ON LI-YORKE MEASURABLE SENSITIVITY

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ABSTRACT. The notion of Li-Yorke sensitivity has been studied extensively in the case of topological dynamical systems. We introduce a measurable version of Li-Yorke sensitivity, for nonsingular (and measure-preserving) dynamical systems, and compare it with various mixing notions. It is known that in the case of nonsingular dynamical systems, a conservative ergodic Cartesian square implies double ergodicity, which in turn implies weak mixing, but the converses do not hold in general, though they are all equivalent in the finite measure-preserving case. We show that for nonsingular systems, an ergodic Cartesian square implies Li-Yorke measurable sensitivity, which in turn implies weak mixing. As a consequence we obtain that, in the finite measure-preserving case, Li-Yorke measurable sensitivity is equivalent to weak mixing. We also show that with respect to totally bounded metrics, double ergodicity implies Li-Yorke measurable sensitivity.

1. INTRODUCTION

The notion of sensitive dependence for topological dynamical systems has been studied by many authors; see, for example, the works [3, 7, 10] and the references therein. Recently, various notions of measurable sensitivity have been explored in ergodic theory; see for example [2, 9, 11–14].

In this paper we are interested in formulating a measurable version of the topological notion of Li-Yorke sensitivity for the case of nonsingular and measure-preserving dynamical systems.

In Section 2, we review some preliminary definitions and introduce the notion of Li-Yorke measurable sensitivity (also called Li-Yorke M-sensitivity); which is based on the topological notion of Li-Yorke sensitivity in [5]; and prove that in the conservative ergodic case it implies W-sensitivity introduced in [11]. In Section 3 we prove that if the Cartesian square is conservative ergodic (a condition stronger than weak mixing in the nonsingular case [1]), then it is Li-Yorke M-sensitive. Section 4 shows that for conservative ergodic nonsingular systems, an ergodic Cartesian square implies Li-Yorke M-sensitivity, which in turn implies weak mixing; a consequence of this is that in the finite measure-preserving case Li-Yorke sensitivity is equivalent to weak mixing. Section 5 studies scrambled sets. In Section 6 we consider conservative ergodic infinite measure-preserving transformations which are

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not W -measurably sensitive. The final sections study entropy and the existence of a W -measurably sensitive μ -compatible metric for any conservative ergodic transformation.

2. PRELIMINARY DEFINITIONS AND MEASURABLE SENSITIVITY

A *nonsingular dynamical system* $(X, \mathcal{S}, \mu, T)$ is a standard Borel space $(X, \mathcal{S})$ with a σ -finite, nonatomic measure μ and a nonsingular endomorphism $T : X \rightarrow X$ (i.e., for all $A \in \mathcal{S}$, $T^{-1}(A) \in \mathcal{S}$ and $\mu A = 0$ if and only if $\mu(T^{-1}A) = 0$). We sometimes take T to be measure-preserving or the measure space to be finite. We sometimes suppress $\mathcal{S}$ and write (X, μ, T) . We say that T is conservative if for all A of positive measure there exists $n > 0$ such that $\mu(T^{-n}A \cap A) > 0$. A set A is positively invariant if $A \subset T^{-1}(A)$ and invariant if $T^{-1}(A) = A$. If T is conservative and A is positively invariant, then A is invariant mod μ (henceforth all equalities will be interpreted as mod μ). A transformation T is ergodic if whenever A is invariant, then $\mu(A) = 0$ or $\mu(X \setminus A) = 0$. A nonsingular transformation T is weakly mixing if whenever f is an L^∞ function such that $f \circ T = zf$ for $z \in \mathbb{C}$, then f is constant a.e. If $T \times T$ is ergodic, then T is weakly mixing, but the converse, while true in the finite measure-preserving case, does not hold in general [1].

We consider metrics d on X . We assume throughout that these are Borel measurable on $X \times X$ and bounded by 1. We say a metric d on X is μ -compatible if μ assigns positive measure to nonempty, open d -balls [11, 14]. It follows from [14, Lemma 1.1] that the topology generated by d is separable. Thus open sets are measurable as they are countable unions of balls. The notion of measurable sensitivity was introduced in [14].

Definition 2.1. We say a nonsingular dynamical system (X, μ, T) is *measurably sensitive* if for every isomorphic mod 0 dynamical system (X_1, μ_1, T_1) and μ_1 -compatible metric d on X_1 there exists a $\delta > 0$ such that for all $x \in X_1$ and $\varepsilon > 0$ there is an $n \in \mathbb{N}$ such that

$$\mu_1\{y \in B_\varepsilon(x) : d(T_1^n(x), T_1^n(y)) > \delta\} > 0.$$

This definition was refined in [11].

Definition 2.2. Let (X, μ, T) be a nonsingular dynamical system and d a μ -compatible metric on X . We say the system is *W -measurably sensitive* with respect to d if there is a $\delta > 0$ such that for each $x \in X$

$$\limsup_{n \rightarrow \infty} d(T^n x, T^n y) > \delta$$

for a.e. $y \in X$. The system is *W -measurably sensitive* if it is W -measurably sensitive with respect to each μ -compatible metric d .

Proposition 7.2 in [11] shows that for conservative ergodic transformations these two notions are equivalent. Requiring that the condition hold for each $x \in X$ and a.e. $y \in X$ is equivalent to requiring that it hold for a.e. pair $(x, y) \in X^2$ (Proposition A.1 in [11]), a notion called pairwise sensitivity introduced in [9]. The following classification result is proved in [11, Theorem 7.1].

Theorem 1. *Let (X, μ, T) be a conservative ergodic nonsingular dynamical system. Then T is W -measurably sensitive or T is isomorphic mod 0 to an invertible minimal uniformly rigid isometry on a Polish space.*

The proof of this theorem shows that the isometric metric is μ -compatible. The following are two technical propositions from [11] that are needed for our work. The proofs follow those in the aforementioned paper and are included for completeness.

