A COMPLETE CLASSIFICATION OF THE SPACES OF COMPACT OPERATORS ON $C([1, \alpha], l_p)$ SPACES, $1 < p < \infty$

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ABSTRACT. We complete the classification, up to isomorphism, of the spaces of compact operators on $C([1, \gamma], l_p)$ spaces, $1 < p < \infty$. In order to do this, we classify, up to isomorphism, the spaces of compact operators $\mathcal{K}(E, F)$, where $E = C([1, \lambda], l_p)$ and $F = C([1, \xi], l_q)$ for arbitrary ordinals λ and ξ and $1 < p \leq q < \infty$.

More precisely, we prove that it is relatively consistent with ZFC that for any infinite ordinals λ, μ, ξ and η the following statements are equivalent:

- (a) $\mathcal{K}(C([1, \lambda], l_p), C([1, \xi], l_q))$ is isomorphic to $\mathcal{K}(C([1, \mu], l_p), C([1, \eta], l_q))$.
- (b) λ and μ have the same cardinality and $C([1, \xi])$ is isomorphic to $C([1, \eta])$ or there exists an uncountable regular ordinal α and $1 \leq m, n < \omega$ such that $C([1, \xi])$ is isomorphic to $C([1, \alpha m])$ and $C([1, \eta])$ is isomorphic to $C([1, \alpha n])$.

Moreover, in ZFC, if λ and μ are finite ordinals and ξ and η are infinite ordinals, then the statements (a) and (b') are equivalent.

- (b') $C([1, \xi])$ is isomorphic to $C([1, \eta])$ or there exists an uncountable regular ordinal α and $1 \leq m, n \leq \omega$ such that $C([1, \xi])$ is isomorphic to $C([1, \alpha m])$ and $C([1, \eta])$ is isomorphic to $C([1, \alpha n])$.

1. INTRODUCTION

We use standard set theory and Banach space theory terminology and notions as can be found in [14] and [15], respectively. Let K be a compact Hausdorff space and X a Banach space. The space $C(K, X)$ denotes the Banach space of all continuous X -valued functions defined on K equipped with the supremum norm, i.e., $\|f\| = \sup_{k \in K} \|f(k)\|_X$. We will write $C([1, \alpha], X) = C(\alpha, X)$ when K is the interval of ordinals $[1, \alpha] = \{\xi : 1 \leq \xi \leq \alpha\}$ endowed with the order topology. These spaces will be denoted by $C(K)$ and $C(\alpha)$, respectively, in the case $X = \mathbb{R}$.

For a set Γ , $c_0(\Gamma, X)$ is the Banach space of all X -valued functions f on Γ with the property that for every $\epsilon > 0$, the set $\{\gamma \in \Gamma : \|f(\gamma)\| \geq \epsilon\}$ is finite, and equipped with the sup norm. This space will be denoted by $c_0(\Gamma)$ in the case $X = \mathbb{R}$ and by c_0 when, in addition, the cardinality of Γ (denoted by $|\Gamma|$) is $\aleph_0$.

Given Banach spaces X and Y , $\mathcal{K}(X, Y)$ denotes the Banach space of compact operators from X to Y . As usual, when $X = Y$, this space will be denoted by

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$\mathcal{K}(X)$. We write $X \sim Y$ when X and Y are isomorphic and $X \hookrightarrow Y$ when Y contains a copy of X , that is, a subspace isomorphic to X .

In this paper, we are mainly interested in completing the isomorphic classification of the spaces $\mathcal{K}(C(\lambda, l_p), C(\xi, l_q))$ obtained partially in [9], [10] and [11], where λ and ξ are arbitrary ordinals and l_p and l_q are the classical Banach spaces of scalar sequences with $1 < p, q < \infty$.

For $p > q$, an isomorphic classification of these spaces was accomplished in [9, Remark 4.1.3] for $\lambda < \omega$ and $\xi \geq \omega$, and in [10, Remark 1.7] for $\lambda \geq \omega$ and $\xi \geq \omega$. We pointed out that, in both these cases, the following geometric property of the spaces of compact operators from l_p to l_q [23, Main Theorem] was crucial:

$$(1.1) \quad c_0 \not\hookrightarrow \mathcal{K}(l_p, l_q).$$

Thus, in the present work, we turn our attention to the case $1 < p \leq q < \infty$. In this case the situation is quite different. Indeed, in contrast to (1.1), by [1, Theorem 5.3], $\mathcal{K}(l_p, l_q)$ is isomorphic to its c_0 -sum or, what is the same, that

$$(1.2) \quad \mathcal{K}(l_p, l_q) \sim \mathcal{K}(l_p, c_0(l_q)).$$

Nevertheless, in [11] it was shown that the following cancellation law holds:

$$(1.3) \quad \mathcal{K}(C(\lambda, l_p), C(\xi, l_q)) \sim \mathcal{K}(C(\mu, l_p), C(\eta, l_q)) \iff C(\xi) \sim C(\eta),$$

whenever ξ and η are infinite countable ordinals and $\lambda\mu < \omega$ or λ and μ have the same cardinality.

The main aim of this paper is to extend the above cancellation law by giving a complete isomorphic classification of the spaces $\mathcal{K}(C(\lambda, l_p), C(\xi, l_q))$, where λ and ξ are arbitrary ordinals.

To do this, first in Section 3, Theorem 3.2, we prove a new geometric property of the spaces of compact operators involving the spaces l_p , l_q and c_0 . Namely,

$$\mathcal{K}(c_0) \not\hookrightarrow \mathcal{K}(c_0(l_p), l_q).$$

Moreover, in order to use the isomorphic classification of some $C(\xi, X)$ spaces obtained in [8], in Section 4 we prove a stability property of certain spaces of compact operators containing copies of some $c_0(\Gamma)$ spaces. Specifically, we show that if X is a Banach space and Γ is a set of cardinality $\aleph_1$, then

$$c_0(\Gamma) \hookrightarrow \mathcal{K}(l_p, X) \iff c_0(\Gamma) \hookrightarrow X.$$

In Section 5 we present our main result (Theorem 5.2). It provides a necessary and sufficient condition for two spaces $\mathcal{K}(C(\lambda, l_p), C(\xi, l_q))$ to be isomorphic, whenever λ and ξ are infinite and the cardinality of λ is strictly less than the least real-valued measurable cardinal m_r .

