SPLITTING OF 3-MANIFOLDS AND RIGIDITY OF AREA-MINIMISING SURFACES

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ABSTRACT. In this paper we modify an argument of Bray, Brendle and Neves to prove an area comparison result (Theorem 2) for certain totally geodesic surfaces in 3-manifolds with a lower bound on the scalar curvature.

This theorem is a variant of a comparison theorem (Theorem 3.2 (d) in the 1978 paper) of Heintze and Karcher for minimal hypersurfaces in manifolds of nonnegative Ricci curvature. Our assumptions on the ambient manifold are weaker, but the assumptions on the surface are considerably more restrictive.

We then use our comparison theorem to provide a unified proof of various splitting theorems for 3-manifolds with lower bounds on the scalar curvature that were first proved by Cai and Galloway; Bray, Brendle and Neves; and Nunes.

1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

In [14], Corollary 3.6.1, Simons observed that there are no closed, stable, minimal, 2-sided hypersurfaces in a manifold of positive Ricci curvature. This is a variant of the classical Synge lemma. An easy, but unstated, extension of Simons's observation is that a closed, stable, minimal, 2-sided hypersurface Σ in a manifold M of nonnegative Ricci curvature is necessarily totally geodesic and the normal Ricci curvature of M must vanish all along Σ . The simplest such example is $\Sigma \times (-\varepsilon, \varepsilon)$, $\varepsilon > 0$, with the product metric $g + dt^2$, where the metric g on Σ has nonnegative Ricci curvature. However, the existence of a closed, stable, minimal, 2-sided hypersurface Σ in a manifold M of nonnegative Ricci curvature does not imply that the metric of M near Σ must split as $g + dt^2$. Indeed, the metric $(1 - t^4)g + dt^2$ on $\Sigma \times (-\varepsilon, \varepsilon)$, $1 > \varepsilon > 0$, has nonnegative Ricci curvature if g has nonnegative Ricci curvature and $\Sigma \times \{0\}$ is stable.¹ However, note that in this example, $\Sigma \times \{0\}$ does not minimise area, not even locally.

So, one might surmise whether the existence of a closed, *area-minimising*, 2-sided hypersurface Σ in a manifold M of nonnegative Ricci curvature implies that the metric of M near Σ must split as $g + dt^2$. This is indeed the case and follows from a special case of Theorem 3.2 (d) in [10], where Heintze and Karcher prove that the exponential map of the normal bundle $\Sigma \times \mathbb{R}$ of Σ in M is volume nonincreasing when M has nonnegative Ricci curvature, Σ is two-sided and minimal in M and $\Sigma \times \mathbb{R}$ is equipped with the product metric. Anderson extended this result to area-minimising integral currents of codimension 1 in a compact manifold of nonnegative

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¹If g is the round metric on the unit sphere S^n , then it is easy to embed this example, for ε sufficiently small, into a metric on S^{n+1} of nonnegative sectional curvature.

Ricci curvature; see Theorem 3 in [1]. Anderson then used this theorem, together with an existence result for area-minimising hypersurfaces, to obtain a different proof of the splitting theorem in [6] of Cheeger and Gromoll in the compact case; see Corollary 3 in [1].

It is easy to construct examples of totally geodesic, area-minimising hypersurfaces in manifolds of positive scalar curvature. Therefore, positive Ricci curvature in Simons's result cannot be relaxed to positive scalar curvature. Nevertheless, in the well-known paper [12], Schoen and Yau obtained topological restrictions on an oriented, two-sided, stable, minimal surface Σ in a 3-manifold M whose scalar curvature R is positive. In particular, they proved that the genus of Σ must be zero. Soon after, Fischer-Colbrie and Schoen studied the case $R \geq 0$ and proved in [7] that, in this case, the genus of Σ must be zero or one, and if it is one, then Σ is totally geodesic and flat and both the normal Ricci curvature of M and R vanish all along Σ .

