

LOCAL EXISTENCE AND UNIQUENESS FOR EXTERIOR STATIC VACUUM EINSTEIN METRICS

MICHAEL T. ANDERSON

(Communicated by Lei Ni)

ABSTRACT. We study solutions to the static vacuum Einstein equations on domains of the form $M \simeq \mathbb{R}^3 \setminus B$ with prescribed Bartnik data (γ, H) on the inner boundary ∂M . It is proved that for any smooth boundary data (γ, H) close to standard round data on the unit sphere $(\gamma_{+1}, 2)$, there exists a unique asymptotically flat solution of the static vacuum Einstein equations realizing the boundary data (γ, H) which is close to the standard flat solution.

1. INTRODUCTION

In this short note, we prove a local existence and uniqueness theorem for solutions of the static vacuum Einstein equations, motivated by a well-known conjecture of Bartnik in general relativity. To describe the result and its setting, recall that the static vacuum Einstein equations on an n -dimensional manifold M are given by

$$(1.1) \quad uRic = D^2u, \quad \Delta u = 0,$$

where g is a Riemannian metric on M and $u : M \rightarrow \mathbb{R}^+$ is a positive (or non-negative) potential function. Here also Ric is the Ricci curvature of g , D^2 is the Hessian, and $\Delta = tr D^2$ is the Laplacian with respect to g . These equations are equivalent to the statement that the manifold $\mathcal{M} = S^1 \times M$, with metric

$$g_{\mathcal{M}} = u^2 d\theta^2 + g,$$

is Ricci-flat, i.e. $Ric_{g_{\mathcal{M}}} = 0$, (the vacuum Einstein equations). In this work, we will only consider the situation where M is diffeomorphic to the exterior of a ball $B \subset \mathbb{R}^n$, so $M \simeq \mathbb{R}^n \setminus B$. Moreover, we consider only solutions which are asymptotically flat (AF), in the sense that the metric g and potential function u satisfy

$$(1.2) \quad g - g_{Eucl} = 1 + O(r^{n-2}), \quad u = 1 + O(r^{n-2}).$$

These conditions imply that the metric and potential in fact have an asymptotic expansion near infinity; cf. [6]. However, this will not be relevant here.

Let $\mathbb{E}^{m,\alpha}$ be the space of all AF static vacuum solutions (g, u) on M which are $C^{m,\alpha}$ smooth up to the boundary ∂M . We assume $m \geq 2$ and $\alpha \in (0, 1)$. This space is given the $C^{m,\alpha}$ topology with weight function $r^{n-2-\varepsilon}$. Let $\text{Diff}_1^{m+1,\alpha}$ be the group of AF diffeomorphisms of M which equal the identity on ∂M . This serves as the “gauge” group of the problem and the quotient space

$$(1.3) \quad \mathcal{E}^{m,\alpha} = \mathbb{E}^{m,\alpha} / \text{Diff}_1^{m+1,\alpha}$$

Received by the editors August 16, 2013 and, in revised form, February 14, 2014.

2010 *Mathematics Subject Classification.* Primary 83C20, 58D29, 58J32.

This work was partially supported by NSF grant DMS 1205947.

is the corresponding moduli space of AF static vacuum solutions. It is proved in [1] (cf. also [3]) that $\mathcal{E}^{m,\alpha}$ is a smooth infinite-dimensional Banach manifold; in particular the group $\text{Diff}_1^{m+1,\alpha}$ acts freely on $\mathbb{E}^{m,\alpha}$.

With each $(g, u) \in \mathbb{E}^{m,\alpha}$ one may associate its Bartnik boundary data (γ, H) , where $\gamma = g_{\partial M}$ is the induced metric on ∂M and H is the mean curvature with respect to the inward unit normal N into M . Here the sign is chosen so that $H = n - 1$ for the round unit sphere $S^{n-1}(1) \subset \mathbb{R}^n$. This choice of boundary data was introduced by Bartnik [4] in connection with his definition of quasi-local mass in general relativity. Note that the boundary data are invariant under the gauge group $\text{Diff}_1^{m+1,\alpha}$.

