *Suppose $G \overset{\eta}{\leq} D(X)$ is any essential completion.*

(1) *$G \in F(\mathbf{frA})$ if and only if there is d such that $\eta(G)$ is closed under $\otimes_d$ in $D(X)$.*

(2) $G \notin F(\mathbf{frA})$ if and only if $\forall p \in P$, $\exists g_1, g_2 \in G$ with $p \cdot \eta(g_1) \cdot \eta(g_2) \notin \eta(G)$.

Proof. (1) restates Corollary 3.5. (2) is the same as (1) using $p = d^{\dashv}$. $\square$

Note. If G is divisible, the condition in Corollary 3.6 (2) is equivalent to: $\forall p$, $\exists g$ with $p \cdot \eta(g)^2 \notin \eta(G)$ (because $(x+y)^2 = x^2 + 2xy + y^2$).

4. Epicomplete $G \notin F(\mathbf{frA})$.

Such G will be of the form $G(\mathcal{W}, \gamma) \leq \mathbb{R}^S$ as in §2. The items $S, \mathcal{W}, P, \ldots$ are as in §2.

Theorem 4.1. *If $|\mathcal{W}| = 2^{\aleph_1}$, then there is $\mathcal{W} \xrightarrow{\delta} P$ for which $G(\mathcal{W}, \delta) \notin F(\mathbf{frA})$.*

Combining Theorem 4.1 with Theorem 2.2, and with other information in §1, gives the full statement of Theorem 1.3.

The definition of the δ requires a lemma.

Lemma 4.2. *Suppose $A \subseteq S$ with $|A| = \aleph_1$, and suppose $f \in P$. Then there is $h \in P$ for which*

(i) $h(x) \geq f(x)$ for each $x \in S$, and

(ii) for any countable $T \subseteq S$ and any $r \in \mathbb{R}$, $h|(A - T) \neq rf|(A - T)$.

Proof. Given the A and f, we define separately $h|(S - A)$ and $h|A$:

$h|(S - A) \equiv (f + 1)|(S - A)$.

If there is no countable T with $f|(A-T)$ constant, then $\forall$ such T, $|f(A-T)| \geq 2$, and again define $h|A = (f+1)|A$. (If $(f+1)|(A-T) = rf|(A-T)$, then $\forall x \in A-T$, $f(x) = \frac{1}{r-1}$, contradicting $|f(A-T)| \geq 2$.)

Suppose there is countable T with $f|(A - T)$ constant, take one-to-one $A \xrightarrow{\beta} \mathbb{R}$, and define $h|A \equiv (f + \beta)|A$. (If there were countable T' and r with $h|(A - T') = rf|(A - T')$, then also $h = rf$ on $A - (T' \cup T)$ where f is constant, say r_0. Then, on $A - (T' \cup T)$, $\beta = (r - 1)r_0$ constant, contradicting β one-to-one.) $\square$

Proof of Theorem 4.1. We shall define δ using Lemma 4.2, which gives $G(\mathcal{W}, \delta) \leq \mathbb{R}^S$ as in Theorem 2.2, and then verify 3.6 to get $G(\mathcal{W}, \delta) \notin F(\mathbf{frA})$. Since $|\mathcal{W}| = 2^{\aleph_1}$, there is a bijection $\mathcal{W} \xleftarrow{\alpha} P$; for each $A \in \mathcal{W}$, there is $v \in P$ with $A = \alpha(v)$. Apply Lemma 4.2 to $f = 1/v$, producing $h \equiv \delta(A)$ for which

(ii) $\forall T, r$, $h|(A - T) \neq \frac{r}{v}|(A - T)$.

Theorem 2.2 applies to $G \equiv G(\mathcal{W}, \delta) \leq \mathbb{R}^S$, which is an essential completion (Corollary 3.3). We use Corollary 3.6 (suppressing the η there). We claim: $\forall v \in P$, $\exists g \in G$ with $vg^2 \notin G$. Take $v \in P$, let $A = \alpha(v)$, and let $g = \chi_A \gamma(A) = \chi_A h$, h satisfying (ii) above. (This g is $u(\{A\}, \chi_A)$.) Then, $g^2 = \chi_A h^2$ and $vg^2 = v\chi_A h^2$.