We recall that a cardinal number m is a real-valued measurable cardinal if there exists a nontrivial real-valued measure defined on all subsets of a set of cardinality m for which points have measure 0 [5, p. 560].

Theorem 5.2 is a complete isomorphic classification of the $\mathcal{K}(C(\lambda, l_p), C(\xi, l_q))$ spaces, where λ and ξ are infinite. Indeed, it is well known that the existence of real-valued measurable cardinals cannot be proved in ZFC [16, pp. 106 and 108]. On the other hand, it is relatively consistent with ZFC that real-valued measurable cardinals

do not exist [6, Theorem 4.14, p. 972]. So it is relatively consistent with ZFC that we obtain a complete isomorphic classification of the spaces $\mathcal{K}(C(\lambda, l_p), C(\xi, l_q))$, where λ and ξ are infinite.

Finally, in Theorem 5.3 we consider the remaining case where λ is finite. Then, by using (1.2) we complete the proof of the isomorphic classification mentioned in the abstract.

2. PRELIMINARY RESULTS

In this section we recall the isomorphic classification of the $C(\xi)$ spaces, the generalization proved in [8], and some properties of the spaces of compact operators.

From now on $|\xi|$ denotes the cardinality of the ordinal ξ . Recall that an ordinal α is said to be regular if its cofinality is itself. Otherwise α is a singular ordinal. The smallest two infinite regular ordinals are ω and ω_1 . We denote by $X \hat{\otimes} Y$ the injective tensor product of the Banach spaces X and Y .

Bessaga and Pelczyński [2] showed that for infinite countable ordinals, $\xi \leq \eta$, $C(\xi)$ is isomorphic to $C(\eta)$ if and only if $\eta < \xi^\omega$. In 1976 the classification was extended to nondenumerable ordinals ξ and η , [12] and [18], independently. The general criterion is the same as that obtained by Bessaga and Pelczynski except in the case that the initial ordinal α of cardinality $|\xi| = |\eta|$ is a nondenumerable, regular ordinal, and $\alpha \leq \xi \leq \eta \leq \alpha^2$. For this case, equality of *cardinalities* of the ordinal quotients, ξ', η' , where $\xi = \alpha\xi' + \delta$, $\eta = \alpha\eta' + \gamma$, and $\delta, \gamma < \alpha$ is the requirement for isomorphism of $C(\xi)$ and $C(\eta)$.

For the spaces $C(\xi, X)$ and $C(\eta, X)$, where X is an infinite dimensional Banach space, some additional difficulties arise in determining the isomorphic classification. Indeed, since $C(\xi, X)$ is isomorphic to $C(\xi) \hat{\otimes} X$, [4, Example 6, p. 224], it follows that

$$(2.1) \quad C(\xi) \sim C(\eta) \implies C(\xi, X) \sim C(\eta, X).$$

Moreover,

$$(2.2) \quad X \sim X \oplus X \implies C(\alpha n, X) \sim C(\alpha, X)$$

for all positive finite ordinals n . Thus for a nondenumerable regular ordinal α the classification is less fine than for the spaces of real-valued functions. However, for some Banach spaces it is possible to determine the isomorphic classification of the spaces $C(\xi, X)$.

To state this result recall that a Banach space X is said to have the Mazur Property, MP for short, if every element of X^{**} which is sequentially weak* continuous is weak* continuous and thus is an element of X . Such spaces were investigated in [5] and [17] and sometimes they are also called d-complete [19] or μ B-spaces [24]. In [17] it was shown (Theorem 4.1) that $C(\xi)$ for $\xi < \omega_1$, $\left(\sum_{\gamma \in \Gamma} X_\gamma\right)_{l_1}$, where $|\Gamma|$ is nonmeasurable and X_γ has MP for all γ , (Theorem 3.1), and $X \hat{\otimes} Y$, where X has MP and Y is separable, (Corollary 5.2.(3)) have MP.

Let $\mathcal{F}$ be the class of Banach spaces containing no copy of $c_0(\Gamma)$, where $|\Gamma| = \aleph_1$, and having the Mazur Property. For this class the main theorem of [8] gives a classification of the spaces $C(\xi, X)$.

Theorem 2.1. *Let $X \in \mathcal{F}$ and α be an initial ordinal and $\xi \leq \eta$ be infinite ordinals. Then*

- (1) *If $C(\xi, X)$ is isomorphic to $C(\eta, X)$, then $|\xi| = |\eta|$.*
- (2) *Suppose that $|\xi| = |\eta| = |\alpha|$ and assume that α is a singular ordinal, or α is a nondenumerable regular ordinal with $\alpha^2 \leq \xi$. Then*

$$C(\xi, X) \sim C(\eta, X) \iff \eta < \xi^\omega.$$

- (3) *Suppose that α is an uncountable regular ordinal, $\xi, \eta \in [\alpha, \alpha^2]$ and let ξ', η', γ and δ be ordinals such that $\xi = \alpha\xi' + \gamma$, $\eta = \alpha\eta' + \delta$, $\xi', \eta' \leq \alpha$ and $\gamma, \delta < \alpha$. Then*

$$C(\xi, X) \sim C(\eta, X),$$

if and only if one of the following statements holds:

- (a) *$|\xi'|, |\eta'| \leq \aleph_0$ and $c_0(J, X)$ is isomorphic to $c_0(I, X)$, where J and I are sets with $|J| = |\xi'|$ and $|I| = |\eta'|$.*
- (b) *$|\xi'| = |\eta'| > \aleph_0$.*
- (4) *Suppose that α is an uncountable regular ordinal and $\alpha \leq \xi < \alpha^2 \leq \eta$. Then*

$$C(\xi, X) \not\sim C(\eta, X).$$

Remark 2.2. Notice that for a fixed $X \in \mathcal{F}$, the isomorphic classification of the spaces $C(\xi, X)$ is the same as that of spaces $C(\xi)$ except in the case (2.4) (3)(a) of the theorem. As noted above this case can occur when $X \sim X \oplus X$.