A closer look at the proof of Schoen-Yau reveals that a lower bound on the scalar curvature R of the 3-manifold M provides a bound on the area of the stable minimal surface Σ . More precisely,

- (1) if $R \geq 2$, then the area of Σ is bounded above by 4π ; and
- (2) if $R \geq -2$, then the area of Σ is bounded below by $4\pi(\gamma - 1)$, where γ is the genus of Σ .

To our knowledge, these bounds first appeared in [13]. (These bounds also appear in the literature of general relativity. See, for example, [8].) Easy examples show that no area bounds are possible for stable minimal tori in flat three-dimensional tori. If the area bound in (1) or (2) is attained and $R_0 := \min_{x \in M} R(x)$, then an analysis similar to that used by Fischer-Colbrie and Schoen in the case of genus one yields that

- (i) the surface is totally geodesic,
- (ii) R is equal to R_0 all along Σ and
- (iii) the normal Ricci curvature of M vanishes all along Σ .

It follows that

- (iv) the Gauss curvature K of Σ is identically equal to $\frac{1}{2}R_0$.

The obvious examples in which the area bound (1) or (2) is attained are provided by $\Sigma \times (-\varepsilon, \varepsilon)$, $\varepsilon > 0$, with the product metric $g + dt^2$, where g has constant Gauss curvature equal to $\frac{1}{2}R_0$. A natural question is whether these are the only examples in which equality is attained in (1) and (2). As in the case of nonnegative Ricci curvature mentioned earlier, this does not hold. Indeed, the scalar curvature R of the metric $(1 - t^4)g + dt^2$ satisfies $R \geq R_0$ and $\Sigma \times \{0\}$ is stable but, once again, does not minimise area, not even locally.

So again, one surmises whether the metric near an *area-minimising* closed surface Σ which satisfies (i), (ii) and (iii), in a 3-manifold M whose scalar curvature R is greater than or equal to R_0 , splits. The examples of totally geodesic, area-minimising hypersurfaces in manifolds of positive scalar curvature that come easily to mind do not satisfy condition (ii).

To make use of the area-minimising property, one has to perturb Σ in a way which decreases its area. When the Ricci curvature of M is nonnegative, Heintze and Karcher (and, to second order, Simons) showed that this is achieved by surfaces that are equidistant from Σ . However, in our case we do not have information on the

Ricci tensor away from Σ , and so this is not a suitable perturbation. We shall see that the right thing to do is to move Σ so that it still has constant mean curvature. This turns out to be possible by Proposition 3.2 in [3], which essentially asserts the existence of a one-parameter family of constant mean curvature surfaces in a neighborhood of a surface Σ which satisfies (i) and (iii); see also [2]. A detailed proof of the following statement, based on the implicit function theorem, can be found in [11]. See also [2] and [4].

Proposition 1. *Let ν be a unit normal field on an oriented, two-sided surface Σ immersed in a 3-manifold M . If Σ satisfies (i) and (iii), then there exists $\varepsilon > 0$ and a smooth function $w: \Sigma \times (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}$ such that, for all $t \in (-\varepsilon, \varepsilon)$, the surfaces*

$$\Sigma_t := \{\exp_x(w(x, t)\nu(x)) : x \in \Sigma\}$$

have constant mean curvature $H(t)$. Moreover, we have

$$w(x, 0) = 0, \quad \left. \frac{\partial}{\partial t} w(x, t) \right|_{t=0} = 1, \quad \text{and} \quad \int_{\Sigma} (w(\cdot, t) - t) dA = 0,$$

for all $x \in \Sigma$ and $t \in (-\varepsilon, \varepsilon)$.

We shall prove the following theorem, which, in light of the preceding discussion, is the optimal analogue of the Heintze-Karcher Theorem 3.2(d) in [10] in the context of 3-manifolds with lower bounds on scalar curvature. The proof relies heavily on the Gauss-Bonnet Theorem, and so it is not at all clear how it may be generalised to dimension 4 or higher.

