EXISTENCE AND NON-EXISTENCE OF FREQUENTLY HYPERCYCLIC SUBSPACES FOR WEIGHTED SHIFTS

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ABSTRACT. We study the existence and the non-existence of frequently hypercyclic subspaces of frequently hypercyclic operators living on Banach spaces. In particular, we give an example of a weighted shift on l^p possessing a frequently hypercyclic subspace and an example of a frequently hypercyclic weighted shift on l^p possessing a hypercyclic subspace but no frequently hypercyclic subspace. The latter example allows us to answer positively Problem 1 posed by Bonilla and Grosse-Erdmann in [Monatsh. Math. 168 (2012)].

1. INTRODUCTION

Let X be a separable infinite-dimensional Banach space. We denote by $L(X)$ the space of continuous linear operators on X .

An operator $T \in L(X)$ is said to be *hypercyclic* if there exists a vector $x \in X$ (called hypercyclic) such that the set $\{T^n x : n \geq 0\}$ is dense in X . The hypercyclicity is an important notion of linear dynamics and we refer to the books [3, 7] for more details about this theory. In 2004, Bayart and Grivaux [1, 2] defined a new notion of hypercyclicity, called frequent hypercyclicity. An operator $T \in L(X)$ is said to be *frequently hypercyclic* if there exists a vector $x \in X$ (also called frequently hypercyclic) such that for any non-empty open set $U \subset X$, $\underline{\text{dens}}\{n \geq 0 : T^n x \in U\} > 0$, where for any subset A of $\mathbb{Z}_+$ (the set of non-negative integers), we define the lower density of A as

$$\underline{\text{dens}} A = \liminf_{N \rightarrow \infty} \frac{\#(A \cap [0, N])}{N + 1}.$$

A useful sufficient condition for frequent hypercyclicity has been given by Bayart and Grivaux [1, 2] and improved by Bonilla and Grosse-Erdmann [5]:

Theorem 1.1 (Frequent Hypercyclicity Criterion). *Let X be a separable infinite-dimensional Banach space and $T \in L(X)$. Suppose that there exist a dense subset X_0 of X and a map $S : X_0 \rightarrow X_0$ such that, for all $x \in X_0$,*

- (1) $\sum_{n=1}^{\infty} T^n x$ converges unconditionally,
- (2) $\sum_{n=1}^{\infty} S^n x$ converges unconditionally,
- (3) $TSx = x$.

Then T is frequently hypercyclic.

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In this paper, we are interested in the existence of infinite-dimensional closed subspaces in which every non-zero vector is frequently hypercyclic. Such a subspace is called a *frequently hypercyclic subspace*. The notion of a frequently hypercyclic subspace was studied for the first time by Bonilla and Grosse-Erdmann [6]. In particular, they obtained the following sufficient condition for the existence of frequently hypercyclic subspaces:

Theorem 1.2 ([6]). *Let X be a separable infinite-dimensional Banach space and $T \in L(X)$. Suppose that*

- (1) *T satisfies the Frequent Hypercyclicity Criterion;*
- (2) *there exists an infinite-dimensional closed subspace M_0 of X such that, for all $x \in M_0$, $T^n x \rightarrow 0$ as $n \rightarrow \infty$.*

Then T has a frequently hypercyclic subspace.

The convergence to 0 along the *whole* sequence (n) for each vector in M_0 is a strong assumption and the examples of frequently hypercyclic operators satisfying this criterion are mainly operators for which there exists some scalar λ with $|\lambda| < 1$ such that $\dim \ker(T - \lambda) = \infty$. In particular, so far no weighted shift with frequently hypercyclic subspaces is known.

On the other hand, an operator T possesses a *hypercyclic subspace* if there exists an infinite-dimensional closed subspace in which every non-zero vector is hypercyclic for T . Obviously, if T does not possess a hypercyclic subspace or if T is not frequently hypercyclic, then T does not possess a frequently hypercyclic subspace. Bonilla and Grosse-Erdmann then asked the following question [6, Problem 1]:

Problem 1. Does there exist a frequently hypercyclic operator that has a hypercyclic subspace but not a frequently hypercyclic subspace?

In Section 2, we state a sufficient condition for the non-existence of frequently hypercyclic subspaces. Then we show that there exists some frequently hypercyclic weighted shift on l^p that possesses a hypercyclic subspace and that satisfies this condition. This allows us to answer Problem 1 positively.