Proposition 2.1. *Suppose T is a nonsingular transformation. If for almost every pair $(x, y) \in X \times X$ there exists $n \geq 0$ such that $d(T^n x, T^n y) \geq \delta$, then for almost every pair $(x, y) \in X \times X$ we have $\limsup_{n \rightarrow \infty} d(T^n x, T^n y) \geq \delta$.*

Proof. Let

$$Z(N, x) = \{y \in X : \exists n, d(T^n(T^N x), T^n y) \geq \delta\}.$$

Then by hypothesis there exists a full measure set A such that $\mu(Z(N, x)^c) = 0$ for each $x \in A$ and for all $N \in \mathbb{N}$. Let

$$Y(N, x) = \{y \in X : \exists n > N, d(T^n x, T^n y) \geq \delta\}.$$

Note that $Y(N, x) = T^{-N}(Z(N, x))$. Since $Z(N, x)$ has full measure and T is nonsingular, we see that $Y(N, x)$ also has full measure. Hence

$$\bigcap_{N>0} Y(N, x)$$

also has full measure. But this says for each $x \in A$ and almost every $y \in X$ that $\limsup_{n \rightarrow \infty} d(T^n x, T^n y) \geq \delta$. $\square$

The proof of the next proposition follows from the same arguments as above.

Proposition 2.2. *Suppose T is a nonsingular transformation. If for almost every pair $(x, y) \in X \times X$ there exists $n \geq 0$ such that $d(T^n x, T^n y) \leq \delta$, then for almost every pair $(x, y) \in X \times X$ we have $\liminf_{n \rightarrow \infty} d(T^n x, T^n y) \leq \delta$.*

The notion of W -measurable sensitivity adapts the notion of sensitivity to initial conditions from topological dynamics to the measurable case. In topological dynamics there is also the notion of Li-Yorke sensitivity, in which points are not only required to separate but also to come back together. We give the definitions of topological Li-Yorke sensitivity as in [5].

We recall the following definition from topological dynamics.

Definition 2.3. Let (X, d, T) be a topological dynamical system. A pair (x, y) is said to be *proximal* if

$$\liminf_{n \rightarrow \infty} d(T^n x, T^n y) = 0.$$

Akin and Kolyada introduced the following in [5].

Definition 2.4. Let (X, d, T) be a topological dynamical system. We call the system *Li-Yorke sensitive* if there exists an $\varepsilon > 0$ such that every $x \in X$ is a limit of points $y \in X$ such that the pair (x, y) is proximal but whose orbits are at least ε apart at arbitrarily large times.

In this paper we consider the measure-theoretic analogue of Li-Yorke sensitivity.

Definition 2.5. Let (X, μ, T) be a nonsingular dynamical system and d a μ -compatible metric on X . We say that a pair (x, y) is a *Li-Yorke pair* if

$$\liminf_{n \rightarrow \infty} d(T^n x, T^n y) = 0 \text{ and } \limsup_{n \rightarrow \infty} d(T^n x, T^n y) > 0.$$

We say (X, μ, T) is *Li-Yorke measurably sensitive for the metric d* if the set of Li-Yorke pairs $(x, y) \in X \times X$ has full measure. We say it is *Li-Yorke measurably*

sensitive (henceforth Li-Yorke M-sensitive) if it is Li-Yorke M-sensitive for all μ -compatible metrics.

Note that atomic measures are never sensitive. We now show that Li-Yorke M-sensitivity is a (measurable) isomorphism invariant. We need the following lemma [11, Lemma 6.1].

Lemma 2.3. *Let $(X, \mathcal{S})$ be a standard Borel space with nonatomic measure μ . Let $U \subset X$ be a Borel subset of full measure and d_U a μ -compatible metric defined on U . Then d_U can be extended to a μ -compatible metric d_X on X such that d_U and d_X agree on a set of full measure contained in $U \times U$.*

The fact that the set is contained in $U \times U$ is not in the statement but follows from the proof of the lemma in the original paper. We now proceed in a way similar to Proposition 6.2 of the same paper.

Proposition 2.4. *Suppose nonsingular dynamical system (X, μ, T) is Li-Yorke M-sensitive. Then any isomorphic system (Y, ν, S) is also Li-Yorke M-sensitive.*

Proof. Suppose (Y, ν, S) is not Li-Yorke M-sensitive. Then there is a ν -compatible metric d_Y on Y for which (Y, ν, S) is not Li-Yorke M-sensitive. As the systems are isomorphic, there are Borel sets $U \subset X$ and $V \subset Y$ of full measure and a bijection $\phi : U \rightarrow V$ such that $\phi \circ T = S \circ \phi$. Define a μ -compatible metric d_U on U by $d_U(x, y) = d_Y(\phi(x), \phi(y))$. Applying Lemma 2.3 extends d_U to a μ -compatible metric d_X on X which agrees with d_U on a set $X_0 \subset U \times U$ of full measure in $X \times X$. By hypothesis T is Li-Yorke M-sensitive, so the set $L \subset X^2$ of Li-Yorke pairs has full measure. It follows that the set $A = \bigcap (T \times T)^{-n}(X_0 \cap L)$ has full measure. Note that if $(x, y) \in A$, it is Li-Yorke, and for all n we have that $d_U(T^n x, T^n y) = d_X(T^n x, T^n y)$ and $\phi(T^n x), \phi(T^n y) \in V$ (since $T^n x, T^n y \in U$). Now $\phi \times \phi(A)$ has full measure in $Y \times Y$, and for all $(\phi(x), \phi(y)) \in \phi \times \phi(A)$ we have for all n that

$$d_Y(S^n \phi(x), S^n \phi(y)) = d_U(T^n x, T^n y) = d_X(T^n x, T^n y).$$

It follows that all pairs in $\phi \times \phi(A)$ are Li-Yorke for d_Y , a contradiction. $\square$

Li-Yorke M-sensitivity is not a priori stronger than W-measurable sensitivity. But this turns out to be the case when the system in question is conservative ergodic.

