A CRITERION FOR COMPLETENESS

PETER SCHENZEL

(Communicated by Irena Peeva)

ABSTRACT. Let $(R, \mathfrak{m})$ denote a local ring with $E = E_R(k)$ the injective hull of $k = R/\mathfrak{m}$, its residue field. Let M denote a finitely generated R -module. By Jensen's result it follows that $\text{Ext}_R^1(F, M) = 0$ for any flat R -module F if and only if M is complete. Let $\underline{x} = x_1, \dots, x_r$ be a system of elements of R such that $\text{Rad } \underline{x}R = \mathfrak{m}$. In the main result it is shown that the vanishing of $\text{Ext}_R^1(F, M)$, $F = \bigoplus_{i=1}^r R_{x_i}$, implies that M is complete. It is known from work of Enochs and Jenda that $\text{Hom}_R(E_R(R/\mathfrak{p}), E) \simeq \widehat{R_{\mathfrak{p}}^{\mu_{\mathfrak{p}}}}$ for a certain finite or infinite number $\mu_{\mathfrak{p}}$. We discuss which $\mu_{\mathfrak{p}}$ might occur for certain primes with $\dim R/\mathfrak{p} = 1$.

1. INTRODUCTION

Let $(R, \mathfrak{m})$ denote a local ring and M a finitely generated R -module. In his paper (see [6, Proposition 3]) C. U. Jensen proved that M is complete if and only if $\text{Ext}_R^1(F, M) = 0$ for all flat (resp. countably generated flat) R -modules F . A different proof for the vanishing of $\text{Ext}_R^i(F, M)$ for all $i \geq 1$ and all flat R -modules follows by results of R.-O. Buchweitz and H. Flenner (see [2, Theorem 2]).

Let $I \subset R$ denote an ideal in a commutative Noetherian ring R . In their paper (see [4]) A. Frankild and S. Sather-Wagstaff proved that M is I -adically complete if and only if $\text{Ext}_R^i(\hat{R}^I, M) = 0$ for all $1 \leq i \leq \dim_R M$, where $\hat{R}^I$ denotes the I -adic completion of R .

The main result of the paper is the construction of a simple flat test module F for the completeness of M in terms of the vanishing of $\text{Ext}_R^1(F, M)$. More precisely, we prove the following result:

Theorem 1.1. *Let M denote a finitely generated R -module. Let $x_i \in \mathfrak{m}$, $i = 1, \dots, r$, be elements such that $(x_1, \dots, x_r)R$ is an $\mathfrak{m}$ -primary ideal. Then the following conditions are equivalent:*

- (i) M is $\mathfrak{m}$ -complete.
- (ii) $\text{Ext}_R^i(F, M) = 0$ for any flat R -module F and all $i \geq 1$.
- (iii) $\text{Ext}_R^1(\bigoplus_{i=1}^r R_{x_i}, M) = 0$.

For the proof of the implication (i) $\implies$ (ii) we use an argument on Matlis Duality different from Jensen's approach. Moreover, note that $\bigoplus_{i=1}^r R_{x_i}$ is a countably generated flat R -module.

Let $E_R(R/\mathfrak{p})$ denote the injective hull of $R/\mathfrak{p}$ for some $\mathfrak{p} \in \text{Spec } R$. Then for its Matlis dual it is known that $\text{Hom}_R(E_R(R/\mathfrak{p}), E) \simeq \widehat{R_{\mathfrak{p}}^{\mu_{\mathfrak{p}}}}$ (see [3, 3.3.10]), where

Received by the editors August 1, 2013 and, in revised form, January 25, 2014.
 2010 *Mathematics Subject Classification.* Primary 13J10; Secondary 13C11, 13D07.
Key words and phrases. Completion, flat module, injective hull, localization.

