A NOTE ON CONDENSATIONS OF FUNCTION SPACES ONTO σ -COMPACT AND ANALYTIC SPACES

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ABSTRACT. Modifying a construction of W. Marciszewski we prove (in ZFC) that there exists a subspace X of the real line $\mathbb{R}$, such that the realcompact space $C_p(X)$ of continuous real-valued functions on X with the pointwise convergence topology does not admit a continuous bijection onto a σ -compact space. This answers a question of Arhangel'skii.

1. INTRODUCTION

In this article, unless otherwise stated, by a space we mean a Hausdorff topological space. For a space X , we denote by $C_p(X)$ the space of all continuous real-valued functions on X equipped with the pointwise convergence topology. It is well known that for a completely regular space X the space $C_p(X)$ is almost never σ -compact. Namely, it is σ -compact if and only if the space X is finite (see [11, 1.2.186]). Thus it is natural to ask which $C_p(X)$ spaces admit a weaker σ -compact (compact) topology, approximating the original one. This general problem posed by Arhangel'skii can be reformulated using a concept of *condensation*, i.e., continuous bijection: *When is there a condensation of $C_p(X)$ onto a σ -compact (compact) space?* (see, e.g., [7, Problem 5.1]).

For example, as was proved by Michalewski in [8], the space $C_p(X)$ condenses onto a (metrizable) compactum whenever X is a metrizable analytic space, i.e., X is a continuous image of the space ω^ω of the irrationals. Let us recall two problems posed by Arhangel'skii concerning the possible generalizations of the result mentioned above.

Problem 1.1. (Arhangel'skii, [2, Problem 4]) Suppose that X is a separable metrizable space. Does $C_p(X)$ condense onto a σ -compact space?

Problem 1.2. (Arhangel'skii, [1, Problem 37]) Suppose that $C_p(X)$ is realcompact. Does $C_p(X)$ condense onto a σ -compact space?

Note that Problem 1.2, which was also asked in the recent book of Tkachuk [11, 4.10.1], is more general than Problem 1.1. Indeed, the space $C_p(X)$ is realcompact provided X is separable (see [11, Problems 418 and 429]).

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A consistent negative answer to Problem 1.1 (and hence also to Problem 1.2) was given by Marciszewski in [6]. He constructed, assuming that $\mathfrak{d} = 2^\omega$, a subspace X of the real line $\mathbb{R}$ such that $C_p(X)$ does not condense onto a σ -compact space. However, the question if one can construct such a space without any additional set-theoretic assumptions remained open (see [7, page 363]).

In this short note we show how to modify the construction from [6] to make it work in ZFC. Thus the answer to both aforementioned problems is in the negative (see Theorem 3.1 and Corollary 3.2 below).

A significant part of the construction is the same as in [6] and therefore some details will be omitted. The results presented in the main text of [6] work for completely regular spaces. However, as was pointed out in [6, Remark], the construction of a space X can be modified to the effect that $C_p(X)$ cannot be condensed onto a σ -compact Hausdorff space. It appears that as far as we are concerned with a condensation onto Tychonoff spaces, even the stronger result is true: the space $C_p(X)$ cannot be condensed onto any Tychonoff analytic space. We refer the interested reader to [7, Chapter 5], [10] and [11] for more information on condensations in C_p -theory and further references.

2. AUXILIARY RESULTS

For a space X and its subset D , we denote by $C_D(X)$ the subspace $\{f \upharpoonright D : f \in C_p(X)\}$ of the space $C_p(D) \subseteq \mathbb{R}^D$. Note that the natural projection $\pi_D : \mathbb{R}^X \rightarrow \mathbb{R}^D$ condenses $C_p(X)$ onto $C_D(X)$, whenever D is dense in X .

Proposition 2.1 given below was formulated in [6] without a proof (see [6, Remark (2)]). Since it is not completely straightforward we decided to enclose a short argument.

Proposition 2.1. *Suppose that a space X is a countable union of metrizable compacta. Then there exists an injective Borel map of X onto a metrizable σ -compact space.*