Proposition 2.5. *Let (X, μ, T) be a conservative ergodic and nonsingular dynamical system. If it is Li-Yorke M-sensitive, then it is W-measurably sensitive.*

Proof. We show the contrapositive. If T is not W-measurably sensitive, then by Theorem 1 it is isomorphic mod 0 to an isometry. But then the isomorphic system is both Li-Yorke M-sensitive and an isometry for a μ -compatible metric, which is impossible. $\square$

3. MIXING CONDITIONS STRONGER THAN LI-YORKE SENSITIVITY

We study not just the $\liminf$ and $\limsup$ but also the set of distances to which T separates pairs. Define

$$\mathcal{N} = \{r : \text{for a.e. } (x, y) \in X^2, \exists \{n_k\} \text{ s.t. } d(T^{n_k} x, T^{n_k} y) \rightarrow r\}.$$

Notice that $0 \in \mathcal{N}$ for Li-Yorke M-sensitive systems. Further, it is easily shown that $\mathcal{N}$ is closed. We then have the following result. Let G be the set of values taken on by the metric d .

Proposition 3.1. *Let $T \times T$ be conservative ergodic nonsingular and d be a μ -compatible metric on (X, μ) . Then for a.e. $(x, y) \in X^2$, we have that $(T^n x, T^n y)$ is dense in the product topology, so that for any $r \in G$ there is an $\{n_k\}$ such that $d(T^{n_k} x, T^{n_k} y) \rightarrow r$.*

Proof. Since (X, d) is separable it has a countable topological basis $\{B_i\}$ of balls. As the metric is μ -compatible, $B_i \times B_j$ has positive measure in the product space for each i and j . It then follows from the ergodicity of $T \times T$ that

$$A = \bigcap_{i,j} \bigcup_n (T \times T)^{-n} (B_i \times B_j)$$

has full measure. By construction $(T^n x, T^n y)$ is dense in the product topology for all $(x, y) \in A$. Now for each k one can choose n_k such that $T^{n_k} x \in B(z_1, 1/k)$ and $T^{n_k} y \in B(z_2, 1/k)$, where $d(z_1, z_2) = r$. $\square$

Corollary 3.2. *Under the hypotheses of Proposition (3.1) $\overline{G} \subset \mathcal{N}$. In particular, T is Li-Yorke M-sensitive and W -measurably sensitive.*

We now investigate another condition under which T is Li-Yorke M-sensitive. The ergodicity of $T \times T$ is a strong condition in general. We can extend the results from the $T \times T$ ergodic case to similar results in the doubly ergodic (i.e., for all sets of positive measure A and B iterates of A intersect both A and B in positive measure) case. The main difference is the necessity of requiring the metric to have some degree of compactness. Define H to be the set of $r \in G$ such that for every ball B there is a point $x \in B$ and $y \in X$ such that $d(x, y) = r$. Note that $0 \in H$.

Proposition 3.3. *Let T be doubly ergodic and let d be a μ -compatible metric such that (X, d) is compact. For each $r \in H$, for all $x \in X$, for a.e. $y \in X$ there is a sequence $\{n_k\}$ such that $d(T^{n_k} x, T^{n_k} y) \rightarrow r$.*

Proof. Suppose the contrary. Then there is some $r \in H$, $x \in X$ and positive measure set $B \subset X$ such that for no y in B is it the case that $d(T^{n_k} x, T^{n_k} y) \rightarrow r$ for any sequence $\{n_k\}$. Then for each $y \in B$ there is an N and an ε such that $|d(T^n x, T^n y) - r| \geq \varepsilon$ for all $n \geq N$. Thus we can find an N and an ε for which this property holds on a positive measure subset of B (which for simplicity we just relabel B). Cover X with a finite number of $\varepsilon/2$ -balls $B_1, \dots, B_k$. As T is doubly ergodic, it is k -ergodic [8], so we can choose an n so that $\mu(T^{-n} B_i \cap B) > 0$ for each $i = 1, \dots, k$. We have that $T^n x$ is contained in some ball B_j , and by hypothesis this ball contains a point x' such that for some y' in the space $d(x', y') = r$. Now y' is contained in some ball B_ℓ . Consider any $y \in T^{-n} B_\ell \cap B$. Then $r - \varepsilon < d(T^n x, T^n y) < r + \varepsilon$ for all $y \in B'$, a contradiction. $\square$

Lemma 3.4. *If T is doubly ergodic, then it does not admit a μ -compatible metric for which it is an isometry.*

Proof. Suppose not and let d be a μ -compatible isometric metric. Choose an $\varepsilon > 0$ such that there exist sets $A, B \subset X$ with $d(A, B) > \varepsilon$. Since d is μ -compatible, $\mu(B_{\varepsilon/2}(x)) > 0$. By double ergodicity there exists an n such that

$$\mu(T^{-n}(B_{\varepsilon/2}(x)) \cap A) > 0 \quad \mu(T^{-n}(B_{\varepsilon/2}(x)) \cap B) > 0.$$

This implies that the diameter of $T^n(B_{\varepsilon/2}(x))$ is strictly greater than ε , contradicting the fact that T is an isometry for d . $\square$

Corollary 3.5. *Let T be doubly ergodic and d be a totally bounded μ -compatible metric. Then T is Li-Yorke M -sensitive with respect to d .*

Proof. By Proposition 6.3 in [11] we know that T is either W -measurably sensitive or is measurably isomorphic to an isometry. Since it is doubly ergodic the preceding lemma shows that it cannot be isomorphic to an isometry and hence is W -measurably sensitive. Thus T separates points.

We now show that T brings points together. Suppose it did not. Proceeding as in the proof of Theorem 1 (Theorem 7.1 in [11]), let (X_1, d_1) be the topological completion of (X, d) . We can extend the measure to this space by letting sets $S \subset X_1$ be measurable when $S \cap X$ is measurable and by defining $\mu_1(S) = \mu(S \cap X)$. Define T_1 on X_1 as T on X and as the identity on $X_1 \setminus X$. This new system is isomorphic mod 0 to (X, μ, T) , so T is doubly ergodic. Since (X, d) is totally bounded, (X_1, d_1) is compact. Then by Proposition 3.3, T_1 brings points together in the metric d_1 . It follows that T brings points together in the metric d . $\square$

4. LI-YORKE M -SENSITIVITY AND WEAK MIXING

We now show that Li-Yorke sensitivity implies weak mixing, which will be a converse to the above results in the finite measure-preserving case. We first show the useful fact that every positive measure subset of a standard Borel space admits a μ -compatible metric, which in fact turns out to be totally bounded. We start with a lemma whose proof is standard.