$\mu_{\mathfrak{p}} = \dim_{k(\mathfrak{p})} \operatorname{Hom}_R(k(\mathfrak{p}), E)$. Under certain additional assumptions on $R/\mathfrak{p}$ the following numbers of $\mu_{\mathfrak{p}}$ occur:

Theorem 1.2. *Let $\mathfrak{p}$ denote a one-dimensional prime ideal of a local ring $(R, \mathfrak{m})$. Let $\mu_{\mathfrak{p}}$ be defined as above. Suppose that $R/\mathfrak{p}$ is a Gorenstein ring and analytically unramified. The cardinal numbers $\mu_{\mathfrak{p}}$ that occur are exactly*

- (1) *any infinite cardinal number,*
- (2) *among the finite cardinal numbers exactly those of the form p^t , p being a prime number and t a non-negative integer.*

When $\mu_{\mathfrak{p}}$ is finite and not equal to 1 $R/\mathfrak{p}$ is necessarily of prime characteristic.

Let $\mathfrak{p} \subset R$ denote an arbitrary one-dimensional prime ideal. In [12] it was shown that $\mu_{\mathfrak{p}} = 1$ if and only if $R/\mathfrak{p}$ is complete. See also Proposition 4.3. In the case of a one-dimensional local domain A with its quotient field $Q = A_x, x \in \mathfrak{m} \setminus \{0\}$, the A -module $\operatorname{Ext}_A^1(Q, A)$ is (by Theorem 1.1) the obstruction for A being complete. Clearly, it is a Q -vector space. Its dimension was investigated by C. U. Jensen (see [7]). These results apply in order to get the information on $\mu_{\mathfrak{p}}$ above.

2. PRELIMINARY RESULTS

In the following $(R, \mathfrak{m})$ denotes a local ring. We begin with the definition of the global transform.

Definition 2.1. Let M denote a finitely generated R -module. Then

$$T(M) = \varinjlim_{\alpha} \operatorname{Hom}_R(\mathfrak{m}^{\alpha}, M)$$

is called the global transform of M . In the case of $M = R$ we also write $T = T(R)$.

Let M denote an R -module. In the following we shall consider the local cohomology modules $H_{\mathfrak{m}}^i(M), i \in \mathbb{Z}$ (see e.g. [5] for definitions and basic results). There is an exact sequence

$$0 \rightarrow H_{\mathfrak{m}}^0(M) \rightarrow M \rightarrow T(M) \rightarrow H_{\mathfrak{m}}^1(M) \rightarrow 0$$

relating the global transform to the local cohomology.

Let $E = E_R(k)$ denote the injective hull of the residue field $k = R/\mathfrak{m}$. Then $D_R(\cdot) = \operatorname{Hom}_R(\cdot, E)$ denotes the Matlis Duality functor. Moreover $\hat{\cdot}$ denotes the completion functor. In particular $\hat{R}$ is the completion of R . In the case of a local ring $(R, \mathfrak{m})$ completion means always completion with respect to the maximal ideal $\mathfrak{m}$. In all other cases the ideal will be specified.

Remark 2.2. (A) Let X be an arbitrary R -module. There is a natural homomorphism

$$X \rightarrow D_R(D_R(X))$$

that is always injective. If $(R, \mathfrak{m})$ is complete it is an isomorphism whenever X is an Artinian R -module resp. a finitely generated R -module (see [8, p. 528] and [8, Corollary 4.3]). Moreover, it follows that the map is an isomorphism if and only if there is a finitely generated R -submodule $Y \subset X$ such that X/Y is an Artinian R -module. For the proof we refer to [14] and also to [1] for a generalization.

(B) Let M denote a finitely generated R -module. Then there is a natural isomorphism $M \otimes_R \hat{R} \simeq D_R(D_R(M))$. That is, M is Matlis reflexive if and only if it is complete.

(C) Let X denote an R -module with $\operatorname{Supp}_R X \subseteq \{\mathfrak{m}\}$. Then M admits the structure

of an $\hat{R}$ -module compatible with its R -module structure such that $X \otimes_R \hat{R} \rightarrow X$ is an isomorphism (see e.g. [14, (2.1)]). Let M denote an R -module and N an $\hat{R}$ module. Then $\text{Ext}_R^i(M, N), i \in \mathbb{Z}$, has the structure of an $\hat{R}$ -module. Moreover, there are natural isomorphisms

$$\text{Ext}_R^i(M, N) \simeq \text{Ext}_{\hat{R}}^i(M \otimes_R \hat{R}, N)$$

for all $i \in \mathbb{Z}$ since $\hat{R}$ is a flat R -module.