Suppose (towards contradiction) that $v\chi_A h^2 \in G$. Then, $v\chi_A h^2 \stackrel{*}{=} u(\mathcal{B}, r)$ for some $(\mathcal{B}, r)$, and using Lemma 2.4, we can suppose $A \in \mathcal{B}$ (by replacing $\mathcal{B}$ by $\mathcal{B} \cup \{A\}$). For $\dot{A} = A - c(\mathcal{B})$, we then have $v\chi_{\dot{A}} h^2 = r\chi_{\dot{A}}\gamma(A) = r\chi_{\dot{A}} h$ (from the form of $u(\mathcal{B}, r)$). Cancelling an h gives $v\chi_{\dot{A}} h = r\chi_{\dot{A}}$, whence $h = \frac{r}{v}$ on $A - c(\mathcal{B})$, contradicting (ii) above. $\square$

5. Compatible truncation

Suppose $G \in \mathbf{Arch}$, and e is a positive weak unit of G. Then G^+ is closed under the "truncation" $g \mapsto e \wedge g \equiv t_e(g)$. Ball [Bal13] has viewed this as a unary operation, abstracted (axiomatized) its features, and defined the category **AT** of Archimedean "truncs": The (G,t), $t : G^+ \to G^+$ satisfying his axioms (which we need not list), with morphisms $(G,t) \xrightarrow{\varphi} (G',t')$ the l-group morphisms with $\varphi(t(g)) = t'(\varphi(g))\ \forall g \in G^+$.

Let $\mathbf{Arch} \xleftarrow{F} \mathbf{AT}$ be the forgetful functor.

Note that any $D(Y)$ (not necessarily an l-group) has $f \wedge 1_Y \in D(Y)\ \forall f \geq 0$, and $f \in D_0(Y)$ implies $f \wedge 1_Y \in D_0(Y)$.

Here is Ball's companion to Johnson's Theorem 1.4.

Theorem 5.1 ([Bal13]). *For each $(G,t) \in \mathbf{AT}$, there is a J-representation $G \overset{\alpha}{\leq} D_0(Y)$ with $\alpha(t(g)) = 1_Y \wedge \alpha(g)$ for each $g \in G^+$.*

At this point, we see that for EC G, $G \in F(\mathbf{AT})$ if and only if $G \in F(\mathbf{frA})$, in consequence of Johnson's 1.4, Ball's 5.1, and the long argument in [BHJK09] that if EC G has any J-representation, then $G \approx D(K,p)$, the latter evidently in $F(\mathbf{frA}) \cap F(\mathbf{AT})$. We described this in §1.

Thus, the $G(\mathcal{W},\delta)$ in Theorem 4.1 has $G(\mathcal{W},\delta) \notin F(\mathbf{AT})$.

We shall directly produce other $G(\mathcal{W},\tau) \leq \mathbb{R}^S$ with $G(\mathcal{W},\tau) \notin F(\mathbf{AT})$, by arguing in parallel to §3 and Theorem 4.1. We do this in the hope of ultimately understanding why, on a $G \in \mathbf{Arch}$, truncations and $\mathbf{frA}$-multiplications behave so similarly: we do not yet.

Corollary 5.2. *Suppose $(G,t) \in \mathbf{AT}$.*

(1) There is $H \in \mathbf{Arch}$ with a weak unit e, and an embedding $(G,t) \overset{\beta}{\leq} (H, t_\beta)$ (i.e., qua $\mathbf{AT}$) with $G \overset{F\beta}{\leq} H$ essential.

(2) There is X compact ED and $(G,t) \overset{\psi}{\leq} (D(X), t_1)$ $(1 = 1_X)$ with $G \overset{F\psi}{\leq} D(X)$ an essential completion.

Proof.

(1) Take $G \overset{\alpha}{\leq} D_0(Y)$ per Theorem 5.1, let $H = jm(\alpha(G) + \mathbb{Z} \cdot 1_Y)$. (Here, $m(\bullet)$ is the collection of all finite meets from $(\bullet)$, and likewise $j(\bullet)$ for joins.) Then, $H \in \mathbf{Arch}$, and $e = 1_Y$ is a weak unit. (See [HJ10], §2.) Take $e = 1_Y$, and β the codomain restriction of the function α. $F\beta$ is essential because α is a Johnson representation.

(2) With $(G,t) \overset{\beta}{\leq} (H,e)$ as in (1), take an essential completion $H \overset{\gamma}{\leq} D(X)$, for which $\beta(e) = 1_X$. Then $\psi = \gamma\beta$.

□

Remark 5.3. (2) also can be proved from the $G \overset{\alpha}{\leq} D_0(Y)$ in Theorem 5.1, and some topology, like this. Let $\beta Y \xleftarrow{a} a\beta Y \equiv X$ be the ED cover (absolute, Gleason cover, projective cover) of the Čech-Stone compactification. We have $D(Y) \approx D(\beta Y) \xrightarrow{\overline{a}} D(X)$ as $\overline{a}(f) = f \circ a$, which produces $G \overset{\psi}{\leq} D(X)$ as $g \mapsto \alpha(g) \mapsto \beta\alpha(g) \mapsto \overline{a}(\beta\alpha(g))$. The embedding is essential because a is irreducible.

