COPRIME ACTIONS WITH ALL ORBIT SIZES SMALL

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ABSTRACT. Let a p' -group G act faithfully on a finite p -group P . Suppose that every G -orbit on P has size at most $p - 1$. We show that G must have a regular orbit on P .

1. INTRODUCTION

Let G be a p' -group which acts faithfully on a finite p -group P . Suppose that all orbit sizes of G on P are “small” in terms of p . Must it follow, somewhat paradoxically, that G has a regular orbit on P ?

I. M. Isaacs [3, Theorem 3.1] showed that if all orbit sizes are at most $(p + 1)/2$, then G must indeed have a regular orbit on P , and he gave an example showing that if there is an orbit of size $p + 1$, then there need not be a regular orbit. He asks if there must be a regular orbit if the bound $(p + 1)/2$ is replaced by $p - 1$. This would be best possible; p cannot be an orbit size because $(|G|, p) = 1$. We answer his question in the affirmative:

Theorem A. *Let G be a p' -group that acts faithfully on a finite p -group P . Suppose that every G -orbit on P has size at most $p - 1$. Then G has a regular orbit on P .*

We remark that the coprimeness hypothesis in Theorem A is appropriate but logically superfluous; if $x \in G$ had order p , then $\langle x \rangle$ would have an orbit of size p , so some G -orbit would have size at least p .

We prove Theorem A by a variety of counting arguments; the simple group classification is not needed. We reduce to the case that P is elementary abelian and irreducible under the action of G . We then show that if the proportion of pseudoreflections in G is at most $1/3$, then G must have a regular orbit on P . We quickly deduce that regular orbits must exist when $|P| > p^3$. More detailed analysis is needed when $|P| \leq p^3$.

2. THE GENERAL CASE

In this section we reduce Theorem A to the case that P is elementary abelian and G -irreducible. We introduce the important pseudoreflection ratio $r(G, V)$ for coprime linear groups. We show that if $r(G, V) \leq 1/3$, then (G, V) is not a counterexample to Theorem A. In particular, (G, V) is not a counterexample to Theorem A if V is an irreducible $\text{GF}(p)[G]$ -module of dimension greater than 3.

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Lemma 2.1. *Suppose G acts faithfully on a finite p -group P . Suppose that P is a nontrivial direct product $P_1 \times P_2$ of G -invariant subgroups. Suppose that G acts nontrivially on P_1 and on P_2 and that every G -orbit on P has size at most $p + 1$. Then G has a regular orbit on P .*

Proof. This is a slightly weaker form of [3, Theorem 3.3]. $\square$

Lemma 2.2. *If Theorem A is false, then there is a counterexample in which P is elementary abelian and irreducible under the action of G .*

Proof. Suppose (G, P) is a counterexample to Theorem A. Let $A = P/\Phi(P)$. Then G acts faithfully on A and each G -orbit on A has size at most $p - 1$. Since G has no regular orbit on P , it follows that G has no regular orbit on A . Thus (G, A) is a counterexample to Theorem A.

We have $A = [A, G] \times C_A(G)$. Now G acts faithfully on $[A, G]$, so $(G, [A, G])$ is a counterexample to Theorem A. We may therefore assume that $A = [A, G]$. Then if A is a nontrivial direct sum of G -invariant subgroups, then G acts nontrivially on both summands. By Lemma 2.1, G has a regular orbit on A , contrary to hypothesis. We conclude that A has no nontrivial G -invariant direct summand. Since $(|G|, |A|) = 1$, we conclude that A is G -irreducible. $\square$

Definition 2.3. Let V be a d -dimensional vector space over $\text{GF}(p)$. We say that a p' -element $g \in \text{GL}(V)$ is a pseudoreflection if $|C_V(g)| = p^{d-1}$. Note that this implies that g is diagonalizable over $\text{GF}(p)$. If G is a subgroup of $\text{GL}(V)$, we define $r(G, V)$ to be $|G|^{-1}$ times the number of pseudoreflections in G .