Lemma 4.1. *Given a measure space $(X, \mathcal{S}, \mu)$ with a σ -finite measure μ , there exists an equivalent probability measure η .*

Proposition 4.2. *Let $(X, \mathcal{B}, \mu)$ be a standard Borel space with nonatomic σ -finite measure μ . Then there exists a totally bounded μ -compatible metric d on X .*

Proof. By Lemma 4.1 there exists an equivalent probability measure η . Now using the isomorphism theorem for measures [15, Theorem 17.41], there is a Borel isomorphism $f : X \rightarrow [0, 1]$ with $f\eta = m$, where m is the Lebesgue measure on $[0, 1]$. Define a metric d on X by

$$d(x, y) = |f(x) - f(y)|.$$

Note that this defines a totally bounded metric. Now we show it is μ -compatible. Given $x \in X$ and $\varepsilon > 0$ consider $B_\varepsilon(x)$. The set $f(B_\varepsilon(x))$ is an open ball of radius ε about $f(x)$ on the interval $[0, 1]$ and thus has positive Lebesgue measure. As $m(f(B_\varepsilon(x))) > 0$, we have that $\eta(B_\varepsilon(x)) > 0$, so that $\mu(B_\varepsilon(x)) > 0$. Hence d is μ -compatible. $\square$

Corollary 4.3. *Let $(X, \mathcal{B}, \mu)$ be a standard Borel space with nonatomic σ -finite measure μ and $Y \in \mathcal{B}$ be a positive measure subset of X . Then there exists a totally bounded μ -compatible metric d on Y .*

Proof. By [15, Corollary 13.4] we know that Y is a standard Borel space and the restriction of μ to Y is a nonatomic nontrivial σ -finite measure. $\square$

We now show the main result of this section.

Proposition 4.4. *Let (X, μ, T) be a conservative ergodic nonsingular dynamical system. If T is Li-Yorke M -sensitive, then it is weakly mixing.*

Proof. We prove the contrapositive. Suppose T is not weakly mixing, so that it has a nonconstant eigenfunction $f : X \rightarrow S^1$. This is a factor map mapping (X, μ, T) to (S^1, ν, R) , where ν is some measure on the circle and R is a rotation. Given $z, z' \in S^1$ let $a(z, z')$ denote the normalized arc length on the circle (so $a(1, -1) = \frac{1}{2}$). Since the map f is nontrivial and measure-preserving, there exists an m such that the set $A \subset X^2$ defined by

$$A = \{(x, y) \in X \times X : a(f(x), f(y)) \geq 1/m\}$$

has positive measure. Let $D_i = f^{-1}([e^{\frac{2\pi(i-1)}{m}}, e^{\frac{2\pi i}{m}}))$ for $i = 1, \dots, m$. Relabel the D_i as

$$\{E_1, \dots, E_k, \dots, E_m\},$$

where $\mu(E_j) = 0$ for $j > k$. Then we have

$$X = (E_1 \sqcup \dots \sqcup E_k) \mod 0.$$

By removing from X the backwards orbits of E_j for $j > k$, a measure zero set, we obtain a T -invariant set of full measure. Thus since Li-Yorke M -sensitivity is preserved under measurable isomorphism we can assume that $X = E_1 \sqcup \dots \sqcup E_k$. By Corollary 4.3 we know that for each $j \leq k$ there exists a metric d_j on E_j which is μ -compatible. Define a metric d on X by

$$d(x, y) = \begin{cases} 1 & \text{if } x \in E_i, y \in E_j, i \neq j, \\ d_i(x, y) & \text{if } x, y \in E_i. \end{cases}$$

Note that d is a μ -compatible metric for X . As A has positive measure in the product space, showing that no $(x, y) \in A$ is proximal suffices to show that T is not Li-Yorke M -sensitive. Note that since R is a rotation it is an isometry under arc length. Then if $(x, y) \in A$,

$$a(f(T^n x), f(T^n y)) = a(R^n f(x), R^n f(y)) = a(f(x), f(y)) \geq 1/m.$$

This implies that $T^n x$ and $T^n y$ cannot lie in the same E_i , for otherwise we would have that $a(f(x), f(y)) < 1/m$. Thus $d(T^n x, T^n y) = 1$ for all $n \geq 0$, which implies that (x, y) is not proximal. $\square$

Collecting our results we see that we have the following chain of implications.

Theorem 2. *Let (X, μ, T) be a conservative ergodic nonsingular dynamical system. Then*

$$T \times T \text{ ergodic} \Rightarrow T \text{ Li-Yorke } M\text{-sensitive} \Rightarrow T \text{ weakly mixing}.$$

In the finite measure-preserving case it is known that weak mixing and ergodicity of the product are equivalent. We thus have the following result.

Theorem 3. *Suppose (X, T, μ) is a finite measure-preserving ergodic dynamical system. Then T is Li-Yorke M -sensitive if and only if it is weakly mixing.*

Note that in this case T is not just Li-Yorke M -sensitive but in fact separates points arbitrarily close to every possible distance that any given μ -compatible metric assumes. We now show the following lemma, which will allow us to relate this result to the classification result on W -measurable sensitivity (Theorem 1).

Lemma 4.5. *Suppose (X, μ, T) is an ergodic finite measure-preserving transformation. Then T is weakly mixing if and only if T has no nontrivial factors which admit a μ -compatible metric for which they are an isometry.*

Proof. If T is weak mixing, then all its factors are doubly ergodic and so by Lemma 3.4 do not admit μ -compatible isometric metrics. If T is not weak mixing, then it has a nontrivial rotation factor. If the rotation is irrational, then the measure in the factor must be a Lebesgue measure, for which arc length is an isometry. Otherwise, the factor is a rotation on n points. Spacing these atoms evenly around the unit circle and using arc length gives an isometric metric. $\square$

Corollary 4.6. *Suppose (X, μ, T) is a finite measure-preserving ergodic dynamical system. Then T is Li-Yorke M -sensitive if and only if no nontrivial factors of T admit a μ -compatible isometric metric.*

5. SCRAMBLED SETS

The notion of a scrambled set in the topological case goes back to Li and Yorke [16]. Akin and Kolyada [5] take the existence of an uncountable scrambled set as indicating that a system is chaotic. Here we show that Li-Yorke M -sensitive systems always contain an uncountable scrambled set.