Proof. Since a finite union of metrizable compacta is a metrizable compactum we have $X = \bigcup_{n \in \omega} K_n$, where $K_0 = \emptyset$, $K_n \subseteq K_{n+1}$ and each K_n is metrizable compact. For $n \in \omega$, let $i_n : K_n \rightarrow [0, 1]^\omega$ be a homeomorphic embedding. Let $f_n : X \rightarrow [0, 1]^\omega$ be a Borel extension of i_n taking $X \setminus K_n$ onto an arbitrary point. The diagonal map $F = \bigtriangleup_{n \in \omega} f_n : X \rightarrow ([0, 1]^\omega)^\omega$ is a Borel injection (every two points in X belong to some K_n and f_n is injective on K_n). We shall prove that $F(X)$ is σ -compact. To this end it suffices to show that for each $n \in \omega$, the map F is continuous on the σ -compact set $K_{n+1} \setminus K_n$. For $n = 0$ this follows from the fact that each f_n is continuous on K_n and $K_1 \subseteq K_n$, for $n \geq 1$. Let $n \geq 1$. For $m \geq n + 1$ the map f_m is continuous on $K_{n+1} \setminus K_n \subseteq K_m$ and for $m < n + 1$ the map f_m is continuous on $K_{n+1} \setminus K_n$ being constant (we assumed that f_m maps $X \setminus K_m$ onto a point). $\square$

Proposition 2.2 ([6, Remark (3)]). *Let E be a countable dense subset of a separable metrizable space X and let $\psi : C_p(X) \rightarrow Y$ be continuous, where Y is an arbitrary second countable space. Then ψ has the following factorization property: There is a countable set $D \subseteq X$ containing E and a continuous map $\xi : C_D(X) \rightarrow Y$ such that $\psi = \xi \circ (\pi_D \upharpoonright C_p(X))$.*

Proof. Let $\{U_n\}_{n \in \omega}$ be a countable base for Y . Since the space X is separable metrizable it has a countable network. It follows that $C_p(X)$ also has a countable

network and hence it is hereditarily Lindelöf. Thus, each open set in $C_p(X)$ is a countable union of basic open sets.

For each $n \in \omega$, we have $\psi^{-1}(U_n) = \bigcup_{m \in \omega} [A_m^n, V_m^n]$, where $[A_m^n, V_m^n]$ is a basic open set in $C_p(X)$ given by a finite set $A_m^n \subseteq X$ and an open set $V_m^n \subseteq \mathbb{R}$. It is easy to check that the set $D = E \cup \bigcup_{n, m \in \omega} A_m^n$ is as promised. The map ξ is uniquely determined by D because density of D implies that π_D is injective on $C_p(X)$. $\square$

The next proposition can be proved by a simple transfinite induction (cf. [9, the proof of 1.5.14]). It is a slight modification of the construction of the Bernstein set.

Proposition 2.3. *There exists a set $\mathbb{K} \subseteq \mathbb{R}$ containing the rationals $\mathbb{Q}$ and such that both $\mathbb{K}$ and $\mathbb{R} \setminus \mathbb{K}$ intersect each copy of the Cantor set in $\mathbb{R}$.*

Since in every Cantor set we can find continuum many pairwise disjoint Cantor sets, both $\mathbb{K}$ and $\mathbb{R} \setminus \mathbb{K}$ have cardinality 2^ω .

The next, easy proposition will be crucial in our construction.

Proposition 2.4. *Let $\mathbb{K} \subseteq \mathbb{R}$ be the set given by Proposition 2.3. Suppose that $G \subseteq \mathbb{R}$ is a G_δ set containing $\mathbb{K}$. Then the set $\mathbb{R} \setminus G$ is countable.*

Proof. Since G is G_δ , we have $\mathbb{R} \setminus G = \bigcup_{n \in \omega} F_n$, where each F_n is closed. If $\mathbb{R} \setminus G$ were uncountable one of the sets F_n , being closed and uncountable, would contain a copy of the Cantor set. This however would contradict the property of $\mathbb{K}$. $\square$

3. THE CONSTRUCTION

In this section we will prove the following:

Theorem 3.1. *There is a space $X \subseteq \mathbb{R}$, containing the rationals $\mathbb{Q}$, such that $C_p(X)$ condenses neither onto a σ -compact (Hausdorff) space, nor onto an analytic Tychonoff space.*

Proof. Let $\mathcal{F}$ be the following family of maps (cf. [6, Remark])

$$\mathcal{F} = \{\varphi : B \rightarrow \mathbb{R}^\omega : B \text{ is an analytic subset of } \mathbb{R}^D, \text{ for some countable } D \subseteq \mathbb{R} \\ \text{with } \mathbb{Q} \subseteq D, \varphi \text{ is Borel}\}.$$

The family $\mathcal{F}$ has cardinality 2^ω and hence we can enumerate it as $\{\varphi_\alpha : B_\alpha \rightarrow \mathbb{R}^\omega : \alpha < 2^\omega\}$ (repetitions allowed) in such a way that B_α is a subset of $\mathbb{R}^{D_\alpha}$. Let $\mathbb{K} \subseteq \mathbb{R}$ be the set given by Proposition 2.3.