Theorem 2. *Let M be a 3-manifold and let $\Sigma \subset M$ be an immersed, 2-sided, closed surface. Denote by R the scalar curvature of M , and let $R_0 := \min_{x \in M} R(x)$. Suppose that Σ has the following properties:*

- (i) Σ is totally geodesic,
- (ii) R is equal to R_0 all along Σ and
- (iii) the normal Ricci curvature of M vanishes all along Σ .

Let Σ_t and ε be as in Proposition 1 and denote by $A(t)$ the area of Σ_t . (In particular, $A(0)$ is the area of Σ .) Then there exists $0 < \delta < \varepsilon$ such that

$$\text{for } |t| < \delta, \quad A(t) \leq A(0).$$

Moreover, Σ has constant Gauss curvature equal to $\frac{1}{2}R_0$, and therefore, by Gauss-Bonnet, $A(0) = \frac{8\pi}{|R_0|}|\gamma - 1|$ if R_0 is nonzero.

Theorem 2 is the key ingredient that we use to provide a unified and more elementary proof of the following splitting and rigidity theorem (Theorem 3), the three cases of which were separately proved, using different techniques, in [3], [5] and [11]. Indeed, our research for Theorem 2 was sparked by a lecture that A. Neves gave at the University of Warwick in which he presented the results in [3]. An interesting connection with the Yamabe problem is made in [11], which appeared on the arXiv preprint website while the present paper was being written.

Theorem 3. *Let (M, g) be a complete Riemannian 3-manifold with scalar curvature R and let $R_0 := \min_{x \in M} R(x)$. Assume that M contains a closed, embedded, oriented, two-sided, area-minimizing surface Σ .*

- (1) Suppose that $R_0 = 2$ and that the area of Σ is equal to 4π . Then Σ has genus zero and it has a neighbourhood which is isometric to the product $g_1 + dt^2$ on $S^2 \times (-\delta, \delta)$, where g_1 is the metric on the Euclidean two-sphere of radius 1.
- (2) Suppose that $R_0 = 0$ and that Σ has genus one. Then Σ has a neighbourhood which is flat and isometric to the product $g_0 + dt^2$ on $T^2 \times (-\delta, \delta)$, where g_0 is a flat metric on the torus T^2 .
- (3) Suppose that $R_0 = -2$ and that Σ has genus $\gamma \geq 2$ and area equal to $4\pi(\gamma - 1)$. Then Σ has a neighbourhood which is isometric to the product $g_{-1} + dt^2$ on $\Sigma \times (-\delta, \delta)$, where g_{-1} is a metric of constant Gauss curvature equal to -1 on Σ .

2. THE PROOFS

Proof of Theorem 2. With the notation as in Proposition 1, let

$$f_t(x) := \exp_x(w(x, t)\nu(x)), \quad x \in \Sigma, \quad t \in (-\varepsilon, \varepsilon).$$

Thus, $f_0 =: f$ is the given totally geodesic embedding. The lapse function $\rho_t: \Sigma \rightarrow \mathbb{R}$ is defined by

$$(1) \quad \rho_t(x) := \langle \nu_t(x), \frac{\partial}{\partial t} f_t(x) \rangle,$$

where ν_t is a unit normal to Σ_t , chosen so as to be continuous in t . It satisfies the following Jacobi equation (cf. [9], eq. (1.2)):

$$(2) \quad H'(t) = -\Delta_{\Sigma_t} \rho_t - (\text{Ric}(\nu_t, \nu_t) + \|B_t\|^2) \rho_t,$$

where B_t is the second fundamental form of Σ_t and $(\cdot)' := \frac{\partial}{\partial t}(\cdot)$. Since $\rho_0(x) = 1$ for all $x \in \Sigma$, we can assume, by decreasing ε if necessary, that $\rho_t(x) > 0$ for all $x \in \Sigma$. So we can divide (2) by ρ_t and, using the Gauss equation

$$(3) \quad \text{Ric}(\nu, \nu) = \frac{1}{2}R - K + \frac{1}{2}H^2 - \frac{1}{2}\|B\|^2,$$