One may thus define a Bartnik boundary map as

$$(1.4) \quad \begin{aligned} \Pi_B : \mathcal{E}^{m,\alpha} &\rightarrow \text{Met}^{m,\alpha}(\partial M) \times C^{m-1,\alpha}(\partial M), \\ \Pi_B(g, u) &= (\gamma, H). \end{aligned}$$

This is a map of Banach manifolds, and again it is proved in [1], [3] that Π_B is a smooth Fredholm map, of Fredholm index 0. This means that, at each $(g, u) \in \mathcal{E}^{m,\alpha}$, the derivative or linearization

$$(1.5) \quad \begin{aligned} D\Pi_B : T\mathcal{E}^{m,\alpha} &\rightarrow T(\text{Met}^{m,\alpha})(\partial M) \times C^{m-1,\alpha}(\partial M), \\ D\Pi_B(h, u') &= (h^T, H'_h), \end{aligned}$$

is a Fredholm linear map, i.e. has finite-dimensional kernel and cokernel, with $\dim \text{Ker}(D\Pi_B) = \dim \text{Coker}(D\Pi_B)$. Here h is a symmetric bilinear form, h^T is the restriction of h to $T(\partial M)$, and H'_h is the variation of the mean curvature H in the direction of the deformation h . The pair (h, u') satisfy the linearization of the static vacuum equations (1.1) and so give an infinitesimal Einstein deformation.

The pure gauge infinitesimal deformations are of the form $\kappa = (\mathcal{L}_Z g, Z(u))$, where Z is a $C^{m+1,\alpha}$ vector field on M with $Z = 0$ on ∂M . These are of course “modded out” so don’t appear in the finite-dimensional kernel $K = \text{Ker}(D\Pi_B)$.

A fundamental conjecture of Bartnik [4], [5], of importance in general relativity, is that in dimension $n = 3$ the map Π_B in (1.4) is a global diffeomorphism, i.e. one has global existence and uniqueness for this nonlinear boundary value problem, at least when the mean curvature $H > 0$. As discussed in [3], this is not true in general and it remains a very interesting problem to understand the existence and uniqueness of solutions to the boundary value problem, i.e. the injectivity and range of the map Π_B . This issue is also closely related to a deeper understanding of the Bartnik quasi-local mass; cf. [4], [5].

Note that when the potential function satisfies $u = 1$ on M , then M is necessarily flat, so that in the context above, $(M, g) \subset (\mathbb{R}^n, g_{\text{Eucl}})$ isometrically. We will call the solution with $u = 1$, $g = g_{\text{Eucl}}$ and $M = \mathbb{R}^n \setminus B^n(1)$, the *standard exterior solution*; thus M is the exterior of the round unit ball in \mathbb{R}^n (up to isometry). The boundary data of the standard exterior solution are given by (γ, H) with $\gamma = \gamma_{+1} = \gamma_{S^{n-1}(1)}$, and $H = n - 1$.

Theorem 1.1. *The standard exterior of the round unit ball $M = \mathbb{R}^3 \setminus B^3(1)$, $g = g_{\text{Eucl}}$, $u = 1$ is a regular point of the boundary map Π_B , so that $D\Pi_B$ is an isomorphism. Thus*

$$(1.6) \quad K = \text{Ker} D\Pi_B = 0,$$

and $D\Pi_B$ is surjective at the standard exterior solution on \mathbb{R}^3 .

In particular, if $\kappa = (k, u') \in T\mathbb{E}^{m,\alpha}$ is tangent to the standard exterior solution and $k^T = H'_k = 0$, then $k = \delta^* Z$ with $Z = 0$ on $\partial M = S^n(1)$.