Proposition 2.4. *Let V be a finite vector space of characteristic p . Let $G \leq \text{GL}(V)$ be a p' -subgroup of order n . Suppose that every G -orbit on V has size at most $p - 1$ and suppose that $r(G, V) \leq 1/3$. Then $n < 3p/2$ and G has a regular orbit on V .*

Proof. Let S be the set of all ordered pairs $(v, g) \in V \times G$ such that $vg = v$. On the one hand,

$$|S| = \sum_{v \in V} |C_G(v)| \geq \sum_{v \in V} n/(p-1) = n|V|/(p-1).$$

On the other hand, since $|S| = \sum_{g \in G} |C_V(g)|$, the hypotheses give

$$|V| + (n/3)|V|/p + ((2n/3) - 1)|V|/p^2 \geq |S| \geq n|V|/(p-1).$$

Hence

$$(*) \quad 1 + n/3p + ((2n/3) - 1)/p^2 \geq n/(p-1).$$

For fixed p , both sides of $(*)$ are linear functions of n . Since $1/3p + 2/3p^2 < 1/(p-1)$ for all p , the desired conclusion that $(*)$ is false for all $n \geq 3p/2$ will follow if we show that $(*)$ is false for $n = 3p/2$.

Suppose then that $n = 3p/2$, so that $(*)$ becomes

$$1 + 1/2 + (p-1)/p^2 \geq 3/2 + 3/2(p-1),$$

which simplifies to $(p-1)^2 \geq 3p^2/2$, which is false for all p , as desired. Hence $n < 3p/2$.

To show that G has a regular orbit on V , it suffices to show $|S| = \sum_{g \in G} |C_V(g)|$ is less than $2|V|$. Since $r(G, V) \leq 1/3$ and $n < 3p/2$, we have

$$|S| < |V|(1 + n/3p + 2n/3p^2) < |V|(1 + 1/2 + 1/p) \leq 2|V|,$$

as desired. $\square$

Corollary 2.5. *Let V be a vector space of dimension $d > 3$ over $GF(p)$. Let $G \leq GL(V)$ be an irreducible p' -group such that every G -orbit on V has size at most $p - 1$. Then G has a regular orbit on V .*

Proof. Suppose, to get a contradiction, that $r(G, V) > 1/4$. Then, in particular, $r(G, V) > 0$, and so $C_V(g)$ is a hyperplane in V for some $g \in G$. Since $V/C_V(g)$ is a vector space over $\text{End}_{GF(p)[G]}(V)$, it follows that $\text{End}_{GF(p)[G]}(V) = GF(p)$, and so V is an absolutely irreducible $GF(p)[G]$ -module. It follows that the Brauer character χ of V is an irreducible complex character of G .

If $g \in G$ centralizes a hyperplane in V , then $|\chi(g)| \geq d - 2$. Since $r(G, V) > 1/4$, we have

$$|G| = \sum_{g \in G} |\chi(g)|^2 > (|G|/4)(d - 2)^2.$$

Since $d \geq 4$, we have $(d - 2)^2 \geq 4$, and so $|G| > |G|$, a contradiction. We conclude that $r(G, V) \leq 1/4$. By Proposition 2.4, G has a regular orbit on V . $\square$

3. LOW DIMENSIONS

Throughout this section we assume that V is a faithful irreducible $GF(p)[G]$ -module of cardinality p^d , with $1 < d \leq 3$. We assume that G is a p' -group and that every G -orbit on V has size at most $p - 1$. Our goal is to show that G has a regular orbit on V .

Lemma 3.1. *Suppose G has no regular orbit on V . Then V is an absolutely irreducible $GF(p)[G]$ -module.*

Proof. Let $E = \text{End}_{GF(p)[G]}(V)$. Then V is an absolutely irreducible $E[G]$ -module. Since d is prime, we have $E = GF(p)$ or $E = GF(p^d)$. In the former case, V is an absolutely irreducible $GF(p)[G]$ -module. In the latter case, V has dimension 1 over E and every nonzero vector in V lies in a regular G -orbit. $\square$

Lemma 3.2. *Suppose G has no regular orbit on V . Then $|Z(G)| < 3$ if $d = 3$ and $|Z(G)| < 6$ if $d = 2$.*

Proof. Let $k = |G : Z(G)|$ and let $\{g_1, \dots, g_k\}$ be a set of coset representatives for $Z(G)$ in G . Fix $i \leq k$. Lemma 3.1 implies that every element of $Z(G)$ acts on V as multiplication by some scalar in $GF(p)^*$. If $d = 3$, it follows that at most one element of the coset $g_i Z(G)$ can have 1 as a double eigenvalue on V . Thus $r(G, V) \leq k/|G| = 1/|Z(G)|$. If $d = 2$, then at most two elements of the coset $g_i Z(G)$ can have 1 as an eigenvalue on $|V|$. Hence G contains at most $2k$ pseudoreflections and so $r(G, V) \leq 2k/|G| = 2/|Z(G)|$. The conclusion of the lemma now follows from Proposition 2.4. $\square$

Lemma 3.3. *Suppose a normal subgroup A of G of index d acts Frobeniusly on V . Then G has a regular orbit on V .*