The following lemma is of some interest in general.

Lemma 2.3. *Let M denote a finitely generated R -module. Then there is a short exact sequence*

$$0 \rightarrow T(M) \rightarrow T(\hat{M}) \rightarrow \hat{M}/M \rightarrow 0.$$

Moreover, $D_{\hat{R}}(D_R(T(M))) \simeq T(M) \otimes_R \hat{R} \simeq T(\hat{M})$.

Proof. The above exact sequence of the global transform induces a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \rightarrow & H_{\mathfrak{m}}^0(M) & \rightarrow & M & \rightarrow & T(M) & \rightarrow & H_{\mathfrak{m}}^1(M) & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & H_{\mathfrak{m}}^0(M) \otimes_R \hat{R} & \rightarrow & M \otimes_R \hat{R} & \rightarrow & T(M) \otimes_R \hat{R} & \rightarrow & H_{\mathfrak{m}}^1(M) \otimes_R \hat{R} & \rightarrow & 0. \end{array}$$

All the vertical maps are injective. Now recall that $H_{\mathfrak{m}}^i(M), i \in \mathbb{Z}$, is an Artinian R -module. Therefore, by view of Remark 2.2 (C) the vertical homomorphisms at the left and right end are isomorphisms. This implies a short exact sequence

$$0 \rightarrow T(M) \rightarrow T(M) \otimes_R \hat{R} \rightarrow \hat{M}/M \rightarrow 0.$$

Since $\hat{R}$ is a flat R -module it commutes with the global transform. This completes the proof of the exact sequence of the statement.

Again by Remark 2.2 (C) there is an isomorphism $D_R(T(M)) \simeq D_{\hat{R}}(T(M) \otimes_R \hat{R})$. Now $T(M) \otimes_R \hat{R}$ is Matlis reflexive as follows by the second row of the above commutative diagram. This finishes the proof. $\square$

Let M denote a finitely generated R -module. We define the multiplicatively closed set

$$S = S(M) = \bigcap_{\mathfrak{p} \in \text{Assh } M} (R \setminus \mathfrak{p}) \text{ where } \text{Assh } M = \{\mathfrak{p} \in \text{Supp } M \mid \dim R/\mathfrak{p} = \dim M\}.$$

In the following proposition we assume that $\dim_R M = 1$.

Proposition 2.4. *Let $(R, \mathfrak{m})$ denote a local ring. Let M be a finitely generated R -module and $\dim_R M = 1$. Then there are isomorphisms*

$$T(M) \simeq M_S \simeq M_x \quad \text{and} \quad (M_S \otimes_R \hat{R})/M_S \simeq \hat{M}/M$$

where $x \in \mathfrak{m}$ denotes a parameter of M and $\hat{M}$ is the completion of M .

Proof. The proof of the first isomorphisms is well known and easy to see by the exact sequence in the proof of the previous statement. The second isomorphism is true by Lemma 2.3. $\square$

The next result is a consequence of [10, Corollary 4.7]. Moreover, it seems to be well known. Here we include a simple direct argument.

Proposition 2.5. *Let $(x_1, \dots, x_r)R$ denote an ideal of a local ring $(R, \mathfrak{m})$ and $I = \text{Rad}(x_1, \dots, x_r)R$. Let M be a finitely generated R -module such that M is complete in the $(x_i)R$ -adic topology for $i = 1, \dots, r$. Then M is I -adically complete.*