In general, we are going to repeat the construction from [6]. The only change we are going to make is the starting point: We will use the set $\mathbb{K}$ instead of $\mathbb{Q}$. This simple idea allows us to drop the set-theoretic assumption $\mathfrak{d} = 2^\omega$ required in [6] (cf. Remark 3.4 at the end of the paper). By induction we choose points $x_\alpha, y_\alpha \in \mathbb{R} \setminus \mathbb{K}$, G_δ -subsets A_α of $\mathbb{R}$ containing $\mathbb{K}$ and continuous functions $f_\alpha, g_\alpha : A_\alpha \rightarrow \mathbb{R}$ such that the following conditions are satisfied (we put $X_\alpha = \mathbb{K} \cup \{x_\beta : \beta < \alpha\}$).

- (i) $x_\beta \neq x_\alpha$, for $\beta < \alpha$,
- (ii) $(X_\alpha \cup \{x_\alpha\}) \cap \{y_\beta : \beta \leq \alpha\} = \emptyset$,
- (iii) $x_\alpha \in (\mathbb{R} \setminus \mathbb{K}) \cap (\bigcap_{\beta \leq \alpha} A_\beta)$,
- (iv) if $D_\alpha \setminus X_\alpha \neq \emptyset$, then $y_\alpha \in D_\alpha$,
- (v) if $D_\alpha \subseteq X_\alpha$ and $C_{D_\alpha}(X_\alpha) \setminus B_\alpha \neq \emptyset$, then $X_\alpha \subseteq A_\alpha$ and $f_\alpha \upharpoonright D_\alpha \notin B_\alpha$,
- (vi) if $D_\alpha \subseteq X_\alpha$, $C_{D_\alpha}(X_\alpha) \subseteq B_\alpha$ and $\varphi_\alpha \upharpoonright C_{D_\alpha}(X_\alpha)$ is not injective, then $X_\alpha \subseteq A_\alpha$, $f_\alpha \neq g_\alpha$ and $\varphi(f_\alpha \upharpoonright D_\alpha) = \varphi(g_\alpha \upharpoonright D_\alpha)$.

Note that for $\alpha < 2^\omega$ the set $(\mathbb{R} \setminus \mathbb{K}) \cap (\bigcap_{\beta \leq \alpha} A_\beta)$ has cardinality 2^ω . Indeed, for each $\beta \leq \alpha$ the set A_β is a G_δ -subset of $\mathbb{R}$ containing $\mathbb{K}$ and hence Proposition 2.4 implies that $|(\bigcap_{\beta \leq \alpha} A_\beta)^c| < 2^\omega$. Moreover, as we have already observed, the set $\mathbb{R} \setminus \mathbb{K}$ has cardinality 2^ω .

Now, the inductive step can be made by considering the same four cases (corresponding to conditions (iv)-(vi)) as in [6].

Put $X = \mathbb{K} \cup \{x_\alpha : \alpha < 2^\omega\}$. We need to show that there is no condensation from $C_p(X)$ onto a space M being σ -compact or Tychonoff and analytic. The argument given in [6] works for M being Tychonoff and σ -compact. If M is just Hausdorff, the proof is a little bit different and for this reason we enclose a justification.

Let M be σ -compact. Suppose that there is a condensation $\psi : C_p(X) \rightarrow M$. Since X is separable metrizable, the space $C_p(X)$ has a countable network and so does M . Thus M is a countable union of metrizable compacta. Since every space with a countable network condenses onto a space with a countable base [3, Ch. 2, Problem 149], without loss of generality we may assume that M has a countable base. By Proposition 2.2, there exists a countable set $D \subseteq X$ containing $\mathbb{Q}$ and a map $\xi : C_D(X) \rightarrow M$ such that $\psi = \xi \circ (\pi_D \upharpoonright C_p(X))$. Proposition 2.1 yields the existence of a metrizable σ -compact space S and an injective Borel map $\eta : M \rightarrow S$ (in the case M is an analytic Tychonoff space, by [11, Problem 156 (iii)] there exists a continuous map η and a space S with a countable base which is analytic as a continuous image of an analytic space M). As S is metrizable, we can assume that $S \subseteq \mathbb{R}^\omega$. We take $\varphi' = \eta \circ \xi : C_D(X) \rightarrow \mathbb{R}^\omega$. Using a theorem of Kuratowski (see [4, §35.VI]) we can find a Borel subset B' of $\mathbb{R}^D$ containing $C_D(X)$ and a Borel extension $\varphi'' : B' \rightarrow \mathbb{R}^\omega$ of a Borel injective map φ' . From now on we proceed as in [6]. Take $B = (\varphi'')^{-1}(S)$ and $\varphi = \varphi'' \upharpoonright B$. The set B is analytic, being a Borel preimage of a σ -compact set (if M were a Tychonoff analytic space, the set B is a Borel preimage of an analytic set; hence, it is analytic) and φ maps injectively $C_D(X)$ onto S . The map φ belongs to $\mathcal{F}$ and thus there is $\alpha < 2^\omega$ such that $\varphi = \varphi_\alpha$, $D = D_\alpha$, $B = B_\alpha$.