In exactly the same way, Bernau's Theorem 3.4 can be proved from Johnson's Theorem 1.4.

In the following, X is compact ED, and P is the set of positive weak units of $D(X)$, and "d" always denotes a $d \in P$. Given d, let $t_d(f) \equiv d \wedge f$ $(f \in D(X)^+)$. Recall $D(X) \xrightarrow{\overline{d}} D(X)$ is $\overline{d}(f) = df$, and $\overline{d}(1_X) = d1_X = d$.

Theorem 5.4. *Suppose $G \overset{\eta}{\leq} D(X)$ is any essential completion.*

*(a) If there is d such that $\eta(G)^+$ is closed under t_d, then $(\eta(G), t_d) \in$ **AT** and $G \in F($**AT**$)$.*

*(b) If there is t with $(G, t) \in$ **AT** (i.e., $G \in F($**AT**$)$), then there is d with $(G, t) \approx (\eta(G), t_d)$.*

Proof. (a) is obvious, and (b) follows from (a) as in Theorem 3.4, but more simply: Consider

$$\begin{tikzcd}
& D(X) & \\
G \arrow[ur, "\eta"] \arrow[dr, "(F\psi =)\ \psi"'] & & D(X) \arrow[ul, dashed, "\overline{d}"'] \\
& D(X) \arrow[uu, dashed, "\mu"] \arrow[ur, dashed, "\overline{\tau}"'] &
\end{tikzcd}$$

where η is the given, $(G, t) \overset{\psi}{\leq} (D(X), t_1)$ is from 5.3(b) $(1 = 1_X)$, μ with $\mu\psi = \eta$ comes from Theorem 3.2 (3), and $\mu = \overline{d}\overline{\tau}$ by Theorem 3.1. Note that $\mu(1) = \overline{d}\overline{\tau}(1) = \overline{d}(1) = d$.

The claim is that $\eta(G)^+$ is closed under t_d, as $d \wedge \eta(g) = \eta(t(g))$:

$$\eta(t(g)) = \mu\psi(t(g)) = \mu(1 \wedge \psi(g)) = \mu(1) \wedge \mu\psi(g) = d \wedge \eta(g).$$

□

Corollary 5.5. *Suppose $G \overset{\eta}{\leq} D(X)$ is any essential completion. $G \notin F($**AT**$)$ if and only if $\forall d \in P$, $\exists g \in G^+$ with $d \wedge \eta(g) \notin \eta(G)$.*

Here is the advertised companion to Theorem 4.1. (The τ appearing has nothing to do with the τ in §3 and the proof of Theorem 5.4.)

Theorem 5.6. *If $|\mathcal{W}| = 2^{\aleph_1}$, then there is $\mathcal{W} \xrightarrow{\tau} P$ for which $G(\mathcal{W}, \tau) \notin F($**AT**$)$.*

Proof. As in Theorem 4.1, take any bijection $\mathcal{W} \xleftarrow{\alpha} P$; for each $A \in \mathcal{W}$, there is $d \in P$ with $A = \alpha(d)$. Apply Lemma 4.2 exactly as stated to $f = d$, producing $h \equiv \tau(A)$ for which

(i) $h(x) \geq d(x)$ $\forall x \in S$, and

(ii) $\forall$ countable $T \subseteq S$, $\forall r \in \mathbb{R}$, $h|(A - T) \neq rd|(A - T)$.

Theorem 2.2 applies to $G \equiv G(\mathcal{W}, \tau) \leq \mathbb{R}^S$, which is an essential completion as in Corollary 5.5 (suppressing the η there). We claim: $\forall d \in P$, $\exists g \in G$ with $d \wedge g \notin G$.

Take $d \in P$, let $A = \alpha(d)$, and let $g = \chi_A \tau(A) = \chi_A h$, with h as above. (This g is $u(\{A\}, \chi_A)$.) Since $h \geq d$, $d \wedge g = d \wedge \chi_A h = d\chi_A$.

Suppose (toward contradiction) that $d\chi_A \in G$. Then, $d\chi_A \overset{*}{=} u(\mathcal{B}, r)$ for some $(\mathcal{B}, r)$, and we can suppose $A \in \mathcal{B}$. For $\dot{A} = A - c(\mathcal{B})$, we then have $d\chi_{\dot{A}} = r_A \chi_{\dot{A}} h$, which means $h = \frac{1}{r_A} d$ on $A - c(\mathcal{B})$, contradicting (ii) above.

□

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