Definition 5.1. A set $A \subset X$ is said to be scrambled if for all $x, y \in A$ with $x \neq y$ the pair (x, y) is Li-Yorke.

Proposition 5.1. *Let $A \subset X^2$ have positive measure and suppose there exists $D \in X$ with positive measure such that $A \subset D \times D$ and A has full measure in $D \times D$. Then for almost every $x \in D$ there exists an uncountable set $B \subset X$ containing x such that $y, z \in B$ with $y \neq z$ implies $(y, z) \in A$.*

Proof. In this context we use the term scrambled set to refer to a set C such that $y, z \in C$ and $y \neq z$ implies $(y, z) \in A$. We can assume without loss of generality that $D^2 = X^2$ and that A has full measure. Then a.e. $x \in X$ has a full measure fiber, i.e. $\mu\{y \in X : (x, y) \in A\} = 1$. Let the set of such x be denoted X_0 . For any $x, y \in X_0$ let P be the collection of scrambled sets contained in X_0 and containing $\{x, y\}$. This is nonempty as $\{x, y\} \in P$. Let P be partially ordered by set inclusion and take Q to be any totally ordered subset. Then $\bigcup_{S \in Q} S$ is contained in P and is an upper bound for Q . Thus every totally ordered subset of P has an upper bound. Now Zorn's Lemma guarantees the existence of an element M such that $M \in P$ and M is not contained in any other element of P . We show M is uncountable. Suppose it were not. Then for $z \in A$ let $C(z) = \{w \in X : (z, w) \text{ is Li-Yorke}\}$. We have $\mu(C(z)^c) = 0$ so that $\mu((\bigcap_{z \in A} C(z))^c) = 0$. In particular $\bigcap_{z \in A} C(z)$ is nonempty and so contains some w . But then $M' = M \cup \{w\}$ is scrambled and $M \subsetneq M'$. $\square$

Corollary 5.2. *Suppose T is a Li-Yorke M -sensitive nonsingular transformation. Then a.e. x is contained in an uncountable scrambled set.*

Thus a Li-Yorke M -sensitive transformation is also chaotic in the sense of [5] when viewed as a topological dynamical system under any μ -compatible metric. Uncountable scrambled sets play a smaller role here than they do in topological dynamics because uncountable sets can still be measure-theoretically small. However, this idea will be important in proving that periodic transformations admit no μ -compatible sensitive metrics.

6. WEAK MIXING AND W-MEASURABLE SENSITIVITY

Weak mixing implies W-measurable sensitivity in the finite measure-preserving case; as in this case weak mixing is equivalent to double ergodicity. We do not know if this is true in the general nonsingular case. Here we study in more detail when the system is not W-measurably sensitive. We start with the following lemma that proves a certain uniformity of the measures for μ -compatible isometries.

Lemma 6.1. *Let (X, μ, T) be a conservative ergodic measure-preserving dynamical system and d a μ -compatible metric on X for which T is an invertible isometry. Then the measure of balls $B_r(x)$ depends only on the radius.*

Proof. By Lemma 5.4 in [11], almost every point of X is transitive. Let x be such a point, and consider $f(r) = \mu B(x, r)$. As μ is nonatomic $f(0) = 0$, and since the function is increasing we may define $R = \sup\{r : \mu B(x, r) < \infty\}$. Then f is continuous for $r < R$ and moreover left-continuous at R if $R < \infty$. Let $y \in X$ and $r, \varepsilon > 0$. There is an n such that $d(T^n x, y) < \varepsilon$. Then $T^n(B_{r-\varepsilon}(x)) \subset B_r(y) \subset T^n(B_{r+\varepsilon}(x))$, so that

$$\mu(B(x, r - \varepsilon)) \leq \mu(B(y, r)) \leq \mu(B(x, r + \varepsilon)).$$

For $r < R$ we then have by continuity that $\mu B(x, r) = \mu B(y, r)$.

If $r = R < \infty$, then applying left-continuity and the $r < R$ case we find that $\mu B(y, R) = \lim_{\varepsilon \rightarrow 0+} \mu B(y, R - \varepsilon) = \lim_{\varepsilon \rightarrow 0+} \mu B(x, R - \varepsilon) = \mu B(x, R)$. If $r > R$, then for some ε we have $\mu(B(y, r)) \geq \mu(B(x, r - \varepsilon)) = \infty$, and hence $\mu(B(y, r)) = \infty = \mu(B(x, r))$. $\square$

The following proposition makes explicit some conclusions of the classification theorem in [11] (Theorem 1) and introduces notation that we use when we prove in the remark below that locally finite measure metrics are not possible when T is infinite measure-preserving. These are basically additional conclusions in Proposition 5.6 of [11] that were not stated there explicitly (namely, that every element of G is an isometry and the group is abelian); as mentioned in [11], Proposition 5.6, is essentially from Akin and Glasner [4].

Proposition 6.2 ([11]). *Let (X, μ, T) be a conservative ergodic nonsingular dynamical system. If T is not W-measurably sensitive, then it is isomorphic mod 0 to a minimal uniformly rigid invertible isometry on a monothetic, hence abelian, Polish group of isometries.*

Proof. Suppose T is not W-measurably sensitive with respect to a μ -compatible metric d . As then d is separable, there is a dense set of points $\{x_i\}$; thus the collection of balls of rational radius about each x_i is a countable collection of Borel sets that separates points. By Blackwell's theorem [20, 4.5.10] it generates the Borel sets of X . Lemma 5.3 in [11] shows that there is a positively invariant Borel set of full measure X_0 and a μ -compatible metric d_T on X_0 for which T is an isometry, where $d_T(x, y) = \sup_{n \geq 0} d(T^n x, T^n y)$. In Theorem 1 from [11], (X_0, μ, d_T, T) is extended to its topological completion (of which it is a Borel subset as X_0 is standard, [19, p. 134]). Relabel this system (X, μ, T) with metric d_X . Proposition 5.6 in [11] and the discussion preceding it demonstrate the following. Let G be the set of continuous transformations on X which commute with T . Then G is a group. For $S_1, S_2 \in G$ define $d_G(S_1, S_2) = \sup_{x \in X} d_X(S_1 x, S_2 x)$. Then d_G is a metric on G and G is equal to the closure of the set $\{I, T, T^2, \dots\}$, so it is

a monothetic group. Moreover, for any $x \in X$ the map $\phi_x : G \rightarrow X$ given by $\phi_x(S) = Sx$ is an isometry. Choose one such map ϕ and use it to induce a measure η on G . Let $T : G \rightarrow G$ denote the group rotation on G by the element T . Then (G, η, T) is isomorphic to (X, μ, T) by ϕ .