Proof. Since A has a regular orbit on V , the hypotheses for this section imply that $|A| \leq p-1$. Suppose first that $d = 2$. Then $|G-A| \leq p-1$, so the number of nonzero vectors in V fixed by some nonidentity element of G is at most $(p-1)(p-1) < p^2-1$. Hence G has a regular orbit on V . Suppose next that $d = 3$. Fix a coset C of A with $C \neq A$. Then every nonzero vector in V that is fixed by a nonidentity element of G is fixed by an element of C . Since $|C| \leq p-1$, the number of nonzero vectors in V fixed by some nonidentity element of G is at most $(p-1)(p^2-1) < p^3-1$. Hence G has a regular orbit on V , as desired. $\square$

Lemma 3.4. *Suppose that G is not solvable. Then $G = QZ(G)$ for a quasisimple group Q . If $d = 2$, then Q is $\mathrm{SL}_2(5)$. If $d = 3$, then Q is $A_5, \mathrm{L}_2(7)$, or $3A_6$.*

Proof. By Lemma 3.1, G acts coprimely and absolutely irreducibly on V . Hence G is isomorphic to a complex linear group of degree $d \leq 3$. An easy abstract argument (see [2, Lemma 1.1]) shows that $G/Z(G)$ is contained in the automorphism group of a simple group. By [5, pp. 213, 250-251], $G/Z(G)$ is in fact simple of order 60, 168, or 360. Hence $G = F^*(G) = QZ(G)$ for a quasisimple group Q . The possibilities for Q can then be determined from [1]. $\square$

Lemma 3.5. *Suppose G is not solvable. Then G has a regular orbit on V .*

Proof. First suppose $d = 3$. Let $G = QZ(G)$ as in Lemma 3.4. By Lemma 3.2, we may assume $|Z(G)| < 3$. Hence G is $A_5, \mathrm{L}_2(7), A_5 \times Z_2$, or $\mathrm{L}_2(7) \times Z_2$. Since A_5 and $\mathrm{L}_2(7)$ are contained in $\mathrm{SL}(V)$, we have $r(A_5, V) = r(\mathrm{L}_2(7), V) = 0$. If $g \in A_5$ and $\langle z \rangle = Z(A_5 \times Z_2)$, then gz is a pseudoreflexion only if g is an involution. Hence $A_5 \times Z_2$ contains at most 15 pseudoreflexions and so $r(A_5 \times Z_2) \leq 15/120 < 1/3$. Similarly, $r(\mathrm{L}_2(7) \times Z_2) \leq 21/336 < 1/3$. In all cases G has a regular orbit on V by Proposition 2.4.

Suppose next that $d = 2$. By Lemma 3.4 and Lemma 3.2, we have $G = \mathrm{SL}_2(5)$ or $G = \mathrm{SL}_2(5) * Z_4$. Since $\mathrm{SL}_2(5) \leq \mathrm{SL}(V)$, we have $r(\mathrm{SL}_2(5), V) = 0$. Suppose then that $G = \mathrm{SL}_2(5) * Z_4$. Let $Z(G) = \langle z \rangle$ and let $g \in \mathrm{SL}_2(5)$. Then gz is a pseudoreflexion if and only if g has order 4. Since $\mathrm{SL}_2(5)$ contains exactly 30 elements of order 4, we have $r(G, V) \leq 30/240 < 1/3$. In either case G has a regular orbit on V by Proposition 2.4. $\square$

Proposition 3.6. *Suppose that G acts primitively on V . Then G has a regular orbit on V .*

Proof. Suppose that G is a counterexample to Proposition 3.6. By Lemma 3.5, G is solvable. Let A be a normal abelian subgroup of G . We claim that $A \leq Z(G)$. If V_A is reducible, then, since d is prime, V_A is a direct sum of one-dimensional submodules. Since V_A is homogeneous and G acts faithfully on V , we have $A \leq Z(G)$, as desired. Hence we may assume that V_A is irreducible. By [4, Theorem 2.1], G is contained in the semilinear group $\Gamma(p^d)$. Thus G has a normal cyclic subgroup of index d that acts Frobeniusly on V . By Lemma 3.3, G has a regular orbit on V , contrary to hypothesis. This proves the claim.