This theorem does not say much about the case when ξ and η are infinite countable ordinals. For this case we use the following result, [11, Theorem 1.3]. Notice that the condition $\xi \leq \eta < \xi^\omega$ is equivalent to $C(\xi) \sim C(\eta)$.

Theorem 2.3. *Let α, β, ξ , and η be ordinals with $\omega \leq \xi \leq \eta < \omega_1$, $\alpha, \beta < \omega$ or else $|\alpha| = |\beta|$, X a Banach space such that X^* is weakly sequentially complete and has the approximation property and Y a Banach space which has an unconditional basis and contains no subspace isomorphic to l_1 . Then*

$$\mathcal{K}(C(\alpha, X), C(\xi, Y)) \sim \mathcal{K}(C(\beta, X), C(\eta, Y)) \iff \eta < \xi^\omega.$$

We are concerned with spaces of compact operators $\mathcal{K}(C(\lambda, Y), C(\xi, Z))$, where Y and Z are Banach spaces which are isomorphic to their squares, i.e. $Y \sim Y \oplus Y$, and such that their duals have the approximation property, and λ and ξ are ordinals.

Because $C(\lambda, Z)$ has the approximation property if Z has the approximation property [13, Theorem 2], by [3, Proposition 5.3] we see that

$$\mathcal{K}(C(\lambda, Y), C(\xi, Z)) \sim C(\lambda, Y)^* \hat{\otimes} C(\xi, Z).$$

Moreover, notice that $C(\lambda, Y)^*$ is isomorphic to $l_1(\Gamma, Y^*)$, where Γ is a set with cardinality $|\lambda|$. (See [4, pp. 181-182].) Thus if we let $\lambda_0 = 1$ for λ is finite, and λ_0 be the smallest ordinal with $|\lambda_0| = |\lambda|$ if λ is infinite,

$$(2.3) \quad \mathcal{K}(C(\lambda, Y), C(\xi, Z)) \sim \mathcal{K}(C(\lambda_0, Y), C(\xi, Z)).$$

In [1] (1.2) was proved and this is equivalent to

$$(2.4) \quad C(\omega, \mathcal{K}(l_p, l_q)) \sim \mathcal{K}(l_p, l_q), \quad 1 < p \leq q < \infty.$$

Because

$$C(\omega, \mathcal{K}(l_p, l_q)) \sim C((\omega) \hat{\otimes} l_{p'} \hat{\otimes} l_q) \sim \mathcal{K}(l_p, C(\omega, l_q)),$$

where $p' = p/(p-1)$, in the case $Y = l_p$ and $Z = l_q$ there is the possibility of additional collapsing of the isomorphism classes. Indeed, in contrast to (1.3), if α is either 1 or an uncountable regular ordinal, then the spaces $C(\alpha\xi)$, $1 \leq \xi \leq \omega$, are isomorphically distinct. However, if $1 \leq \lambda < \omega$, then by (2.4) and the fact that l_p and $l_{p'} \hat{\otimes} l_q$ are isomorphic to their squares,

$$\mathcal{K}(C(\lambda, l_p), C(\alpha\xi, l_q)) \sim l_{p'} \hat{\otimes} C(\alpha\xi) \hat{\otimes} l_q \sim C(\alpha, C(\xi, l_{p'} \hat{\otimes} l_q)) \sim C(\alpha, l_{p'} \hat{\otimes} l_q),$$

and therefore

$$\mathcal{K}(C(\lambda, l_p), C(\alpha\xi, l_q)) \sim \mathcal{K}(l_p, C(\alpha, l_q)).$$

The next proposition summarizes the effects of the isomorphic classification of the spaces $C(\alpha)$, (2.3) and (2.4). Below, if γ is an ordinal, $\gamma_0 = 1$, if $1 \leq \gamma < \omega$, and γ_0 is the smallest ordinal with the same cardinality as γ , if $\gamma \geq \omega$.

Proposition 2.4. *Suppose that Y and Z are Banach spaces, have the approximation property, are isomorphic to their squares, and $\mathcal{K}(Y, Z)$ is isomorphic to $C(\omega, \mathcal{K}(Y, Z))$. Let λ and ξ be nonzero ordinals.*

- (1) *If $\lambda \geq \omega$, ξ_0 is an uncountable regular ordinal, $\xi < \xi_0^2$, and $\xi = \xi_0\xi' + \delta$, $\delta < \xi_0$, then*

$$\mathcal{K}(C(\lambda, Y), C(\xi, Z)) \sim \mathcal{K}(C(\lambda_0, Y), C(\xi_0\xi'_0, Z)).$$

- (2) *If $\lambda < \omega$, ξ_0 is an uncountable regular ordinal, $\xi_0\omega_1 \leq \xi < \xi_0^2$, and $\xi = \xi_0\xi' + \delta$, $\delta < \xi_0$, then*

$$\mathcal{K}(C(\lambda, Y), C(\xi, Z)) \sim \mathcal{K}(C(\lambda_0, Y), C(\xi_0\xi'_0, Z)).$$

- (3) *If $\lambda < \omega$, ξ_0 is an uncountable regular ordinal, $\xi_0 \leq \xi < \xi_0\omega_1$, and $\xi = \xi_0\xi' + \delta$, $\delta < \xi_0$, then*

$$\mathcal{K}(C(\lambda, Y), C(\xi, Z)) \sim \mathcal{K}(C(\lambda_0, Y), C(\xi_0, Z)).$$

- (4) *If ξ_0 is an uncountable regular ordinal, $\xi_0^2 \leq \xi$, and γ is the smallest ordinal such that $\gamma^\omega > \xi$, then*

$$\mathcal{K}(C(\lambda, Y), C(\xi, Z)) \sim \mathcal{K}(C(\lambda_0, Y), C(\max\{\xi_0^2, \gamma\}, Z)).$$

- (5) *If $\xi_0 < \omega$ or $\lambda < \omega$ and $\xi < \omega^\omega$, then*

$$\mathcal{K}(C(\lambda, Y), C(\xi, Z)) \sim \mathcal{K}(C(\lambda_0, Y), C(1, Z)).$$

- (6) *In all other cases if γ is the smallest ordinal such that $\gamma^\omega > \xi$, then*

$$\mathcal{K}(C(\lambda, Y), C(\xi, Z)) \sim \mathcal{K}(C(\lambda_0, Y), C(\gamma, Z)).$$

Our goal in the next few sections is to show that for $Y = l_p$ and $Z = l_q$ there is no further collapsing of isomorphism classes, i.e., the representatives in the previous proposition are from distinct isomorphism classes. Observe that it is only for the conditions in (3) and (5) that the representative of the isomorphism class is not $\mathcal{K}(C(\lambda_0, Y), C(\rho, Z))$ where $C(\rho)$ is a representative of the isomorphism class of $C(\xi)$.