In Section 3, we state a slight improvement of Theorem 1.2 and we give an example of a weighted shift on l^p possessing a frequently hypercyclic subspace.

2. NON-EXISTENCE OF FREQUENTLY HYPERCYCLIC SUBSPACES

In the case of hypercyclic subspaces, we know a simple criterion [9, 10] such that if an operator T satisfies this criterion, then T does not possess any hypercyclic subspace.

Theorem 2.1. *Let X be a separable infinite-dimensional Banach space and $T \in L(X)$. If there exist $C > 1$, a closed subspace E of finite codimension in X and a positive integer $n \geq 1$ such that*

$$\|T^n x\| \geq C\|x\| \quad \text{for any } x \in E,$$

then T does not possess any hypercyclic subspace.

In fact, one can show that if T satisfies the assumption of Theorem 2.1, then each infinite-dimensional closed subspace M of X contains a vector x such that $\|T^n x\| \geq 1$ for any n . The construction of such a vector x relies on the following lemma.

Lemma 2.2 ([7, Lemma 10.39]). *Let X be a separable infinite-dimensional Banach space. For any finite-dimensional subspace F of X , any $\varepsilon > 0$, there exists a closed subspace E of finite codimension in X such that for any $x \in E$ and any $y \in F$,*

$$\|x + y\| \geq \max\left(\frac{\|x\|}{2 + \varepsilon}, \frac{\|y\|}{1 + \varepsilon}\right).$$

We would like a similar criterion for the notion of frequently hypercyclic subspaces. Obviously, if T does not possess any hypercyclic subspace, then T does not possess any frequently hypercyclic subspace. Nevertheless, in order to prove that an operator does not possess any frequently hypercyclic subspace, it is sufficient to prove that each infinite-dimensional closed subspace M contains a vector x such that the upper density of $\{n \geq 0 : \|T^n x\| \geq C\}$ is equal to 1 for some $C > 0$, i.e.

$$\limsup_{N \rightarrow \infty} \frac{\#\{n \leq N : \|T^n x\| \geq C\}}{N + 1} = 1.$$

Indeed, such a vector x cannot be frequently hypercyclic since the lower density of $\{n \geq 0 : \|T^n x\| < C\}$ would be equal to 0.

Theorem 2.3. *Let X be a separable infinite-dimensional Banach space and $T \in L(X)$. If there exists $C > 0$ such that for any $K \geq 1$, any infinite-dimensional closed subspace M of X , any $x \in X$, there exists $x' \in M$ such that $\|x'\| \leq \frac{1}{K}$ and*

$$\sup_{k > K} \frac{\#\{n \leq k : \|T^n(x + x')\| \geq C\}}{k + 1} > 1 - \frac{1}{K},$$

then T does not possess any frequently hypercyclic subspace.

Proof. Let $\tilde{M}$ be an infinite-dimensional closed subspace of X . We show that there exists a vector $x \in \tilde{M}$ such that

$$\limsup_{N \rightarrow \infty} \frac{\#\{n \leq N : \|T^n x\| \geq \frac{C}{2}\}}{N + 1} = 1.$$

To this end, we recursively construct a sequence $(x_n)_{n \geq 0} \subset \tilde{M}$ with $x_0 = 0$ and an increasing sequence $(k_n)_{n \geq 0}$ with $k_0 = 0$ such that for any $n \geq 1$,

- (1) $\|x_n\| \leq \frac{1}{n^2}$;
- (2) we have

$$\frac{\#\{j \leq k_n : \|T^j(\sum_{k=1}^n x_k)\| \geq C\}}{k_n + 1} > 1 - \frac{1}{n^2};$$

- (3) for any $j \leq k_{n-1}$, $T^j x_n \in \bigcap_{k \leq n-1} E_{k,j}$, where $E_{k,j}$ is the closed subspace of finite codimension given by Lemma 2.2 for $F_{k,j} = \text{span}(T^j x_0, \dots, T^j x_k)$ and $\varepsilon = 1$.

At each step, we obtain such a vector x_n by using our assumption for $K = \max\{n^2, k_{n-1}\}$, $M = \tilde{M} \cap \bigcap_{j \leq k_{n-1}} T^{-j}(\bigcap_{k \leq n-1} E_{k,j})$ and $x = \sum_{k=1}^{n-1} x_k$.