Proof. Let $\hat{M}^I$ denote the I -adic completion of M . Because the I -adic topology and the topology defined by $\{(x_1^\alpha, \dots, x_r^\alpha)R\}_{\alpha \geq 0}$ are equivalent it follows $\hat{M}^I \simeq \varprojlim_{\alpha} M/(x_1^\alpha, \dots, x_r^\alpha)M$. Therefore an element $m \in \hat{M}^I$ has the form $m = \sum_{\alpha \geq 0} \sum_{i=1}^r x_i^\alpha m_{\alpha,i}$ with $m_{\alpha,i} \in M$. Since M is $(x_i)R$ -adically complete $M \simeq \varprojlim_{\alpha} M/x_i^\alpha M$ and $m_i = \sum_{\alpha} x_i^\alpha m_{\alpha,i}$ for some $m_i \in M$. Then $m = \sum_{i=1}^r m_i \in M$ and M is I -adically complete. $\square$

The following result is helpful in order to compute projective limits of Ext-modules. Here $\varprojlim^1$ denotes the first right derived functor of the projective limit (see [13]).

Theorem 2.6 ([11, Lemma 2.6]). *Let $\{M_\alpha\}_{\alpha \in \mathbb{N}}$ be a direct system of R -modules. Let N denote an arbitrary R -module. Then there is a short exact sequence*

$$0 \rightarrow \varprojlim^1 \text{Ext}_R^{i-1}(M_\alpha, N) \rightarrow \text{Ext}_R^i(\varprojlim M_\alpha, N) \rightarrow \varprojlim \text{Ext}_R^i(M_\alpha, N) \rightarrow 0$$

for all $i \in \mathbb{Z}$.

Let M denote an R -module and $x \in R$. As an application we consider the following projective system $\{M, x\}$, where $M_\alpha = M$ for all $\alpha \in \mathbb{N}$ and the transition map $M_{\alpha+1} \xrightarrow{x} M_\alpha$ is the multiplication by x .

Lemma 2.7. *With the previous notation it follows that*

$$\varprojlim^1 \{M, x\} \simeq \text{Ext}_R^1(R_x, M) \quad \text{and} \quad \text{Ext}_R^i(R_x, M) = 0 \text{ for all } i \geq 2.$$

Moreover, $\text{Hom}_R(R_x, M) \simeq \varprojlim \{M, x\}$. If M is finitely generated and $x \in \mathfrak{m}$, then $\varprojlim \{M, x\} = 0$.

Proof. Let $\{R, x\}$ denote the direct system with $R_\alpha = R$ for all $\alpha \in \mathbb{N}$ and the transition map $R_\alpha \xrightarrow{x} R_{\alpha+1}$ is the multiplication by x . Then $R_x \simeq \varprojlim_{\alpha} \{R, x\}$ as it is well known. Therefore, $\text{Hom}_R(\{R, x\}, M) \simeq \{M, x\}$ and the results of the first part are consequences of Theorem 2.6. If M is a finitely generated R -module and $x \in \mathfrak{m}$ the vanishing of $\text{Hom}_R(R_x, M)$ is an easy consequence of Krull's Intersection Theorem. $\square$

3. COMPLETION

In this section let $(R, \mathfrak{m})$ denote a local ring. The following result is part of Jensen's result (see [6, Proposition 3]) for local rings. In a particular situation it provides a slightly stronger vanishing statement.

Proposition 3.1. *Let M denote a finitely generated R -module with $\hat{M}$ its completion. Let F denote a flat R -module. Then $\text{Ext}_R^i(F, \hat{M}) = 0$ for all $i > 0$. Moreover, $\text{Hom}_R(F, \hat{M}) = 0$ provided $F \otimes_R k = 0$.*

Proof. Put $N = \text{Hom}_R(M, E)$. Then N is an Artinian R -module and in particular $\text{Supp}_R N \subseteq \{\mathfrak{m}\}$. Moreover, by Remark 2.2 it follows that $\hat{M} \simeq D_R(N)$. Therefore there are isomorphisms

$$\text{Ext}_R^i(F, \hat{M}) \simeq \text{Ext}_R^i(F, \text{Hom}_R(N, E)) \simeq \text{Hom}_R(\text{Tor}_i^R(F, N), E).$$

Whence it will be enough to show the vanishing of $\text{Tor}_i^R(F, N)$. This is clear for $i > 0$ since F is a flat R -module.