(2) becomes

$$(4) \quad H'(t) \frac{1}{\rho_t} = -\frac{1}{\rho_t} \Delta_{\Sigma_t} \rho_t - \frac{1}{2}R_t + K_t - \frac{1}{2}H(t)^2 - \frac{1}{2}\|B_t\|^2,$$

where $R_t(x) := R(f_t(x))$ and $K_t(x) := K(f_t(x))$ is the Gauss curvature of Σ_t at $f_t(x)$. We now make essential use of the hypotheses (i), (ii) and (iii), which, via the Gauss equation (3), imply that

$$(5) \quad K_0 \equiv \frac{1}{2}R_0 \leq \frac{1}{2}R_t(x) \quad \forall x \in \Sigma, \quad t \in (-\varepsilon, \varepsilon),$$

and therefore (4) can be rewritten as

$$(6) \quad \begin{aligned} H'(t) \frac{1}{\rho_t} &= -\frac{1}{\rho_t} \Delta_{\Sigma_t} \rho_t + \frac{1}{2}(R_0 - R_t) + (K_t - K_0) - \frac{1}{2}H(t)^2 - \frac{1}{2}\|B_t\|^2 \\ &\leq -\frac{1}{\rho_t} \Delta_{\Sigma_t} \rho_t + (K_t - K_0). \end{aligned}$$

We integrate (6) over Σ_t (by parts in the first term on the right), and, keeping in mind that H does not depend on x (hence the importance of using a constant mean

curvature perturbation), we obtain

$$(7) \quad \begin{aligned} H'(t) \int_{\Sigma} \frac{1}{\rho_t} dA_t &\leq - \int_{\Sigma} \frac{\|\nabla_t \rho_t\|^2}{\rho_t^2} dA_t + \int_{\Sigma} (K_t - K_0) dA_t \\ &\leq 4\pi(1 - \gamma) - K_0 A(t), \end{aligned}$$

where dA_t denotes the area element of Σ_t with respect to f_t^*g and we have used the Gauss-Bonnet Theorem in the last inequality.

Claim 1. There exists a positive real number $\delta < \varepsilon$ such that $H(t) \leq 0$ for all $t \in [0, \delta)$.

Proof of Claim 1. There are three cases to consider.

Case 1. $R_0 > 0$. By scaling, we can arrange $R_0 = 2$. Then by (5), we have that $K_0 \equiv 1$, and Σ has genus zero and $A(0) = 4\pi$. Therefore inequality (7) becomes

$$(8) \quad \begin{aligned} H'(t) \int_{\Sigma} \frac{1}{\rho_t} dA_t &\leq 4\pi - A(t) = A(0) - A(t) = - \int_0^t A'(s) ds \\ &= - \int_0^t \{H(s) \int_{\Sigma} \rho_s dA_s\} ds, \end{aligned}$$

where, in the last equality, we have used the first variation of area formula,

$$(9) \quad A'(t) = \int_{\Sigma} H(t) \rho_t dA_t = H(t) \int_{\Sigma} \rho_t dA_t.$$

Let $\phi(t) := \int_{\Sigma} \frac{1}{\rho_t} dA_t$ and $\xi(t) := \int_{\Sigma} \rho_t dA_t$. Since ϕ is strictly positive for all $t \in (-\varepsilon, \varepsilon)$, inequality (8) becomes

$$(10) \quad H'(t) \leq - \frac{1}{\phi(t)} \int_0^t H(s) \xi(s) ds.$$

As mentioned above, $\rho_0 \equiv 1$ and, by continuity, we may assume that $\frac{1}{2} < \rho_t(x) < 2 \forall t \in (-\varepsilon, \varepsilon)$ and $x \in \Sigma$. Integrating over Σ_t yields $\frac{1}{2}A(t) < \xi(t) < 2A(t)$. On the other hand, by choosing $\varepsilon > 0$ small enough, we may assume that $\frac{1}{2}A(0) < A(t) < 2A(0)$, and hence $\frac{1}{4}A(0) < \xi(t) < 4A(0) \forall t \in (-\varepsilon, \varepsilon)$. A similar argument holds for $\phi(t)$. In particular, we have

$$(11) \quad \frac{1}{\phi(t)} < \frac{4}{A(0)} \quad \text{and} \quad \xi(t) < 4A(0), \quad \forall t \in (-\varepsilon, \varepsilon).$$