We consider the following four complementary cases:

Case 1. $D_\alpha \setminus X_\alpha \neq \emptyset$.

By (iv), $y_\alpha \in D_\alpha = D$ and $y_\alpha \notin X$ by (ii). Hence $D \setminus X \neq \emptyset$, a contradiction.

Case 2. $D_\alpha \subseteq X_\alpha$ and $C_{D_\alpha}(X_\alpha) \setminus B_\alpha \neq \emptyset$.

By (v), $X_\alpha \subseteq A_\alpha$ and by (iii) $x_\beta \in A_\alpha$ for $\beta \geq \alpha$, so $X \subseteq A_\alpha$ and $f_\alpha \upharpoonright X \in C_p(X)$. Condition (v) implies that $f_\alpha \upharpoonright D \in C_D(X) \setminus B$, a contradiction.

Case 3. $D_\alpha \subseteq X_\alpha$, $C_{D_\alpha}(X_\alpha) \subseteq B_\alpha$ and $\varphi_\alpha \upharpoonright C_{D_\alpha}(X_\alpha)$ is not injective.

Similarly as in Case 2, condition (vi) implies that $X \subseteq A_\alpha$ and $f_\alpha \upharpoonright X, g_\alpha \upharpoonright X \in C_p(X)$. Then $f_\alpha \upharpoonright D, g_\alpha \upharpoonright D$ are distinct elements of $C_D(X)$ and $\varphi(f_\alpha \upharpoonright D) = \varphi(g_\alpha \upharpoonright D)$, a contradiction.

Case 4. $D_\alpha \subseteq X_\alpha$, $C_{D_\alpha}(X_\alpha) \subseteq B_\alpha$ and $\varphi_\alpha \upharpoonright C_{D_\alpha}(X_\alpha)$ is injective.

By (i), X_α is a proper subset of X and hence $C_D(X)$ is a proper subset of $C_D(X_\alpha)$ (see [6, Proposition 2.5]). Since φ is injective on $C_D(X_\alpha)$ and $\varphi(C_D(X_\alpha)) \subseteq \varphi(B) = S$, we have $S \setminus \varphi(C_D(X)) \neq \emptyset$, a contradiction. $\square$

Theorem 3.1 gives a negative answer to Problem 1.1. As we mentioned in the Introduction it also immediately implies the following negative answer to Problem 1.2.

Corollary 3.2. *There is a space $C_p(X)$ which is realcompact and does not admit a condensation onto a σ -compact space.*

Remark 3.3. It seems that the answer to Problem 1.2 has been known before. As R. Pol observed it is enough to consider $X = \omega \cup \{\infty\}$, where the points of ω are isolated and the neighborhoods of ∞ are given by an analytic non-Borel filter on ω (see [5, Theorem 4.1]). Then $C_p(X)$ is clearly realcompact and does not condense onto any Borel subset of a Polish space.

Remark 3.4. We feel that the reader deserves a better explanation why the proof of Theorem 3.1, although very similar to the construction from [6], does not require the set-theoretic assumption $\mathfrak{d} = 2^\omega$ which was vital in [6]. Recall that $\mathfrak{d} = 2^\omega$ is equivalent to the following statement:

- (*) The intersection of less than continuum many G_δ -subsets of $\mathbb{R}$ containing $\mathbb{Q}$ has cardinality 2^ω .

Now, if we start the construction of the space X from $\mathbb{Q}$ and try to add points $x_\alpha \in \mathbb{R} \setminus \mathbb{Q}$ inductively, as it was done in [6], we will need (*) to fulfill conditions (i)-(iii) of the construction. That is, we will need (*) to make sure that the intersection $(\mathbb{R} \setminus \mathbb{Q}) \cap \bigcap_{\beta \leq \alpha} A_\beta$ is big enough (has cardinality 2^ω) to choose a point x_α belonging to it and distinct from points already chosen.

If we start the construction of X from $\mathbb{K}$ (as it was done in the proof of Theorem 3.1), we force the intersection $(\mathbb{R} \setminus \mathbb{K}) \cap \bigcap_{\beta \leq \alpha} A_\beta$ to be big, simply by making sure that it contains the set $\mathbb{K}$. Thus we no longer need the statement (*).

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