Now suppose in addition that $F \otimes_R k = 0$. Then induction implies that $F \otimes_R X = 0$ for any R -module X of finite length. To this end use the short exact sequence $0 \rightarrow k = R/\mathfrak{m} \rightarrow X \rightarrow X' \rightarrow 0$ and the fact $\ell_R(X') < \ell_R(X)$, where $\ell_R(\cdot)$ denotes the length.

Now suppose that X is an R -module with $\text{Supp}_R X \subseteq V(\mathfrak{m})$. Then X is the direct limit of its finitely generated submodules. Since the tensor product commutes with direct limits it implies that $F \otimes_R X = 0$. Because of $\text{Supp}_R N \subseteq \{\mathfrak{m}\}$ this finishes the proof of the second part of the statement. $\square$

Corollary 3.2. *Let M be a finitely generated R -module. Let F denote a flat R -module. Then there are an exact sequence*

$$0 \rightarrow \text{Hom}_R(F, M) \rightarrow \text{Hom}_R(F, \hat{M}) \rightarrow \text{Hom}_R(F, \hat{M}/M) \rightarrow \text{Ext}_R^1(F, M) \rightarrow 0$$

and isomorphisms $\text{Ext}_R^i(F, \hat{M}/M) \simeq \text{Ext}_R^{i+1}(F, M)$ for all $i \geq 1$.

Proof. We have the following short exact sequence of R -modules $0 \rightarrow M \rightarrow \hat{M} \rightarrow \hat{M}/M \rightarrow 0$. Then apply the functor $\text{Hom}_R(F, \cdot)$. The long exact cohomology sequence yields the statements by view of the vanishing of $\text{Ext}_R^i(F, \hat{M})$ for all $i > 0$ as shown in Proposition 3.1. $\square$

We will continue with a result on the vanishing of certain Ext-modules. As a step towards our main result we shall prove a partial result in order to characterize the completion.

Theorem 3.3. *Let $(R, \mathfrak{m})$ denote a local ring. Let M denote a finitely generated R -module. For an element $x \in R$ the following conditions are equivalent:*

- (i) *M is $(x)R$ -adically complete.*
- (ii) *$\text{Ext}_R^1(R_x, M) = 0$.*

Proof. Let $N = \bigcup_{\alpha \geq 1} 0 :_M x^\alpha$. Then $N = 0 :_M x^\beta$ for a $\beta \gg 0$ since M is finitely generated. In both of the statements we may replace x by x^β . That is, without loss of generality we may assume $xN = 0$. Then there is a commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 \rightarrow & M/N & \xrightarrow{x^{\alpha+1}} & M & \rightarrow & M/x^{\alpha+1}M & \rightarrow 0 \\ & \downarrow x & & \parallel & & \downarrow & \\ 0 \rightarrow & M/N & \xrightarrow{x^\alpha} & M & \rightarrow & M/x^\alpha M & \rightarrow 0. \end{array}$$

By passing to the inverse limit it provides an exact sequence

$$0 \rightarrow \varprojlim \{M/N, x\} \rightarrow M \rightarrow \hat{M}^x \rightarrow \varprojlim^1 \{M/N, x\} \rightarrow 0.$$

By view of Lemma 2.7 it yields that $\varprojlim^i \{M/N, x\} \simeq \text{Ext}_R^i(R_x, M/N)$ for $i = 0, 1$. Therefore $\varprojlim \{M/N, x\} = 0$ since M/N is a finitely generated R -module. Furthermore, the short exact sequence $0 \rightarrow N \rightarrow M \rightarrow M/N \rightarrow 0$ induces an isomorphism $\text{Ext}_R^1(R_x, M) \simeq \text{Ext}_R^1(R_x, M/N)$. This follows by the long exact cohomology sequence and $\text{Ext}_R^i(R_x, N) = 0$ for all $i \in \mathbb{Z}$. For this vanishing, note that the multiplication by x acts on R_x as an isomorphism and on N as the zero map.