3. ON THE SUBSPACES OF $\mathcal{K}(c_0(X), l_q(Y))$ SPACES, $1 \leq q < \infty$

The main aim of this section is to prove Theorem 3.2, which is the key ingredient for completing the isomorphic classification of the spaces of compact operators considered in this work. To do this, we need an auxiliary result. Let X and Y be Banach spaces and $T \in \mathcal{K}(c_0(X), l_q(Y))$, with $1 \leq q < \infty$. Represent T as a matrix with entries in $\mathcal{K}(X, Y)$. By [20, Proposition 1.c.8 and following Remarks] for any blocking of the c_0 -sum and l_q -sum the operator given by the block diagonal of the matrix is a bounded linear operator with norm no larger than $\|T\|$. The next result gives a little more information about $\|T\|$.

Lemma 3.1. *Let X and Y be Banach spaces and let $1 \leq q < \infty$. Then the norm of a block diagonal operator of an operator in $\mathcal{K}(c_0(X), l_q(Y))$ is the l_q norm of the norms of the operators on the blocks and the block diagonal operator is compact.*

Proof. For each $j \in \mathbb{N}$ let D_j be the j th block of the block diagonal operator D and let $(x_j)_j$ be a sequence in $c_0(X)$ such that for each j , x_j is the supported block corresponding to D_j . Then

$$\begin{aligned} \|(D_j x_j)_j\|_{l_q(Y)} &= \left(\sum_j \|D_j x_j\|_{l_q(Y)}^q \right)^{1/q} \leq \left(\sum_j \|D_j\|^q \|x_j\|_{c_0(X)}^q \right)^{1/q} \\ &\leq (\sup_j \|x_j\|_{c_0(X)}) (\|D_j\|_j)_{l_q}. \end{aligned}$$

Suppose $1 > \epsilon > 0$, $x_j \in c_0(X)$, supported in the j th block, with $\|x_j\|_{c_0(X)} = 1$ and

$$\|D_j x_j\| \geq (1 - \epsilon) \|D_j\|,$$

$j = 1, 2, \dots$. For each N , let $z_N = (x_1, x_2, \dots, x_N, 0, 0, \dots)$. Then

$$\begin{aligned} \|D z_N\| &= \|(D_j x_j)_{j \leq N}\|_{l_q(Y)} = \left(\sum_{j \leq N} \|D_j x_j\|_{l_q(Y)}^q \right)^{1/q} \\ &\geq \left(\sum_{j \leq N} (1 - \epsilon) \|D_j\|^q \right)^{1/q}. \end{aligned}$$

Taking the supremum over ϵ and N gives

$$\|D\| \geq \|(\|D_j\|)_j\|_{l_q}.$$

It remains to show that the block diagonal of a compact operator is compact. Let Q_n , respectively, R_n , be the projection onto the first n blocks of $l_q(Y)$, respectively, $c_0(X)$. It is easy to see that because $q < \infty$ then for any $T \in \mathcal{K}(c_0(X), l_q(Y))$ and $\epsilon > 0$ there is an N such that for all $n \geq N$,

$$\|(I - Q_n)T\| < \epsilon.$$

Observe that for $j > n$,

$$D_j = (Q_j - Q_{j-1})T(R_j - R_{j-1}) = (Q_j - Q_{j-1})(I - Q_n)T(R_j - R_{j-1}).$$

Thus for $j > n$, the block diagonal element of T , D_j , is also the block diagonal element of $(I - Q_n)T$. By an application of Tong's result [20, Proposition 1.c.8] to

$(I - Q_n)T$, it follows that

$$\left\| \sum_{j \in \mathbb{N}} D_j - \sum_{j \leq N} D_j \right\| < \epsilon.$$

Thus the block diagonal of T is the limit of compact operators and therefore is compact. $\square$

The lemma will be applied near the end of the proof of the next result with $X = l_p$ and $Y = l_q$.

Theorem 3.2. *Suppose that $1 < p \leq q < \infty$. Then*

$$\mathcal{K}(c_0) \not\hookrightarrow \mathcal{K}(c_0(l_p), l_q).$$

Proof. First of all notice that $\mathcal{K}(c_0)$ is isomorphic to

$$l_1 \hat{\otimes} c_0 = c_0(l_1) = \left(\sum_{n=1}^{\infty} l_1 \right)_{c_0}.$$

Let P_n denote the standard projection from $c_0(l_p)$ onto $(\sum_{k=1}^n l_p)_{\infty}$ and let (Q_n) be the basis projections for l_q .

Suppose that $(T_{s,n})$ is a normalized sequence in $\mathcal{K}(c_0(l_p), l_q)$, which is equivalent to the unit vector basis of $(\sum_{n=1}^{\infty} l_1)_{c_0}$, i.e., there are $\delta > 0$ and $K < \infty$ such that

$$\delta \max_s \sum_n |a_{s,n}| \leq \left\| \sum_s \sum_n a_{s,n} T_{s,n} \right\| \leq K \max_s \sum_n |a_{s,n}|$$

for all finitely nonzero sequences $(a_{s,n})$.