By [4, Corollary 1.10(i,x)], we therefore have $F(G) = EZ(G)$, where $E \triangleleft G$. By the preceding paragraph, we may assume that E has no cyclic Sylow subgroups. Hence E is extraspecial by [4, Corollary 1.10(ii)]. Since E has a faithful irreducible representation of degree d , we have $|E| = d^3$. Suppose $d = 3$. Then $Z(E) \leq \mathrm{GF}(p)^*$, since E is absolutely irreducible on V . Hence $Z(E) \leq Z(G)$ and so $|Z(G)| \geq 3$. By Lemma 3.2, G has a regular orbit on V , contrary to hypothesis.

Hence $d = 2$. If G were nilpotent, then $G = F(G) = EZ(G)$ would have a normal abelian subgroup A of index 2. As above, $A \leq Z(G)$ and so $|G : Z(G)| = 2$, which is impossible. Hence G is not nilpotent. It follows that $E \cong Q_8$ and $G/Z(G)$ is isomorphic to A_4 or S_4 .

By Lemma 3.2, we have $|Z(G)| < 6$. Hence $|Z(G)| = 2$ or $|Z(G)| = 4$. Since $(3, |Z(G)|) = 1$, G contains an element of order 3 which acts nontrivially on E . Hence G contains an $SL_2(3)$ subgroup and G contains a subgroup S of index 1 or 2 with $S \cong SL_2(3)$ if $|Z(G)| = 2$ and $S \cong SL_2(3) * Z_4$ if $|Z(G)| = 4$.

Suppose that $G = S$. We claim that $r(G, V) \leq 1/3$. Indeed if $|Z(G)| = 2$, so that $S \cong SL_2(3)$, then only elements of order 3 in G can be pseudoreflections. Since G has exactly 8 such elements, the claim follows. If $|Z(G)| = 4$, so that $G \cong SL_2(3) * Z_4$, then $SL_2(3)$ contains at most 8 pseudoreflections as above. If $g \in SL_2(3)$ and $z \in Z(G)$ has order 4, then gz can be a pseudoreflection only if g has order 4. Hence the coset $SL_2(3)z$ contains at most 6 pseudoreflections. Thus $r(G, V) \leq 14/48 < 1/3$. This proves the claim. By Proposition 2.4, G has a regular orbit on V , contrary to hypothesis.

Thus $|G : S| = 2$. Since every S -orbit on V has size at most $p - 1$, the preceding paragraph implies that S has a regular orbit on V . In particular, $|S| \leq p - 1$, and so $p > |S|$. On the other hand, every nonidentity element of $G/Z(G)$ fixes at most two of the $p + 1$ one-dimensional subspaces of V . Thus at most 46 such lines are fixed by noncentral elements of G . If $|G| = 96$, then $p > 48$ and so some such line is fixed by no noncentral element of G . It follows that G has a regular orbit on V , contrary to assumption. Hence $|G| = 48$, and so $p > 24$. We refine the argument above by observing that the number of lines in V fixed by noncentral elements of G is at most twice the number of prime order subgroups of $G/Z(G)$. Since $G/Z(G) \cong S_4$ has 13 prime order subgroups, we see that at most $26 < p + 1$ lines are fixed by some noncentral element of G . As above, this implies that G has a regular orbit on V , the final contradiction. $\square$

Proposition 3.7. *Suppose that G acts imprimitively on V . Then G has a regular orbit on V .*

Proof. We have $V = V_1 \oplus \dots \oplus V_d$, where G permutes the V_i and $\dim(V_i) = 1$ for all i . Let $C = N_G(V_1) \cap \dots \cap N_G(V_d)$. Let $V_i = \langle v_i \rangle$ for $1 \leq i \leq d$.

Suppose that $d = 2$. If every element $x \in G - C$ has order greater than 2, then x^2 is a nontrivial element of C for all such x . It follows that $v_1 + v_2$ lies in a regular G -orbit. We may therefore assume that G contains an involution t that interchanges V_1 and V_2 . After replacing v_1 and v_2 by suitable scalar multiples, we may assume that t interchanges v_1 and v_2 .

Suppose, for some $\alpha \in \text{GF}(p)^*$, that $v_1 + \alpha v_2$ does not lie in a regular G -orbit. Then some $y_\alpha \in C$ sends $v_1 + \alpha v_2$ to $\alpha v_1 + v_2$, so that $y_\alpha t$ fixes $v_1 + \alpha v_2$. Hence y_α sends v_1 to αv_1 and v_2 to $\alpha^{-1} v_2$. Thus G has a regular orbit on V or $\{y_\alpha : \alpha \in \text{GF}(p)^*\}$ is a cyclic subgroup C_0 of C with $|C_0| = p - 1$. Since $v_1 + v_2$ is a regular vector for C , our hypotheses imply that $|C| \leq p - 1$. Thus $C = C_0$. It follows that v_1 lies in a regular G -orbit.