That is, if the natural homomorphism $M \rightarrow \hat{M}^x$ is an isomorphism, then $\text{Ext}_R^1(R_x, M) = 0$ which proves (i) $\implies$ (ii). The converse holds by similar arguments. $\square$

Proof of Theorem 1.1. The implication (i) $\implies$ (ii) is a consequence of Proposition 3.1. Moreover, the implication (ii) $\implies$ (iii) is trivial. In order to prove (iii) $\implies$ (i) we first note that $\text{Ext}_R^1(R_{x_i}, M) = 0$ for all $i = 1, \dots, r$. By Theorem 3.3 this implies that M is $(x_i)R$ -adically complete for $i = 1, \dots, r$. Since $\underline{x} = x_1, \dots, x_r$ generates an $\mathfrak{m}$ -primary ideal it follows that M is $\mathfrak{m}$ -adically complete by Proposition 2.5. This completes the proof. $\square$

4. APPLICATIONS

Let $(R, \mathfrak{m})$ denote a domain and $\dim R = 1$. Let $Q = \mathbb{Q}(R)$ denote its quotient field. Then there is the following consequence of our previous investigations.

Proposition 4.1. *Let $(R, \mathfrak{m})$ denote a one-dimensional domain. There are the following isomorphisms*

$$\hat{R}/R \simeq (Q \otimes_R \hat{R})/Q \simeq \text{Ext}_R^1(Q, R).$$

In particular, $\hat{R}/R$ admits the structure of a Q -vector space.

Proof. The first isomorphism is as a particular case a consequence of Proposition 2.4. In the proof of Theorem 3.3 there is the short exact sequence

$$0 \rightarrow R \rightarrow \hat{R}^x \rightarrow \varprojlim^1 \{R, x\} \simeq \text{Ext}_R^1(Q, R) \rightarrow 0,$$

where $x \in \mathfrak{m}$ denotes a parameter and $R_x \simeq Q$. Since $(x)R$ is $\mathfrak{m}$ -primary it follows $\hat{R}^x = \hat{R}$ which finishes the proof. $\square$

Remark 4.2. Let $(R, \mathfrak{m})$ denote a local domain and $\dim R = 1$. Let $Q = \mathbb{Q}(R)$ its field of quotients. Then $\hat{R}/R$ may be viewed as a Q -vector space (see Proposition 4.1). That is, $\hat{R}/R \simeq \text{Ext}_R^1(Q, R) \simeq Q^d$ for some finite or infinite cardinal number.

Suppose that R is analytically unramified, i.e. $\hat{R}$ is reduced. In his paper (see [7, Theorem 1]) C. U. Jensen has shown:

The cardinal numbers d which appear above are

- (1) any infinite cardinal number,
- (2) among the finite cardinal numbers exactly those of the form $p^t - 1$, p being a prime number and $t \geq 0$ an integer.

When d is finite and not zero, R is necessarily of prime characteristic. Moreover, any d from (1) and (2) can occur for principal ideal domains; in fact, even for valuation rings. In his construction C. U. Jensen used modifications of Nagata's "bad" local rings (see [9]).

Let $\mathfrak{p}$ denote a prime ideal in a local ring $(R, \mathfrak{m})$. Let $E_R(R/\mathfrak{p})$ denote the injective hull of $R/\mathfrak{p}$. Then for its Matlis dual it follows that $\text{Hom}_R(E_R(R/\mathfrak{p}), E) \simeq \widehat{R_{\mathfrak{p}}^{\mu_{\mathfrak{p}}}}$ (see [3, 3.3.10]), where $\mu_{\mathfrak{p}} = \dim_{k(\mathfrak{p})} \text{Hom}_R(k(\mathfrak{p}), E) = \dim_{k(\mathfrak{p})} \text{Hom}_{R/\mathfrak{p}}(k(\mathfrak{p}), E_{R/\mathfrak{p}}(k))$ (see [12, Lemma 4.1]).

As a main result in [12] it is shown that $\mu_{\mathfrak{p}} = 1$ if and only if $R/\mathfrak{p}$ is complete. First we want to add a proof in the frame of this paper.