For fixed s, r , $(T_{s,n} P_r)$ is (equivalent to) a sequence in $\mathcal{K}(\sum_{n=1}^r l_p, l_q)$, which is isomorphic to $\mathcal{K}(l_p, l_q)$, and thus by [22, Theorem 3.1] cannot be equivalent to the unit vector basis of l_1 . Hence for each s there exists a normalized blocking of $(T_{s,n})_n$, $(U_{s,m})$, and an increasing sequence $(r_{s,m})_m$ such that

$$\|U_{s,m} P_{r_{s,m}}\| \rightarrow 0.$$

Further, because $U_{s,m}$ is compact, for fixed s, m as $r \rightarrow \infty$,

$$\|U_{s,m}(I - P_r)\| \rightarrow 0.$$

(Below we pass to subsequences of the double sequence by choosing infinitely many m for each s and using a pattern like that for enumerating $\mathbb{N} \times \mathbb{N}$.) Thus by a perturbation argument and passing to subsequences we may assume that $(r_{s,m})$ is strictly increasing in some order and that

$$U_{s,m}(P_{r_{s,m}} - P_{r_{s',m'}}) = U_{s,m},$$

for all s, m , where $r_{s',m'}$ is the maximal element strictly smaller than $r_{s,m}$.

For fixed s, m as $k \rightarrow \infty$,

$$\|(I - Q_k)U_{s,m}\| \rightarrow 0,$$

because $q < \infty$ and $U_{s,m}$ is compact. If $(s_t, m_t)_t$ is a sequence with $(s_t)_t$ strictly increasing, $(U_{s_t, m_t})_t$ is equivalent to the usual unit vector basis of c_0 . For fixed k we claim that

$$\|Q_k U_{s_t, m_t}\| \rightarrow 0.$$

Suppose not and let $x_t \in c_0(l_p)$, with $\|x_t\| = 1$ and

$$\|Q_k U_{s_t, m_t} x_t\| > \|Q_k U_{s_t, m_t}\|/2,$$

for all t . Because k is fixed, by passing to a subsequence and a perturbation argument, we may assume that $(Q_k U_{s_t, m_t} x_t)_t$ is independent of t . We may also assume by the choice of $r_{s, m}$ that

$$(P_{r_{s_t, m_t}} - P_{r_{s'_t, m'_t}})x_t = x_t,$$

for all t . Thus for any finite j , $\|\sum_{i=1}^j x_i\| = 1$ but

$$K \geq \left\| \sum_t Q_k U_{s_t, m_t} \sum_{i=1}^j x_i \right\| = \left\| \sum_{t=1}^j Q_k U_{s_t, m_t} x_t \right\| > j \|Q_k U_{s_1, m_1}\|/2.$$

This contradicts $K < \infty$ and establishes the claim.

By again passing to a suitable subsequence and a perturbation we can assume that we have $(U_{s_t, m_t})_t$ with $(s_t)_t$ strictly increasing and $(k_t)_t$ strictly increasing such that

$$U_{s_t, m_t} = (Q_{k_t} - Q_{k_{t-1}})U_{s_t, m_t}.$$

However, by Lemma 3.1 this implies that $(U_{s_t, m_t})_t$ is equivalent to the basis of l_q . $\square$

Remark 3.3. The proof of Theorem 3.2 uses l_p and l_q in a very minor way. l_p could be replaced by a space Y which is isomorphic to its square, and l_q by a space Z with an unconditional basis that has a nontrivial upper and lower estimate such that l_1 is isomorphic to no subspace of $\mathcal{K}(Y, Z)$.

Corollary 3.4. *If $\lambda, \xi \geq \omega$, and $1 < p \leq q < \infty$, then $\mathcal{K}(C(\lambda, l_p), C(\xi, l_q))$ is not isomorphic to a subspace of $\mathcal{K}(C(\lambda, l_p), l_q)$.*

Proof. Observe that $\mathcal{K}(C(\lambda, l_p), C(\xi, l_q))$ is isomorphic to

$$l_1(\lambda, l_{p'}) \hat{\otimes} C(\xi, l_q) \sim l_1(\lambda, l_{p'}) \hat{\otimes} C(\xi) \hat{\otimes} l_q \sim C(\xi, l_1(\lambda, l_{p'}) \hat{\otimes} l_q).$$

Thus, it follows that $c_0(l_1)$ is isomorphic to a subspace of $\mathcal{K}(C(\lambda, l_p), C(\xi, l_q))$. If $\mathcal{K}(C(\lambda, l_p), C(\xi, l_q))$ is isomorphic to a subspace of $\mathcal{K}(C(\lambda, l_p), l_q)$, then $c_0(l_1)$ is also. Because $c_0(l_1)$ is separable, this would imply that $c_0(l_1)$ is isomorphic to a subspace of $\mathcal{K}(C(\mu, l_p), l_q)$ for some countable ordinal μ . Furthermore, by considerations after Remark 2.2, $\mu_0 = \omega$ and the latter space is isomorphic to $\mathcal{K}(C(\omega, l_p), l_q)$. Hence $c_0(l_1)$ would be isomorphic to a subspace of $\mathcal{K}(C(\omega, l_p), l_q)$, contradicting Theorem 3.2. $\square$

4. COPIES OF $c_0(\Gamma)$ IN $\mathcal{K}(l_p, X)$ SPACES, $1 < p < \infty$

Before proving our main theorem we need another preliminary result which will yield additional spaces that are in the class $\mathcal{F}$ and thus Theorem 2.1 will be applicable. In particular we will need to know that the spaces $\mathcal{K}(l_p, X)$, with $1 < p < \infty$, contain a copy of $c_0(\Gamma)$ with $|\Gamma| = \aleph_1$ if and only if the Banach space X has the same property. This is an immediate consequence of the following proposition. Below $\mathcal{B}(Y, X)$ is the space of bounded linear operators from Y to X .