We let $x := \sum_{n=1}^{\infty} x_n$ and $J_n = \{j \leq k_n : \|T^j(\sum_{k=1}^n x_k)\| \geq C\}$. We know that $x \in \tilde{M}$ and moreover, for any $n \geq 1$, any $j \in J_n$, we have

$$\begin{aligned} \|T^j x\| &= \left\| \sum_{k=1}^{\infty} T^j x_k \right\| \\ &\geq \frac{1}{2} \left\| \sum_{k=1}^n T^j x_k \right\| \quad \text{because } \sum_{k=n+1}^{\infty} T^j x_k \in E_{n,j} \text{ and } \sum_{k=1}^n T^j x_k \in F_{n,j} \\ &\geq \frac{C}{2} \quad \text{because } j \in J_n. \end{aligned}$$

We deduce that for any $n \geq 1$,

$$\frac{\#\{j \leq k_n : \|T^j x\| \geq \frac{C}{2}\}}{k_n + 1} \geq \frac{\#J_n}{k_n + 1} \geq 1 - \frac{1}{n^2} \xrightarrow{n \rightarrow \infty} 1.$$

The result follows. $\square$

In the case of weighted shifts, Theorem 2.3 gives us a simple sufficient condition for the non-existence of frequently hypercyclic subspaces. We recall that a weighted shift $B_w : l^p \rightarrow l^p$ is defined by

$$B_w((x_0, x_1, x_2, \dots)) = (w_1 x_1, w_2 x_2, w_3 x_3, \dots),$$

where $w = (w_n)_{n \geq 1}$ is a bounded sequence of non-zero scalars. In other words, we have $B_w e_k = w_k e_{k-1}$, where $(e_k)_{k \geq 0}$ is the canonical basis of l^p and $e_{-1} = 0$.

Theorem 2.4. *Let $B_w : l^p \rightarrow l^p$ be a weighted shift ($1 \leq p < \infty$). If there exists a sequence $(C_k)_{k \geq 0}$ of positive numbers such that $\sum_{k \geq 0} (1/C_k)^p < \infty$ and*

$$(2.1) \quad \frac{\#\{n \geq 0 : \|B_w^n e_k\| \geq C_k\}}{k+1} \xrightarrow{k \rightarrow \infty} 1,$$

then B_w does not possess any frequently hypercyclic subspace.

Proof. Let $K \geq 1$, M be an infinite-dimensional closed subspace of l^p and $x \in l^p$. In view of Theorem 2.3, it suffices to show that there exists $x' \in M$ such that $\|x'\| \leq \frac{1}{K}$ and

$$\sup_{k > K} \frac{\#\{n \leq k : \|B_w^n(x + x')\| \geq 1\}}{k+1} > 1 - \frac{1}{K}.$$

We first remark that for any $x' \in l^p$, any $k \geq 0$,

$$\frac{\#\{n \leq k : \|B_w^n(x + x')\| \geq 1\}}{k+1} \geq \frac{\#\{n \geq 0 : \|(x_k + x'_k)B_w^n e_k\| \geq 1\}}{k+1}$$

and thus, if $|x_k + x'_k| \geq 1/C_k$, we have

$$\frac{\#\{n \leq k : \|B_w^n(x + x')\| \geq 1\}}{k+1} \geq \frac{\#\{n \geq 0 : \|B_w^n e_k\| \geq C_k\}}{k+1}.$$

By (2.1), we know that there exists k_0 such that for any $k \geq k_0$

$$\frac{\#\{n \geq 0 : \|B_w^n e_k\| \geq C_k\}}{k+1} > 1 - \frac{1}{K}.$$

We deduce that it suffices to find a vector $x' \in M$ such that $\|x'\| \leq \frac{1}{K}$ and $|x_k + x'_k| \geq 1/C_k$ for some $k \geq \max(k_0, K + 1)$. Let $k_1 \geq \max(k_0, K + 1)$ such that

$$(2.2) \quad \left(\sum_{k=k_1}^{\infty} |x_k|^p \right)^{\frac{1}{p}} \leq \frac{1}{2K} \quad \text{and} \quad \left(\sum_{k=k_1}^{\infty} \frac{1}{C_k^p} \right)^{1/p} \leq \frac{1}{2K}.$$