Suppose, for a contradiction, that there exists $t_+ \in (0, \delta)$ such that $H(t_+) > 0$. By continuity, $\exists t_- \in [0, t_+)$ such that $H(t_-) \leq H(t) \forall t \in [0, t_+]$. Note that by (10) we must have $H(t_-) < 0$. By the mean value theorem, $\exists t_1 \in (t_-, t_+)$ such that

$$H'(t_1) = \frac{H(t_+) - H(t_-)}{t_+ - t_-}.$$

So, by (10) and (11), we have

$$\frac{H(t_+) - H(t_-)}{t_+ - t_-} = H'(t_1) \leq - \frac{4}{A(0)} H(t_-) (4A(0)) t_1.$$

It follows that

$$H(t_+) \leq H(t_-)(1 - 16\delta^2),$$

which is a contradiction if $0 < \delta < \frac{1}{4}$ because $H(t_+) > 0$ and $H(t_-) < 0$.

Case 2. $R_0 = 0$. By (5), we have that $K_0 \equiv 0$ and Σ has genus one. So, inequality (7) becomes $H'(t) \leq 0 \ \forall t \in [0, \varepsilon)$, and therefore, since $H(0) = 0$, $H(t) \leq 0 \ \forall t \in [0, \varepsilon)$.

Case 3. $R_0 < 0$. By scaling, we can arrange $R_0 = -2$. Then by (5), we have that $K_0 \equiv -1$, and Σ has genus $\gamma > 1$ and $A(0) = 4\pi(\gamma - 1)$. Therefore inequality (7) becomes

$$(12) \quad \begin{aligned} H'(t) \int_{\Sigma} \frac{1}{\rho_t} dA_t &\leq A(t) - A(0) = \int_0^t A'(s) ds \\ &= \int_0^t \{H(s) \int_{\Sigma} \rho_s dA_s\} ds. \end{aligned}$$

Assume, for a contradiction, that there exists $t_0 \in (0, \delta)$ such that $H(t_0) > 0$. Let

$$I := \{t \in [0, t_0] : H(t) \geq H(t_0)\}.$$

Claim 2. $\inf I = 0$.

Proof of Claim 2. Let $t^* := \inf I$ and assume, for a contradiction, that $t^* > 0$. By the mean value theorem, $\exists t_1 \in (0, t^*)$ such that

$$(13) \quad H(t^*) = H'(t_1)t^*,$$

since $H(0) = 0$. From (12), (11) and (13) we have

$$(14) \quad \begin{aligned} H(t^*) &\leq \frac{t^*}{\phi(t_1)} \int_0^{t_1} H(s)\xi(s) ds \\ &\leq \frac{t^*}{\phi(t_1)} \int_0^{t_1} H(t^*)\xi(s) ds \leq \frac{4t^*}{A(0)} H(t^*)(4A(0)t_1) \end{aligned}$$

$$(15) \quad < 16H(t^*)\delta^2,$$

which is a contradiction if $\delta < \frac{1}{4}$, and Claim 2 has been proved. $\square$

Since $\inf I = 0$, it follows from the definition of I that $H(0) \geq H(t_0)$, and since, by assumption, $H(t_0) > 0$, we conclude that $H(0) > 0$. This contradicts the hypothesis that Σ is totally geodesic, and the proof of Claim 1 is complete. $\square$

We can now easily complete the proof of Theorem 2. We have that $H(t) \leq 0 \ \forall t \in [0, \delta)$, and therefore (9) implies that $A'(t) \leq 0$. Hence $A(t) \leq A(0) \ \forall t \in [0, \delta)$. We can argue similarly for $t \in (-\delta, 0]$ to complete the proof of Theorem 2. $\square$

The proof of Theorem 3 now follows easily by a slight variation of arguments that appear in [3] and [11].