Proposition 4.1. *If X and Y are Banach spaces and Y is separable, then $c_0(\Gamma)$, where $|\Gamma| = \aleph_1$, is isomorphic to a subspace of $\mathcal{B}(Y, X)$ if and only if $c_0(\Gamma)$ is isomorphic to a subspace of X .*

Proof. Suppose that S is an isomorphism from $c_0(\Gamma)$ onto a subspace of $\mathcal{B}(Y, X)$ and denote by (e_γ) the coordinate vectors in $c_0(\Gamma)$. Let (y_j) be a sequence which is dense in the unit sphere of Y . For $n, j \in \mathbb{N}$ let

$$N_{j,n} = \{\gamma \in \Gamma : \|Se_\gamma(y_j)\| > 1/n\}.$$

Since $\Gamma = \bigcup_{j,n} N_{j,n}$ there is some j_0, n_0 so that $|N_{j_0, n_0}| = \aleph_1$. Now consider the operator U from $c_0(N_{j_0, n_0})$ to X such that $U(e_\gamma) = S(e_\gamma)(y_{j_0})$ for every $\gamma \in N_{j_0, n_0}$.

Consequently,

$$\|U(e_\gamma)\| > 1/n_0,$$

for every $\gamma \in N_{j_0, n_0}$. Thus, by Rosenthal's disjointness lemma [21, Theorem 3.4] and the remark following, there is a subset N' of N_{j_0, n_0} of the same cardinality such that $U|_{c_0(N')}$ is an isomorphism onto its image.

The converse is obvious. $\square$

Corollary 4.2. *Let V and W be Banach spaces such that c_0 is isomorphic to no subspace of V , V has MP, and W^* is separable and has the approximation property. If $|\lambda|$ is nonmeasurable, then $\mathcal{K}(W, l_1(\lambda, V))$ is in the class $\mathcal{F}$.*

Proof. Because V does not contain a subspace isomorphic to c_0 , it follows from a gliding hump argument that $l_1(\lambda, V)$ has the same property. By Proposition 4.1, $c_0(\Gamma)$ is not isomorphic to a subspace of $\mathcal{K}(W, l_1(\lambda, V))$. To see that $\mathcal{K}(W, l_1(\lambda, V))$ has MP we note that $\mathcal{K}(W, l_1(\lambda, V))$ is isomorphic to $W^* \hat{\otimes} l_1(\lambda, V)$ by [3, Proposition 4.3]. By the results of Kappeler [17, Theorem 3.1 and Corollary 5.2.(3)], $l_1(\lambda, V)$ has MP and consequently $W^* \hat{\otimes} l_1(\lambda, V)$ has MP. $\square$

5. A CLASSIFICATION OF $\mathcal{K}(C(\lambda, l_p), C(\xi, l_q))$ SPACES, $1 < p \leq q < \infty$

In this section we classify, up to isomorphism, the spaces of compact operators $\mathcal{K}(C(\lambda, l_p), C(\xi, l_q))$, with $1 < p \leq q < \infty$. This classification will be established through Theorems 5.1, 5.2 and 5.3. Recall that the density character of a Banach space X is the smallest cardinal number δ such that there exists a set of cardinality δ dense in X . As in Section 2, if λ is an ordinal, $\lambda_0 = 1$, if $1 \leq \lambda < \omega$, and λ_0 is the smallest ordinal with the same cardinality as λ , if $\lambda \geq \omega$.

Theorem 5.1. *Suppose that $1 < p \leq q < \infty$ and ξ, η, λ and μ are nonzero ordinals. If $\mathcal{K}(C(\lambda, l_p), C(\xi, l_q))$ is isomorphic to $\mathcal{K}(C(\mu, l_p), C(\eta, l_q))$, then $\lambda_0 = \mu_0$.*

Proof. Assume that

$$(5.1) \quad \mathcal{K}(C(\lambda, l_p), C(\xi, l_q)) \sim \mathcal{K}(C(\mu, l_p), C(\eta, l_q)),$$

for some nonzero ordinals ξ, η, λ and μ . We can suppose without loss of generality that $|\lambda| \leq |\mu|$. Hence if $\mu < \omega$, then $\lambda < \omega$, $\mu_0 = 1 = \lambda_0$, and we are done.

Next we consider the case $\mu > \omega$. Let p' be the real number satisfying $1/p + 1/p' = 1$. We know that $C(\beta, l_p)$ has the approximation property for every ordinal β . So according to [3, Proposition 5.3] we see that

$$(5.2) \quad \mathcal{K}(C(\lambda, l_p), C(\xi, l_q)) \sim l_1(\lambda, l_{p'}) \hat{\otimes} C(\xi, l_q) \sim C(\xi, l_1(\lambda, l_{p'}) \hat{\otimes} l_q),$$

and

$$(5.3) \quad \mathcal{K}(C(\mu, l_p), C(\eta, l_q)) \sim l_1(\mu, l_{p'}) \hat{\otimes} C(\eta, l_q) \sim C(\eta, l_1(\mu, l_{p'}) \hat{\otimes} l_q).$$

Consequently, by (5.1) we have

$$(5.4) \quad l_1(\mu) \hookrightarrow \mathcal{K}(C(\mu, l_p), C(\eta, l_q)) \sim C(\xi, l_1(\lambda, l_{p'}) \hat{\otimes} l_q).$$

Since $l_1(\mu)$ contains no subspace isomorphic to c_0 and $l_1(\lambda, l_{p'}) \hat{\otimes} l_q$ is isomorphic to its square, we deduce by (5.4) and [7, Theorem 2.3] that

$$(5.5) \quad l_1(\mu) \hookrightarrow l_1(\lambda, l_{p'}) \hat{\otimes} l_q.$$

Therefore, if $\lambda < \omega$, then by (5.5) we would conclude that $l_{p'} \hat{\otimes} l_q$ contains a copy of l_1 , which is absurd by [22, Theorem 3.1]. Hence $\lambda \geq \omega$. In this case, observe that the density character of $l_1(\mu)$ is $|\mu|$ and the density character of $l_1(\lambda, l_{p'}) \hat{\otimes} l_q$ is $|\lambda|$. Thus, it follows from (5.5) that $|\mu| \leq |\lambda|$. So $|\lambda| = |\mu|$. $\square$

Theorem 5.2. *Suppose that $1 < p \leq q < \infty$ and ξ, η, λ and μ are infinite ordinals with $\lambda_0 = \mu_0 < m_r$. Then the following statements are equivalent:*

- (a) $\mathcal{K}(C(\lambda, l_p), C(\xi, l_q))$ is isomorphic to $\mathcal{K}(C(\mu, l_p), C(\eta, l_q))$.
- (b) $C(\xi)$ is isomorphic to $C(\eta)$ or there exists an uncountable regular ordinal α and $1 \leq m, n < \omega$ such that $C(\xi)$ is isomorphic to $C(\alpha m)$ and $C(\eta)$ is isomorphic to $C(\alpha n)$.