Since M is an infinite-dimensional closed subspace, we know that for any $k \geq 0$, M contains a vector y with valuation $v(y) := \inf\{j \geq 0 : y_j \neq 0\}$ larger than k . We can thus choose $x' \in M$ such that

$$v(x') \geq k_1 \quad \text{and} \quad \|x'\| = 1/K.$$

Using (2.2), we deduce that

$$\begin{aligned} \left(\sum_{k=k_1}^{\infty} |x'_k + x_k|^p \right)^{1/p} &\geq \|x'\| - \left(\sum_{k=k_1}^{\infty} |x_k|^p \right)^{1/p} \\ &\geq \frac{1}{K} - \frac{1}{2K} = \frac{1}{2K} \\ &\geq \left(\sum_{k=k_1}^{\infty} \frac{1}{C_k^p} \right)^{1/p}, \end{aligned}$$

and thus, there exists $k \geq k_1$ such that

$$|x'_k + x_k| \geq \frac{1}{C_k},$$

which concludes the proof. $\square$

We can now prove the existence of a frequently hypercyclic weighted shift possessing a hypercyclic subspace and no frequently hypercyclic subspace. Thanks to Bayart and Ruzsa [4], we know a characterization of frequently hypercyclic weighted shifts on l^p . A weighted shift $B_w : l^p \rightarrow l^p$ is frequently hypercyclic if and only if B_w satisfies the Frequent Hypercyclicity Criterion, and if and only if

$$(2.3) \quad \sum_{k=1}^{\infty} \frac{1}{\prod_{\nu=1}^k |w_{\nu}|^p} < \infty.$$

We also know a characterization of weighted shifts on l^p with hypercyclic subspaces [8, 10]. A hypercyclic weighted shift $B_w : l^p \rightarrow l^p$ possesses a hypercyclic subspace if and only if

$$(2.4) \quad \sup_{n \geq 1} \inf_{k \geq 0} \prod_{\nu=1}^n |w_{k+\nu}| \leq 1.$$

In conclusion, it just remains to show that there is some bounded weighted sequence w satisfying (2.3), (2.4) and the condition of Theorem 2.4.

Theorem 2.5. *There exists some frequently hypercyclic weighted shift on l^p possessing a hypercyclic subspace and no frequently hypercyclic subspace.*

Proof. Let $(a_n)_{n \geq 0}$ be a strictly increasing sequence of integers such that $a_0 = 1$. We consider the weighted sequence w given by $w_k = 2$ if $k \in [a_{2n}, a_{2n+1}[$ for some $n \geq 0$, and $w_k = 1$ otherwise.

Since the sequence w is bounded, the weighted shift B_w is an operator on l^p . We first show that if

$$\sum_{n=0}^{\infty} \frac{a_{2n+2} - a_{2n+1}}{(2^{a_{2n+1} - a_{2n}})^p} < \infty,$$

then B_w is frequently hypercyclic. We remark that for any $k \geq a_{2n+1}$, we have $\prod_{\nu=1}^k |w_\nu| \geq 2^{a_{2n+1} - a_{2n}}$ and if $(n_j)_{j \geq 1}$ is the increasing enumeration of the set $\bigcup_{n=0}^{\infty} [a_{2n}, a_{2n+1}[$, then $\prod_{\nu=1}^{n_j} |w_\nu| = 2^j$. We therefore deduce that

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{1}{\prod_{\nu=1}^k |w_\nu|^p} &= \sum_{n=0}^{\infty} \left(\sum_{k \in [a_{2n}, a_{2n+1}[} \frac{1}{\prod_{\nu=1}^k |w_\nu|^p} + \sum_{k \in [a_{2n+1}, a_{2n+2}[} \frac{1}{\prod_{\nu=1}^k |w_\nu|^p} \right) \\ &= \sum_{j=1}^{\infty} \frac{1}{\prod_{\nu=1}^{n_j} |w_\nu|^p} + \sum_{n=0}^{\infty} \sum_{k \in [a_{2n+1}, a_{2n+2}[} \frac{1}{\prod_{\nu=1}^k |w_\nu|^p} \\ &\leq \sum_{j=1}^{\infty} \left(\frac{1}{2^j} \right)^p + \sum_{n=0}^{\infty} \frac{a_{2n+2} - a_{2n+1}}{(2^{a_{2n+1} - a_{2n}})^p} \end{aligned}$$

and, in view of (2.3), we conclude that if

$$\sum_{n=0}^{\infty} \frac{a_{2n+2} - a_{2n+1}}{(2^{a_{2n+1} - a_{2n}})^p} < \infty,$$

then B_w is frequently hypercyclic.