Since ϕ_x is an isometry for all x , $d_X(S_1x, S_2x) = d_G(S_1, S_2) = d_X(S_1y, S_2y)$ for all $x, y \in X$. Then a right cancellation law holds: $d_G(S_1S, S_2S) = d_G(S_1Sx, S_2Sx) = d_G(S_1(Sx), S_2(Sx)) = d_G(S_1, S_2)$, for $S, S_1, S_2 \in G$. For every map $S \in G$ there is a sequence T^{n_k} which converges uniformly to S . It follows that S is an isometry on (G, d_G) . That G is abelian is a direct consequence since it is monothetic: Let $S_1, S_2 \in G$ and $\varepsilon > 0$. Choose k so that $d_G(T^k S_1, S_2) < \varepsilon/2$. Then $d_G(S_1S_2, S_2S_1) \leq d_G(T^k S_1S_2, S_2S_2) + d_G(S_2S_2, S_2T^k S_1) = 2d_G(T^k S_1, S_2) < \varepsilon$. $\square$

We will need the following lemma, that may be standard, to show local compactness. Say a metric d is *locally measure finite* if each point of the space is contained in an open d -ball of finite measure.

Lemma 6.3. *Let (X, d) be a complete metric space. Let μ be a Borel measure on X such that all open balls of the same radius have the same measure, and such that d is locally measure finite. Then (X, d) is locally compact.*

Proof. For x in X there exists $r > 0$ so that for all $0 < \varepsilon < r/2$ the closure of the ball $B(x, r + \varepsilon)$, denoted E , has finite measure. As all balls have the same radius we may assume r works for all x . Let $0 < \varepsilon < r/2$ and K be the set of all positive integers k so that there are k disjoint balls each of radius ε contained in E . As all balls have the same measure, K is finite. Let ℓ be its maximum and $B(y_1, \varepsilon), \dots, B(y_\ell, \varepsilon)$ be balls that attain this maximum. Then $E \setminus \bigcup_{i=1}^\ell B(y_i, \varepsilon)$ does not contain any ball of radius ε . It follows that $\bigcup_{i=1}^\ell B(y_i, 2\varepsilon)$ covers the closure of $B(x, r)$, denoted E_0 . For if not there would exist y in E_0 such that $d(y, y_i) \geq 2\varepsilon$ for all $i = 1, \dots, \ell$, which would imply that $B(y, \varepsilon)$ is in $E \setminus \bigcup_{i=1}^\ell B(y_i, \varepsilon)$, a contradiction. So E_0 is totally bounded, and since it is closed and X is complete, it is compact. $\square$

Proposition 6.4. *Let T be a conservative ergodic, infinite measure-preserving transformation on (X, μ) . If T is not W -measurably sensitive for a μ -compatible metric d on (X, μ) , then nonempty d -open sets have infinite measure.*

Proof. Let d_T be obtained from d as in the proof of Proposition 6.2 (and Theorem 1 from [11]). Recall that d_T is an isometry on a positively invariant set of full measure in X . Again as in Proposition 6.2 relabel the topological completion (X, μ, T) with metric d_X . We first show that the metric d_X is not locally measure finite. Suppose it were. Let (G, η, T) and d_G also be as in the proof of Proposition 6.2. As ϕ is an isometry and η is the image measure, d_G on (G, η) is also locally measure finite. By Lemma 6.1, balls of the same radius in (G, η, T, d_G) have the same measure, so by Lemma 6.3 (G, d_G) is locally compact. As T is minimal, $\mathbb{Z}$ is dense in the locally compact group G , and since G is not discrete, then G is compact; see, for example [18, p. 71] (this is just a folk theorem that there is no minimal map on a noncompact, locally compact space). The local measure finiteness property on the compact space implies that η is a finite measure, but this contradicts that T is infinite measure-preserving. Thus d_X is not locally measure finite on X . Since d_X -balls of the same radius have the same measure, all nonempty d_X -balls have infinite measure. The same is true for d_T -balls before the topological completion. Since $B^{d_T}(x, r) \subset B^d(x, r)$, all nonempty d -open sets have infinite measure. $\square$

7. ENTROPY AND SENSITIVITY

Sensitivity is only one way to capture the notion of chaos. It is generally accepted that entropy should be stronger than sensitivity notions. We show that positive entropy implies WM-sensitivity, improving a result of Cadre and Jacob [9]. The preceding sections suggest that Li-Yorke M-sensitivity is a fundamentally stronger concept than W-measurable sensitivity. Here we demonstrate examples of transformations which are W-measurable sensitive but not Li-Yorke M-sensitive.