Proof. Initially we will prove that statement (b) implies statement (a). Since l_p is isomorphic to its square, as observed in Section 2, it follows from (b) that $C(\xi, l_p)$ is isomorphic to $C(\eta, l_p)$. Therefore, by using (5.2) and (5.3) and the fact that $|\lambda| = |\mu|$ we have that $\mathcal{K}(C(\lambda, l_p), C(\xi, l_q))$ is isomorphic to $\mathcal{K}(C(\mu, l_p), C(\eta, l_q))$.

Conversely, suppose that the statement (a) holds. We may assume that $\lambda = \lambda_0 = \mu_0 = \mu$ by Theorem 5.1 and (2.3). Put

$$X = l_1(\lambda, l_{p'}) \hat{\otimes} l_q = l_1(\mu, l_{p'}) \hat{\otimes} l_q.$$

Hence by (5.2) and (5.3) we have that

$$C(\xi, X) \sim \mathcal{K}(C(\lambda, l_p), C(\xi, l_q)) \sim \mathcal{K}(C(\mu, l_p), C(\eta, l_q)) \sim C(\eta, X).$$

Moreover, because X is isomorphic to $\mathcal{K}(l_{q'}, l_1(\lambda, l_{p'}))$, by Corollary 4.2, X is in the class $\mathcal{F}$. So by Theorem 2.1 (1), we have $|\eta| = |\xi|$. If $|\xi| = \aleph_0$, we have $C(\xi)$ isomorphic to $C(\eta)$ by Theorem 2.3. If $|\xi_0| > \aleph_0$, Theorem 2.1 and Remark 2.2 imply that $C(\xi)$ is isomorphic to $C(\eta)$ except possibly under the conditions of part (3)(a) of Theorem 2.1. In this remaining case, ξ_0 is an uncountable regular ordinal and $\xi, \eta \in [\xi_0, \xi_0^2]$. (Note that (4) of Theorem 2.1 excludes the possibility $\xi < \xi_0^2 \leq \eta$.) By Proposition 2.4 (1) we may assume that $\xi = \xi_0 \xi'$ and $\eta = \xi_0 \eta'$ with $\xi', \eta' \in \{1, \omega\}$. By Corollary 3.4 we see that

$$(5.6) \quad C(\omega, X) \not\sim X.$$

Therefore, either $\xi' = \eta' = 1$ or $|\xi'| = |\eta'| = \aleph_0$. It follows from the isomorphic classification of the spaces $C(\alpha)$ (see Section 2) that in the first case $C(\xi) \sim C(\xi_0 \xi')$ and $C(\eta) \sim C(\xi_0 \eta')$ and in the second case that $C(\xi) \sim C(\eta)$. $\square$

Finally, the next theorem completes the isomorphic classification, up to isomorphism, of the spaces of compact operator $\mathcal{K}(C(\lambda, l_p), C(\xi, l_q))$ for ordinals $\lambda \neq 0$ and $\xi \geq \omega$, and $1 < p \leq q < \infty$.

Theorem 5.3. *Suppose that $1 < p \leq q < \infty$ and ξ, η, λ and μ are ordinals with $\omega \leq \xi \leq \eta$ and $\lambda_0 = \mu_0 = 1$. Then the following statements are equivalent:*

- (a) $\mathcal{K}(C(\lambda, l_p), C(\xi, l_q))$ is isomorphic to $\mathcal{K}(C(\mu, l_p), C(\eta, l_q))$.
- (b) $C(\xi)$ is isomorphic to $C(\eta)$ or there exists an uncountable regular ordinal α and $1 \leq m, n \leq \omega$ such that $C(\xi)$ is isomorphic to $C(\alpha m)$ and $C(\eta)$ is isomorphic to $C(\alpha n)$.

Proof. Just proceed as in the proof of Theorem 5.2 but taking into account that according to (1.2), (5.6) is replaced by

$$C(\omega, X) \sim X.$$

□

If we allow ξ and η to be any nonzero ordinals, we have a little more work to do to complete the isomorphic classification.

- (1) If $\mathcal{K}(C(\lambda, l_p), C(\xi, l_q))$ is isomorphic to $\mathcal{K}(C(\mu, l_p), C(\eta, l_q))$ and both ξ and η are finite, then by Proposition 2.4 (5), $\mathcal{K}(C(\lambda, l_p), C(\xi, l_q))$ is isomorphic to $\mathcal{K}(C(\lambda_0, l_p), l_q)$.
- (2) Suppose that $\lambda, \mu < \omega$ and $\xi < \omega^\omega$. $\mathcal{K}(C(\lambda, l_p), C(\xi, l_q))$ is isomorphic to $\mathcal{K}(C(\mu, l_p), C(\eta, l_q))$, if and only if $\eta < \omega^\omega$. Indeed, we have by Theorem 2.3 and (2.4) that $\mathcal{K}(C(\lambda, l_p), C(\xi, l_q))$ is isomorphic to $\mathcal{K}(C(\lambda, l_p), C(\omega, l_q))$. By the proof of Theorem 5.3 and case (1) of Theorem 2.1, if $\eta \geq \omega$, $|\omega| = |\eta|$ and by Theorem 2.3 $\eta < \omega^\omega$. The converse follows from Theorem 2.3 and (2.4).
- (3) Suppose that $\lambda, \mu \geq \omega$ and $\xi < \omega$. $\mathcal{K}(C(\lambda, l_p), C(\xi, l_q))$ is isomorphic to $\mathcal{K}(C(\mu, l_p), C(\eta, l_q))$, if and only if $|\lambda| = |\mu|$ and $\eta < \omega$. The sufficiency of the two conditions and the necessity of $|\lambda| = |\mu|$ have already been established. Corollary 3.4 completes the argument.