Suppose now that $d = 3$. We proceed to estimate the number of pseudoreflections x in each of the 3 or 6 cosets of C in G . We note first that Cx cannot have order 3 in G/C . Indeed the Brauer character χ of V is induced from a linear character of $N_G(V_1)$. Hence $\chi(x) = 0$ and so 1 cannot be a double eigenvalue of x on V . Suppose next that Cx has order 2 in G/C ; say x fixes V_3 and interchanges V_1

and V_2 . Clearly x must centralize v_3 . Suppose that x sends v_1 to βv_2 and v_2 to γv_1 for $\beta, \gamma \in \text{GF}(p)^*$. Then x^2 has $\beta\gamma$ as a double or triple eigenvalue. Since 1 is a double or triple eigenvalue of x^2 , we have $\gamma = \beta^{-1}$ and so $\det(x) = -1$. Now if $y \in Cx$ is another pseudoreflection, then $v_3y = v_3$ and $\det(y) = -1$. Thus $yx^{-1} \in D := C \cap C_G(v_3) \cap \text{SL}(V)$. We have shown that Cx contains at most $|D|$ pseudoreflections. If $g \in G$ does not fix V_3 , then $D \cap D^g = 1$, and so $|D|^2 = |DD^g| \leq |C|$. Hence Cx contains at most $\sqrt{|C|}$ pseudoreflections. Since $v_1 + v_2 + v_3$ lies in a regular C -orbit, our hypotheses imply that $|C| \leq p-1$ and so $|D| \leq \sqrt{p-1}$.

We claim that C contains either 0 or 3 pseudoreflections. Indeed if $t \in C$ is a pseudoreflection whose matrix with respect to the ordered basis (v_1, v_2, v_3) is $\text{diag}(\lambda, 1, 1)$, then it is easy to see that $\text{diag}(1, \lambda, 1)$ and $\text{diag}(1, 1, \lambda)$ are G -conjugates of t . Hence $\text{diag}(\lambda, \lambda, \lambda) \in Z(G)$. By Lemma 3.2, however, $|Z(G)| \leq 2$, forcing $\lambda = -1$. This proves the claim. We note that if C contains 3 pseudoreflections, then they generate a subgroup of order 8, so that $|C| \geq 8$.

If $|G : C| = 3$, then $r(G, V) = 0$ or $r(G, V) = 3/|G| = 1/|C|$, with the latter possibility occurring only if $|C| \geq 8$. Hence $r(G, V) \leq 1/3$, and so G has a regular orbit on V by Proposition 2.4.

We may now assume that $|G : C| = 6$. Let $C < N < G$ with $|G : N| = 2$. Then N has a regular orbit on V by the preceding paragraph, and so our hypotheses imply that $|N| \leq p-1$. Thus $|G| \leq 2(p-1)$ and $|C| \leq (p-1)/3$. Now G contains at most $3\sqrt{|C|} + 3$ pseudoreflections. The number of nonzero vectors in V centralized by some nonidentity element of G is therefore less than

$$2(p-1)(p-1) + (3\sqrt{(p-1)/3} + 3)(p^2 - 1).$$

We claim that this quantity is less than $|V| - 1$ for $p \geq 11$. That is, we claim that

$$2(p-1) + (3\sqrt{(p-1)/3} + 3)(p+1) < p^2 + p + 1$$

for $p \geq 11$. To verify this inequality, it suffices to show that $3\sqrt{(p-1)/3} + 3 \leq p-2$ for $p \geq 11$. But the last inequality is equivalent to $3(p-1) \leq (p-5)^2$, which indeed holds for $p \geq 11$. Hence G has a regular orbit on V if $p \geq 11$.

Assume then that $p \leq 7$. Then $|G| \leq 2(p-1) \leq 12$. Since $|G : C| = 6$, either $|G| = 12$ or $|G| = 6$. Since the Brauer character χ of V is an irreducible complex character of G , we have $|G| = 12$. But then $|C| = 2$, so $|G| = 12, |Z(G)| \geq 2$, and some $\chi \in \text{Irr}(G)$ has degree 3. This is impossible. Hence $p \geq 11$ and G has a regular orbit on V , as desired. $\square$

Proof of Theorem A. By Lemma 2.2, we may assume $P = V$, a G -irreducible elementary abelian group of cardinality p^d . By Corollary 2.5, we may assume that $1 < d \leq 3$. Now Theorem A follows from Proposition 3.6 and Proposition 3.7. $\square$

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