Proposition 7.1. *Let T on (X, μ) be an ergodic finite measure-preserving transformation. If T has positive entropy, it is W-measurably sensitive.*

Proof. We show the contrapositive. If T is not W-measurably sensitive, then we know it is measurably isomorphic to a compact group rotation [11, Theorem 2]. By the exact same argument as in the proof of Proposition 6.2 this group is abelian. It is well known that compact abelian group rotations have zero entropy [21, Theorem 4.25]. $\square$

Corollary 7.2. *All finite measure-preserving ergodic transformations which have positive entropy but which are not weak mixing are W-measurably sensitive and not Li-Yorke M-sensitive.*

Remark. We now construct examples of W-measurably sensitive but not Li-Yorke M-sensitive transformations with zero entropy. Let (X, μ, T) be a finite measure-preserving doubly ergodic transformation with zero entropy and let (Y, ν, R) be the rotation on two points. Consider the dynamical system $(X \times Y, \mu \times \nu, T \times R)$. This clearly has zero entropy. We show that this system is W-measurably sensitive and not Li-Yorke M-sensitive. Let $S = T \times R$. To see that it is not Li-Yorke M-sensitive, one could note that it has a rotation factor and apply Theorem 3. More explicitly, one could consider any μ -compatible metric d_X on X and the μ -compatible metric d_Y on Y which has $d_Y(1, 2) = 1$. Then define d on $X \times Y$ by

$$d((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2).$$

It is easy to see that d is a μ -compatible metric. If $x_1, x_2 \in X$, then

$$d(S^n(x_1, 1), S^n(x_2, 2)) \geq 1$$

for all $n \geq 0$. Hence the pair $((x_1, 1), (x_2, 2))$ is not proximal and so not Li-Yorke. Thus the set $(X \times \{1\}) \times (X \times \{2\}) \subset (X \times Y)^2$ contains no Li-Yorke pairs and has positive measure, so S is not Li-Yorke M-sensitive.

Now we show that the system is W-measurably sensitive. By Proposition 7.2 in [11] it suffices to show that S is measurably sensitive. Let d be an arbitrary μ -compatible metric on $X \times Y$. Then choose $\delta > 0$ such that $2\delta < \text{diam}(X \times \{i\})$. Choose a point $z = (x, i) \in X \times Y$ and consider $B_\varepsilon(z)$. Then we know that there must exist an i such that $\mu(B_\varepsilon(z) \cap (X \times \{i\})) > 0$. Let E be the projection of $B_\varepsilon(z) \cap (X \times \{i\})$ onto X . Note that E has positive measure. Now choose sets $A, B \subset X$ of positive measure such that $d(A, B) > 2\delta$. Since T is doubly ergodic we can find an n such that

$$\mu(T^{-n}(A) \cap E) > 0,$$

$$\mu(T^{-n}(B) \cap E) > 0.$$

Then we see that $\text{diam}(T^n(E)) > 2\delta$ so that by the triangle inequality there must exist a $z' \in E$ such that $d(T^n z, T^n z') > \delta$. Hence T is measurably sensitive and thus W -measurably sensitive.

8. SENSITIVE METRICS

We show that under mild hypotheses, every nonsingular transformation admits a μ -compatible metric that is W -measurably sensitive, and also that a transformation where the set of periodic points has positive measure never admits a μ -compatible metric that is sensitive. This motivates the definition requiring that sensitivity be exhibited for all μ -compatible metrics. We note that the existence of a μ -compatible metric for every transformation follows from Proposition 4.2. After the research for this work was completed, we learned of the recent work of Morales [17], where he obtains results related to Corollary 8.3 and Proposition 8.5 below, though the methods and context differ: where Morales investigates partitions, we study metrics. Using [17, Theorem 2.6], it can be shown that the existence of a sensitive metric implies the existence of a sensitive partition. Measure expansive, being defined in terms of partitions, is then related but not equivalent to W -measurable sensitivity, as irrational rotations are measure expansive [17, Example 2.2] but not W -measurably sensitive. It would be interesting to investigate further connections.

We start with a generalization, suggested by the referee, of our original proof that assumed conservative ergodic; this needs a preliminary definition. We say a transformation T has *spillover* if there exists a countable collection of Borel sets A_n such that $A_n \subset A_{n+1}$, $\bigcup_{n=1}^{\infty} A_n = X$, and $(A_n)_T$ has measure zero for each n , where for a set A we write $A_T = \bigcap_{i=0}^{\infty} T^{-i}(A)$ (the largest positively-invariant set in A). Clearly, when T is conservative ergodic then it satisfies spillover, and so does $T \times S$ for any nonsingular S .

Proposition 8.1. *Let (X, μ, T) be a nonsingular system that satisfies the spillover condition. If T admits a μ -compatible metric d' for which T is an isometry, then T admits a μ -compatible metric d for which T is W -measurably sensitive.*

Proof. We may assume μ is a probability measure and the sets A_n have been chosen so that $\mu(X \setminus A_n) < \frac{1}{2^n}$. For each n cover X with a countable collection of d' -balls of radius less than $1/2^n$. Then there exist finitely many such balls, denote them $A_{n,1}, \dots, A_{n,k_n}$, such that $\mu(X \setminus \bigcup_{i=1}^{k_n} A_{n,i}) < \frac{1}{2^n}$. Write $A'_{n,i} = A_n \cap A_{n,i}$. Then the union $\bigcup_{n,i} A_{n,i}$ has full measure; let its measure zero complement be E_0 . Rename the sets $A'_{n,i}$ as $D'_0, D'_1, \dots$ and then disjointify them by defining $D''_0 = D'_1, D''_n = D'_n \setminus (D'_0 \cup \dots \cup D'_{n-1})$. We may assume D''_0 has positive measure and is in A_1 . While the metric d' may not be μ -compatible when restricted to each D''_n of positive measure, it is separable, so it is μ -compatible after removing a set of measure zero from each D''_n of positive measure. Let E_1 be the union of all the measure zero sets that have been removed, and all of the D''_n of measure zero. Let $Z = \bigcup (A_n)_T$. Adjoin the measure zero sets E_0, E_1, Z to D''_0 and rename it D_0 . Rename the remaining positive measure sets D''_n by $D_1, D_2, \dots$. Now the sets D_n form a positive measure partition of X , d' is μ -compatible when restricted to each $D_n, n > 0$, and the d' diameter of D_n decreases to 0. Choose a μ -compatible metric d_0 on D_0 . When $x \in D_n$ write $N(x) = n$. Then define a μ -compatible metric d on X to be d_i for two points in D_i and one otherwise.