In view of Theorems 5.2 and 5.3, the following question arises naturally.

Problem 5.4. Does the above isomorphic classification of the spaces of compact operator $\mathcal{K}(C(\lambda, l_p), C(\xi, l_q))$, with $\lambda \geq \omega$ and $1 < p \leq q < \infty$, remain true in ZFC?

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REFERENCES

- [1] Dale E. Alspach and Elói Medina Galego, *Geometry of the Banach spaces $C(\beta\mathbb{N} \times K, X)$ for compact metric spaces K* , *Studia Math.* **207** (2011), no. 2, 153–180, DOI 10.4064/sm207-2-4. MR2864387
- [2] C. Bessaga and A. Pełczyński, *Spaces of continuous functions. IV. On isomorphical classification of spaces of continuous functions*, *Studia Math.* **19** (1960), 53–62. MR0113132 (22 #3971)
- [3] Andreas Defant and Klaus Floret, *Tensor norms and operator ideals*, North-Holland Mathematics Studies, vol. 176, North-Holland Publishing Co., Amsterdam, 1993. MR1209438 (94e:46130)

- [4] J. Diestel and J. J. Uhl Jr., *Vector measures*, American Mathematical Society, Providence, R.I., 1977. With a foreword by B. J. Pettis; Mathematical Surveys, No. 15. MR0453964 (56 #12216)
- [5] G. A. Edgar, *Measurability in a Banach space. II*, Indiana Univ. Math. J. **28** (1979), no. 4, 559–579, DOI 10.1512/iumj.1979.28.28039. MR542944 (81d:28016)
- [6] R. J. Gardner and W. F. Pfeffer, *Borel measures*, Handbook of set-theoretic topology, North-Holland, Amsterdam, 1984, pp. 961–1043. MR776641 (86c:28031)
- [7] Elói Medina Galego, *On subspaces and quotients of Banach spaces $C(K, X)$* , Monatsh. Math. **136** (2002), no. 2, 87–97, DOI 10.1007/s006050200036. MR1914222 (2003g:46038)
- [8] Elói Medina Galego, *An isomorphic classification of $C(2^m \times [0, \alpha])$ spaces*, Bull. Pol. Acad. Sci. Math. **57** (2009), no. 3-4, 279–287, DOI 10.4064/ba57-3-9. MR2581093 (2011h:46013)
- [9] Elói Medina Galego, *On isomorphic classifications of spaces of compact operators*, Proc. Amer. Math. Soc. **137** (2009), no. 10, 3335–3342, DOI 10.1090/S0002-9939-09-09828-1. MR2515403 (2010d:46010)
- [10] Elói Medina Galego, *Complete isomorphic classifications of some spaces of compact operators*, Proc. Amer. Math. Soc. **138** (2010), no. 2, 725–736, DOI 10.1090/S0002-9939-09-10117-X. MR2557189 (2010j:46018)
- [11] Elói Medina Galego, *On spaces of compact operators on $C(K, X)$ spaces*, Proc. Amer. Math. Soc. **139** (2011), no. 4, 1383–1386, DOI 10.1090/S0002-9939-2010-10544-0. MR2748430 (2012a:46011)
- [12] S. P. Gul’ko and A. V. Os’kin, *Isomorphic classification of spaces of continuous functions on totally ordered bicompa* (Russian), Funkcional. Anal. i Priložen. **9** (1975), no. 1, 61–62. MR0377489 (51 #13661)
- [13] S. Heĭnrih, *Approximation properties in tensor products* (Russian), Mat. Zametki **17** (1975), 459–466. MR0412835 (54 #956)
- [14] Thomas Jech, *Set theory*, Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1978. Pure and Applied Mathematics. MR506523 (80a:03062)
- [15] William B. Johnson and Joram Lindenstrauss, *Basic concepts in the geometry of Banach spaces*, Handbook of the geometry of Banach spaces, Vol. I, North-Holland, Amsterdam, 2001, pp. 1–84, DOI 10.1016/S1874-5849(01)80003-6. MR1863689 (2003f:46013)
- [16] A. Kanamori and M. Magidor, *The evolution of large cardinal axioms in set theory*, Higher set theory (Proc. Conf., Math. Forschungsinst., Oberwolfach, 1977), Lecture Notes in Math., vol. 669, Springer, Berlin, 1978, pp. 99–275. MR520190 (80b:03083)
- [17] Thomas Kappeler, *Banach spaces with the condition of Mazur*, Math. Z. **191** (1986), no. 4, 623–631, DOI 10.1007/BF01162352. MR832820 (87h:46040)
- [18] S. V. Kisljakov, *A classification of the spaces of continuous functions on the ordinals* (Russian), Sibirsk. Mat. Ž. **16** (1975), 293–300, 420. MR0377490 (51 #13662)
- [19] Denny H. Leung, *On Banach spaces with Mazur’s property*, Glasgow Math. J. **33** (1991), no. 1, 51–54, DOI 10.1017/S0017089500008028. MR1089953 (92b:46021)
- [20] Joram Lindenstrauss and Lior Tzafriri, *Classical Banach spaces. I*, Springer-Verlag, Berlin, 1977. Sequence spaces; Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 92. MR0500056 (58 #17766)
- [21] Haskell P. Rosenthal, *On relatively disjoint families of measures, with some applications to Banach space theory*, Studia Math. **37** (1970), 13–36. MR0270122 (42 #5015)
- [22] C. Samuel, *Sur la reproductibilité des espaces l_p* (French), Math. Scand. **45** (1979), no. 1, 103–117. MR567436 (81e:46062)
- [23] C. Samuel, *Sur les sous-espaces de $l_p \hat{\otimes} l_q$* (French), Math. Scand. **47** (1980), no. 2, 247–250. MR612698 (82i:46029)
- [24] Albert Wilansky, *Mazur spaces*, Internat. J. Math. Math. Sci. **4** (1981), no. 1, 39–53, DOI 10.1155/S0161171281000021. MR606656 (82f:46001)

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