Proposition 4.3. *Let $(R, \mathfrak{m})$ be a local ring. Let $\mathfrak{p}$ denote a prime ideal and $\dim R/\mathfrak{p} = 1$. Then $R/\mathfrak{p}$ is complete if and only if $\mu_{\mathfrak{p}} = 1$.*

Proof. Let $A = R/\mathfrak{p}$ and $Q = \mathbb{Q}(A)$ be its quotient field. Then $Q \simeq E_A(A)$, the injective hull of A . By view of [12, Lemma 4.1] it will be enough to investigate $\dim_Q \operatorname{Hom}_A(Q, E)$, where E denotes the injective hull of the residue field of A . Now note that $Q = T(A)$ since A is a one-dimensional domain. Then A is complete if and only if $\dim_Q Q \otimes_A \hat{A} = 1$ as follows by view of Lemma 2.3.

There is the following short exact sequence $0 \rightarrow A \rightarrow Q \rightarrow H_{\mathfrak{m}}^1(A) \rightarrow 0$. If A is complete, then $Q \simeq \operatorname{Hom}_R(\operatorname{Hom}_R(Q, E), E)$ (see Remark 2.2) and therefore $\dim_Q \operatorname{Hom}_R(Q, E) = 1$. Conversely, if $\dim_Q(Q \otimes_A \hat{A}) = 1$, then $0 = \hat{A}/A = (Q \otimes_A \hat{A})/Q$ and $A = \hat{A}$. $\square$

Corollary 4.4. *Let $\mathfrak{p}$ denote a one-dimensional prime ideal of a local ring $(R, \mathfrak{m})$. Suppose in addition that $R/\mathfrak{p}$ is analytically unramified and $1 < \mu_{\mathfrak{p}} < \infty$. Then $R/\mathfrak{p}$ is necessarily of prime characteristic.*

Proof. By virtue of [12, Lemma 4.1] it is known that $\mu_{\mathfrak{p}} = \dim_{k(\mathfrak{p})} \operatorname{Hom}_{R/\mathfrak{p}}(k(\mathfrak{p}), E_{R/\mathfrak{p}}(k))$. But $A = R/\mathfrak{p}$. Then $\mu_{\mathfrak{p}} = \dim_Q \operatorname{Hom}_A(Q, E)$. By applying the functor $D_A(D_A(\cdot))$ to the short exact sequence $0 \rightarrow A \rightarrow Q \rightarrow Q/A \rightarrow 0$ it induces an injection

$$0 \rightarrow \hat{A}/A \rightarrow D_A(D_A(Q))/Q$$

by the snake lemma. Suppose that $1 < \mu_{\mathfrak{p}} < \infty$. Then $\operatorname{Hom}_A(Q, E)$ and also $D_A(D_A(Q))$ is of finite Q -dimension. Whence $0 \neq \hat{A}/A \simeq \operatorname{Ext}_A^1(Q, A)$ is also of finite Q -dimension. So the statement is a consequence of C. U. Jensen's result (see Remark 4.2). $\square$

An explicit description of the possible values of $\mu_{\mathfrak{p}}$ can be obtained in the case where $R/\mathfrak{p}$ is a one-dimensional analytically unramified Gorenstein ring.

Proof of Theorem 1.2. As above we use the reduction to $A = R/\mathfrak{p}$ with $Q = \mathbb{Q}(A)$ its quotient field. Then $\mu_{\mathfrak{p}} = \dim_Q \operatorname{Hom}_R(Q, E)$, where E denotes the injective hull of the residue field. Since A is a Gorenstein ring there is a short exact sequence $0 \rightarrow A \rightarrow Q \rightarrow E \rightarrow 0$ since $H_{\mathfrak{m}}^1(A) \simeq E$. By applying the Matlis Duality functor $\operatorname{Hom}_A(\cdot, E)$ and the tensor product $\hat{A} \otimes_A \cdot$ it induces a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \rightarrow & \hat{A} & \rightarrow & \operatorname{Hom}_A(Q, E) & \rightarrow & E \rightarrow 0 \\ & & \parallel & & \downarrow & & \parallel \\ 0 & \rightarrow & \hat{A} & \rightarrow & Q \otimes_A \hat{A} & \rightarrow & E \rightarrow 0. \end{array}$$