Note that $Z_0 = \bigcup_{i \geq 0} T^{-i}(E_0 \cup E_1)$ is of measure zero. Let $x \notin Z \cup Z_0$. As $x \notin Z$, the sequence $N(T^i(x))$ is unbounded. Let $y \in X, y \neq x$ and set $\delta = d'(x, y) > 0$. There exists K so that for all $n > K$ the d' -diameter of D_n is less than δ . Since $d'(T^i(x), T^i(y)) = d'(x, y)$, when $N(T^i(x)) > K$, then $T^i(x)$ and $T^i(y)$ cannot be in the same D_n for $n > K$. Thus $d'(T^i(x), T^i(y)) = 1$ for infinitely many i . Therefore T is W -measurably sensitive for d . $\square$

Recall that a transformation is *aperiodic* if its set of periodic points has measure zero. In the following lemma we need Rohlin's Lemma for nonsingular transformations that are not necessarily invertible. We use a version proved in [6, Theorem 2 and Remark 1]: If T is nonsingular and forward measurable, for each $\varepsilon > 0$ and each $n \in \mathbb{N}$, there exists a measurable set C with $\mu(C) < 1/n + \varepsilon$ and such that $C, T^{-1}(C), \dots, T^{-n+1}(C)$ are disjoint and of measure $> 1 - \varepsilon$.

Lemma 8.2. *Let T be a nonsingular transformation and suppose it is forward measurable. Then T is aperiodic if and only if T satisfies the spillover condition.*

Proof. Clearly, if the set of periodic points has positive measure, the transformation does not satisfy the spillover condition. Suppose, now, that T is aperiodic. Let $k \in \mathbb{N}$. By Rohlin's Lemma, for $\varepsilon = 1/2^k$ and $n = 2^k$ there exists a set C_k such that $C_k, T^{-1}(C_k), \dots, T^{-n+1}(C_k)$ are disjoint with

$$\mu\left(\bigcup_{i=0}^{n-1} T^{-i}(C_k)\right) > 1 - \frac{1}{2^k} \text{ and } \mu(C_k) < \frac{2}{2^k}.$$

Let

$$B_k = \bigcup_{i=2}^{n-1} T^{-i}(C_k).$$

Then if $x \in B_k$ there is some j so that $T^j(x) \in C_k$, which is disjoint from B_k , so $\mu([B_k]_T) = 0$. Also $\mu(B_k) > 1 - \frac{3}{2^k}$. Finally set

$$A_m = \bigcap_{k \geq m} B_k.$$

Then one can verify that $A_m \subset A_{m+1}$ and $\mu([A_m]_T) = 0$. Furthermore,

$$\begin{aligned} \mu(A_m) &= \mu\left(\bigcap_{k \geq m} B_k\right) = \mu\left(X \setminus \bigcup_{k \geq m} (B_k)^c\right) \\ &> 1 - \left(\frac{3}{2^m} + \frac{3}{2^{m+1}} + \dots\right) = 1 - \frac{3}{2^{m-1}}. \end{aligned}$$

Therefore $\bigcup_{m \geq 1} A_m = X \bmod \mu$ and T satisfies spillover. $\square$

Corollary 8.3. *Let (X, μ, T) be a nonsingular system. If T is aperiodic, forward measurable and admits a μ -compatible isometry, or if T is conservative ergodic, then there exists a μ -compatible metric d for which T is W -measurably sensitive.*

Proof. The first part follows from Proposition 8.1 and Lemma 8.2. When T is conservative ergodic, if it is W -measurably sensitive, then we are done; so suppose it is not. Then by Theorem 1, there exists a μ -compatible metric d' on X for which T is an invertible isometry. As T must satisfy the spillover condition, Proposition 8.1 completes the proof. $\square$

We have shown the existence of a sensitive metric for a large class of transformations. We now prove that there exists a class of transformations which do not admit any μ -compatible metric. We are indebted to the referee who suggested to us an argument simpler than our original proof. We start with a lemma from [11, 14] whose proof we include for completeness; a similar lemma can be found in [22], which we learned of only recently.

Lemma 8.4. *Let (X, μ) be a standard space with a nonatomic σ -finite metric. If d is a separable Borel metric on X , then there exists a measure zero set Z such that d is μ -compatible on $X \setminus Z$. If T is nonsingular on (X, μ) , then Z can be chosen so that $X \setminus Z$ is positively invariant.*

Proof. Let Z consist of the union of all measure zero d -open sets. As the metric is separable, it is Lindelöf, and so is Z . Thus Z is a countable union of null sets, so it is null. When T is nonsingular, let $Z_1 = \bigcup_{n=0}^{\infty} T^{-n}(Z)$. So Z_1 is null and $X \setminus Z_1$ is positively invariant. $\square$

Proposition 8.5. *Let (X, μ, T) be a nonsingular system. If T is not aperiodic, then there exists no μ -compatible metric d on X for which T is W -measurably sensitive.*

Proof. First we show there exists a positively invariant set of positive measure X_1 and a positive integer N such that T^N restricted to X_1 is the identity. Let Per denote the set of periodic points of T . For each $n \in \mathbb{N}$ write $P_n = \{x \in X : T^{n!}(x) = x\}$. Then each P_n is positively invariant, $P_n \subset P_{n+1}$ and $Per = \bigcup_{n \geq 1} P_n$. As $\mu(Per) = 0$, then some P_n has positive measure and $T^{n!}$ is the identity on P_n . Let $X_1 = P_n$ and $N = n!$.

Suppose now there exists a μ -compatible metric d for which T is W -measurably sensitive with sensitivity constant $\delta > 0$. While d may not be μ -compatible on X_1 , it is separable. It follows that the metric d_N defined by

$$d_N(x, y) = \sum_{i=0}^{N-1} d(T^i x, T^i y)$$

is separable. By Lemma 8.4, after deleting a set of measure zero, we may assume d_N is μ -compatible on a positively invariant subset of full measure of X_1 , that we may rename X_1 . Let $x \in X_1$ and let $0 < \varepsilon < \delta/2$. Then the d_N ball $B = B^{d_N}(x, \varepsilon)$ has positive measure. So for all $y, z \in B$,

$$d(T^i y, T^i z) \leq d_N(T^i y, T^i z) = d_N(y, z) < 2\varepsilon < \delta.$$

Therefore T is not W -measurably sensitive for d , a contradiction. Thus no sensitive metric exists. $\square$

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