Proof of Theorem 3. The conclusion of Theorem 2 and the assumption that Σ is area-minimising imply that, for the CMC family of surfaces Σ_t provided by Proposition 1, $A(t) = A(0) \ \forall t \in (-\delta, \delta)$. In particular, each Σ_t is area-minimising and, if $\gamma \neq 1$, the area of each Σ_t is equal to $4\pi|\gamma - 1|$. It follows from (i) in the Introduction that each Σ_t is totally geodesic. This holds when $\gamma = 1$ as well. Equation (2) then tells us that the lapse function ρ_t is harmonic and therefore is constant on Σ_t ; i.e. ρ_t is a function of t only.

Claim 3. The vector field ν_t is parallel.

Proof of Claim 3. Σ_t is totally geodesic, and therefore $\nabla_{\frac{\partial f_t}{\partial x^i}} \nu_t = 0$.

$$\begin{aligned} 0 &= \frac{\partial}{\partial x^i} \rho_t = \langle \nabla_{\frac{\partial f_t}{\partial x^i}} \nu_t, \frac{\partial f_t}{\partial t} \rangle + \langle \nu_t, \nabla_{\frac{\partial f_t}{\partial x^i}} \frac{\partial f_t}{\partial t} \rangle \\ &= \langle \nu_t, \nabla_{\frac{\partial f_t}{\partial x^i}} \frac{\partial f_t}{\partial t} \rangle \quad (\Sigma_t \text{ is totally geodesic}) \\ &= \frac{\partial}{\partial t} \langle \nu_t, \frac{\partial f_t}{\partial x^i} \rangle - \langle \nabla_{\frac{\partial f_t}{\partial t}} \nu_t, \frac{\partial f_t}{\partial x^i} \rangle \\ &= -\langle \nabla_{\frac{\partial f_t}{\partial t}} \nu_t, \frac{\partial f_t}{\partial x^i} \rangle. \end{aligned}$$

Hence $\nabla_{\frac{\partial f_t}{\partial t}} \nu_t = 0$. This, together with the fact that Σ_t is totally geodesic, implies that the vector field ν_t is parallel. $\square$

It follows that the integral curves of ν_t are geodesics and that the flow Φ of ν_t is just the exponential map, i.e. $\Phi(t, x) = \exp_x(t\nu(x)) \forall x \in \Sigma$. Furthermore, since ν_t is, in particular, a Killing field, this exponential map $\exp_{(\cdot)}(t\nu(\cdot))$ is an isometry for all $t \in (-\delta, \delta)$. In other words, if g_Σ is the restriction of g to Σ , then the exponential map of the δ -neighbourhood $\Sigma \times (-\delta, \delta)$ of the zero section of the normal bundle of Σ in M with the metric $g_\Sigma + dt^2$ is an isometry onto its image. $\square$

Remark 4. It is straightforward to show that $w(x, t) \equiv t$ and that $\rho_t \equiv 1$. Let $S_t := \{\exp_x(t\nu(x)) : x \in \Sigma\}$. Pick $t_0 \in (-\delta, \delta)$ and $x_0 \in \Sigma$ and set $w_0 := w(x_0, t_0)$. Then Σ_{t_0} and S_{w_0} are both totally geodesic and touch at $\exp_{x_0}(t_0\nu(x_0))$. Therefore, they coincide. In particular, w is a function of t only, and, since by Proposition 1, $\int_\Sigma (w(\cdot, t) - t) dA = 0$, we have $w(x, t) \equiv t$. It follows that $\nu_t(x) = \frac{\partial}{\partial t} f_t(x)$ and that $\rho_t \equiv 1$.

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