Thanks to (2.4), we also know that if $a_{2n+2} - a_{2n+1}$ tends to infinity, then B_w possesses a hypercyclic subspace and, finally, we remark that we have, for any $l \geq a_{2n+1}$,

$$\#\{k \geq 0 : \|B_w^k e_l\| \geq 2^{a_{2n+1} - a_{2n}}\} \geq a_{2n}.$$

Thus, if for any $a_{2n+1} \leq l < a_{2n+3}$, we let $C_l = 2^{a_{2n+1} - a_{2n}}$, then we have

$$\frac{\#\{k \geq 0 : \|B_w^k e_l\| \geq C_l\}}{l+1} \geq \frac{a_{2n}}{a_{2n+3}} \quad \text{and} \quad \sum_{l=a_1}^{\infty} \frac{1}{C_l^p} = \sum_{n=0}^{\infty} \frac{a_{2n+3} - a_{2n+1}}{(2^{a_{2n+1} - a_{2n}})^p}.$$

Therefore, if

$$\sum_{n=0}^{\infty} \frac{a_{2n+3} - a_{2n+1}}{(2^{a_{2n+1} - a_{2n}})^p} < \infty \quad \text{and} \quad \frac{a_{2n}}{a_{2n+3}} \rightarrow 1,$$

we deduce from Theorem 2.4 that B_w does not possess any frequently hypercyclic subspace.

We now show that if we let

$$a_{2n} = 1 + n(n+1) \quad \text{and} \quad a_{2n+1} = 1 + (n+1)^2,$$

then each of the properties above is satisfied, which will conclude the proof. We first remark that

$$a_{2n+1} - a_{2n} = n+1 = a_{2n+2} - a_{2n+1}.$$

The operator B_w is thus frequently hypercyclic because

$$\sum_{n=0}^{\infty} \frac{a_{2n+2} - a_{2n+1}}{(2^{a_{2n+1} - a_{2n}})^p} = \sum_{n=0}^{\infty} \frac{n+1}{(2^{n+1})^p} < \infty,$$

and, as $a_{2n+2} - a_{2n+1}$ tends to infinity, the operator B_w possesses a hypercyclic subspace. Moreover, we remark that

$$\frac{a_{2n}}{a_{2n+3}} = \frac{1 + n(n+1)}{1 + (n+2)^2} \rightarrow 1$$

and

$$\sum_{n=0}^{\infty} \frac{a_{2n+3} - a_{2n+1}}{(2^{a_{2n+1} - a_{2n}})^p} = \sum_{n=0}^{\infty} \frac{(n+2)^2 - (n+1)^2}{(2^{n+1})^p} = \sum_{n=0}^{\infty} \frac{2n+3}{(2^{n+1})^p} < \infty.$$

The operator B_w is thus a frequently hypercyclic weighted shift possessing a hypercyclic subspace and no frequently hypercyclic subspace. $\square$

3. EXISTENCE OF FREQUENTLY HYPERCYCLIC SUBSPACES

Let A be a set of positive lower density and (n_k) the increasing enumeration of A . It is obvious that if J is a set of positive lower density, then $\{n_k : k \in J\}$ is still a set of positive lower density. Using this argument, it is not difficult to adapt the proof of [6, Lemma 1] in order to force the sets $E(l, j)$ (given by this lemma) to be included in A . Indeed, we can see in the proof of this lemma that the sets $E(l, j)$ just have to be sets of positive lower density satisfying convenient separation properties. In view of the previous argument, we can therefore replace the sets $E(l, j)$ by the sets $\{n_k : k \in E(l, j)\} \subset A$. Thanks to this improvement, the proof of Theorem 1.2 can then be adapted in order to obtain the following generalization.

Theorem 3.1. *Let X be a separable infinite-dimensional Banach space and $T \in L(X)$. If T satisfies the Frequent Hypercyclicity Criterion and if there exist an infinite-dimensional closed subspace $M_0 \subset X$ and a set A of positive lower density such that for any $x \in M_0$,*

$$T^n x \xrightarrow[n \rightarrow \infty]{n \in A} 0,$$

then T possesses a frequently hypercyclic subspace.

We can now state a sufficient condition for a frequently hypercyclic weighted shift on l^p to possess a frequently hypercyclic subspace.