By the snake lemma it follows that $\operatorname{Hom}_A(Q, E) \simeq Q \otimes_A \hat{A}$. By view of Proposition 4.1, $\operatorname{Ext}_A^1(Q, A) \simeq (Q \otimes_A \hat{A})/Q$ and C. U. Jensen's result (see Remark 4.2), this completes the proof. $\square$

NOTE ADDED IN PROOF

See also the author's paper in Arch. Math. **102**, no 1. (2014), 25-33, for an extension of Theorem 1.1.

REFERENCES

- [1] Richard G. Belshoff, Edgar E. Enochs, and Juan Ramon García Rozas, *Generalized Matlis duality*, Proc. Amer. Math. Soc. **128** (2000), no. 5, 1307–1312, DOI 10.1090/S0002-9939-99-05130-8. MR1641645 (2000j:13015)

- [2] Ragnar-Olaf Buchweitz and Hubert Flenner, *Power series rings and projectivity*, Manuscripta Math. **119** (2006), no. 1, 107–114, DOI 10.1007/s00229-005-0608-8. MR2194381 (2007a:13025)
- [3] Edgar E. Enochs and Overtoun M. G. Jenda, *Relative homological algebra*, de Gruyter Expositions in Mathematics, vol. 30, Walter de Gruyter & Co., Berlin, 2000. MR1753146 (2001h:16013)
- [4] Anders J. Frankild and Sean Sather-Wagstaff, *Detecting completeness from Ext-vanishing*, Proc. Amer. Math. Soc. **136** (2008), no. 7, 2303–2312, DOI 10.1090/S0002-9939-08-09199-5. MR2390496 (2009c:13016)
- [5] Robin Hartshorne, *Local cohomology*, A seminar given by A. Grothendieck, Harvard University, Fall, vol. 1961, Springer-Verlag, Berlin, 1967. MR0224620 (37 #219)
- [6] C. U. Jensen, *On the vanishing of $\varinjlim^{(i)}$* , J. Algebra **15** (1970), 151–166. MR0260839 (41 #5460)
- [7] C. U. Jensen, *On $\text{Ext}_R^1(A, R)$ for torsion-free A* , Bull. Amer. Math. Soc. **78** (1972), 831–834. MR0311657 (47 #219)
- [8] Eben Matlis, *Injective modules over Noetherian rings*, Pacific J. Math. **8** (1958), 511–528. MR0099360 (20 #5800)
- [9] Masayoshi Nagata, *Local rings*, Interscience Tracts in Pure and Applied Mathematics, No. 13, Interscience Publishers a division of John Wiley & Sons New York-London, 1962. MR0155856 (27 #5790)
- [10] Peter Schenzel, *Proregular sequences, local cohomology, and completion*, Math. Scand. **92** (2003), no. 2, 161–180. MR1973941 (2004f:13023)
- [11] Peter Schenzel, *On the structure of the endomorphism ring of a certain local cohomology module*, J. Algebra **344** (2011), 229–245, DOI 10.1016/j.jalgebra.2011.07.014. MR2831938 (2012j:13026)
- [12] P. Schenzel, *A note on the Matlis dual of a certain injective hull*, J. Pure Appl. Algebra, **219**, no 3, (2015), 666–671.
- [13] Charles A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994. MR1269324 (95f:18001)
- [14] Thomas Zink, *Endlichkeitsbedingungen für Moduln über einem Noetherschen Ring* (German), Math. Nachr. **64** (1974), 239–252. MR0364223 (51 #478)

INSTITUT FÜR INFORMATIK, MARTIN-LUTHER-UNIVERSITÄT HALLE-WITTENBERG, D — 06 099 HALLE (SAALE), GERMANY

E-mail address: peter.schenzel@informatik.uni-halle.de