Theorem 3.2. *Let $B_w : l^p \rightarrow l^p$ be a frequently hypercyclic weighted shift and*

$$G(k, C) := \{n \geq 0 : \|B_w^n e_k\| \leq C\}, \quad k \geq 0, \quad C > 0.$$

If there exist a strictly increasing sequence $(k_l)_{l \geq 1}$ of non-negative integers and $C > 0$ such that $\bigcap_{l \geq 1} G(k_l, C)$ is of positive lower density, then B_w possesses a frequently hypercyclic subspace.

Proof. Let $M_0 = \overline{\text{span}}\{e_{k_l} : l \geq 1\}$. If $x \in M_0$ and $n \in \bigcap_{l \geq 1} G(k_l, C)$, we have, by definition of $G(k, C)$,

$$\|B_w^n x\|^p = \sum_{l=1}^{\infty} |x_{k_l}|^p \|B_w^n e_{k_l}\|^p = \sum_{k_l \geq n} |x_{k_l}|^p \|B_w^n e_{k_l}\|^p \leq \sum_{k_l \geq n} C^p |x_{k_l}|^p \xrightarrow[n \rightarrow \infty]{} 0.$$

Since each frequently hypercyclic weighted shift B_w on l^p satisfies the Frequent Hypercyclicity Criterion and since $\bigcap_{l \geq 1} G(k_l, C)$ is of positive lower density, we conclude by using Theorem 3.1. $\square$

Example 3.3 (Weighted shift with a frequently hypercyclic subspace). Let $w = (4, 1^{a_1}, 4, 1^{a_2}, 4, 1^{a_3}, 4, 1^{a_4}, 4, \dots)$ where (a_n) is a sequence of positive integers satisfying $a_{n+1} \geq \sum_{k=1}^n a_k + (n+1)$ and 1^{a_n} designates a block of 1 of length a_n . For any value of a_n , the weighted shift B_w is an operator on l^p , and by (2.3), B_w is frequently hypercyclic if and only if

$$(3.1) \quad \sum_{k=1}^{\infty} \frac{a_k}{(4^k)^p} < \infty.$$

Moreover, we see that

$$G\left(\sum_{k=1}^n a_k + n, 1\right) = [0, a_n] \cup \left[\sum_{k=1}^n a_k + (n+1), \infty\right[,$$

and since (a_n) satisfies $a_{n+1} \geq \sum_{k=1}^n a_k + (n+1)$, we get

$$\begin{aligned} A := \bigcap_{n=1}^{\infty} G\left(\sum_{k=1}^n a_k + n, 1\right) &= \bigcap_{n=1}^{\infty} \left([0, a_n] \cup \left[\sum_{k=1}^n a_k + (n+1), \infty\right[\right) \\ &= [0, a_1] \cup \bigcup_{n=1}^{\infty} \left[\sum_{k=1}^n a_k + (n+1), a_{n+1}\right]. \end{aligned}$$

In particular, we deduce that if

$$(3.2) \quad \liminf_n \frac{a_n - \sum_{k=1}^{n-1} a_k - n}{\sum_{k=1}^n a_k + (n+1)} > 0,$$

then A is a set of positive lower density. We remark that, if we let $a_n = 3^n$, then $a_{n+1} \geq \sum_{k=1}^n a_k + (n+1)$ because

$$\sum_{k=1}^n 3^k + (n+1) = \frac{3}{2}(3^n - 1) + (n+1) \leq \frac{3^{n+1}}{2} + (n+1) \leq 3^{n+1}.$$

Moreover, if $a_n = 3^n$, then (3.1) and (3.2) are satisfied since we have

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{3^k}{(4^k)^p} &< \infty \quad \text{and} \\ \liminf_n \frac{3^n - \sum_{k=1}^{n-1} 3^k - n}{\sum_{k=1}^n 3^k + (n+1)} &= \liminf_n \frac{3^n - \frac{3}{2}(3^{n-1} - 1) - n}{\frac{3}{2}(3^n - 1) + (n+1)} = \frac{1}{3}. \end{aligned}$$

Therefore, we deduce from Theorem 3.2 that if $a_n = 3^n$, then B_w possesses a frequently hypercyclic subspace. $\square$

In view of Theorem 2.5 and Example 3.3, we deduce on the one hand that there exist some frequently hypercyclic weighted shifts on l^p with a frequently hypercyclic subspace and some others without a frequently hypercyclic subspace, and, on the other hand, that the existence of a frequently hypercyclic subspace is not equivalent to the existence of a hypercyclic subspace and the frequent hypercyclicity. Therefore, we pose the following question:

Question. Which weighted shifts on l^p possess a frequently hypercyclic subspace?

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