A standard application of the inverse function theorem in Banach spaces gives the following:

Corollary 1.2. *There is a neighborhood $\mathcal{U} \subset \text{Met}^{m,\alpha}(S^2) \times C^{m,\alpha}(S^2)$ of the standard flat boundary data $(\gamma_{+1}, 2)$ such that for any $(\gamma, H) \in \mathcal{U}$, there is a unique solution (M, g, u) , $M = \mathbb{R}^3 \setminus B^3(1)$, of the static vacuum Einstein equations (1.1), up to isometry in $\text{Diff}_1^{m+1,\alpha}$, for which*

$$\Pi_B(g, u) = (\gamma, H).$$

Corollary 1.2 generalizes a result of Miao [9], who proved this result for boundary data and static vacuum solutions which are invariant under reflection in the standard coordinate planes in \mathbb{R}^3 , i.e. $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ invariant. Note that this condition does not allow one to imply that the standard exterior solution is a regular point of Π_B . This is discussed further in Remark 2.2. Theorem 1.1 gives the first example of a regular point of Π_B , with the exception of solutions near to the exterior of the horizon of the Schwarzschild metric discussed in [3].

2. PROOFS AND REMARKS

In this section we prove Theorem 1.1 and Corollary 1.2, and make several further remarks. The proofs are very simple, given the setting from the Introduction. To prove Theorem 1.1, it suffices to prove (1.6), since $D\Pi_B$ is Fredholm, of Fredholm index 0.

We work with n arbitrary, although at the end of the proof we will require $n = 3$.

We begin with the Gauss equation or scalar constraint equation for the boundary $\partial M \subset M$. For a general Riemannian metric, this reads

$$|A|^2 - H^2 + R_\gamma = R_g - 2\text{Ric}(N, N),$$

where A is the 2nd fundamental form, N is the unit normal pointing into M and R_γ, R_g are the scalar curvatures of γ and g respectively. For static solutions $R_g = 0$ and $-\text{Ric}(N, N) = -u^{-1}NN(u) = u^{-1}(\Delta_\gamma u + HN(u))$, where the second equality is a standard consequence of the relation $\Delta_M u = 0$ in adapted (Fermi) coordinates about ∂M . (Here Δ_γ is the Laplacian on $(\partial M, \gamma)$). Hence one has

$$(2.1) \quad u(|A|^2 - H^2 + R_\gamma) = 2(\Delta_\gamma u + HN(u)).$$

Consider the linearization of (2.1) in the direction of an infinitesimal static Einstein deformation $\kappa = (k', u') \in K = \text{Ker} D\Pi$ about the standard round solution with $u = 1$. For the standard round boundary data on $S^{n-1}(1) \subset \mathbb{R}^n$, one has $|A|^2 = n - 1$, $H^2 = (n - 1)^2$ and $R_\gamma = (n - 1)(n - 2)$, so that $|A|^2 - H^2 + R_\gamma = 0$. We have $k^T = k|_{\partial M} = 0$ and $A = \gamma$ so that $\text{tr} A'_k = H'_k = 0$. It follows that

$$(2.2) \quad 2(\Delta_\gamma u' + (n - 1)N(u')) = 2\langle A'_k, A \rangle - 2HH'_k = 2H'_k - 2(n - 1)H'_k = 0,$$

on ∂M . Hence,

$$\Delta_\gamma u' + (n - 1)N(u') = 0,$$

on ∂M .

Consider in general the equation

$$(2.3) \quad \Delta_\gamma v + (n - 1)N(v) = 0,$$

on ∂M , where v is harmonic on (M, g_{Eucl}) with $v \rightarrow 0$ at infinity. Note that the variation u' satisfies these conditions, since

$$(2.4) \quad 0 = (\Delta u)' = \Delta' u + \Delta u' = \Delta u',$$

again since $u = 1$ on M . Hence u' is harmonic on M . It is clear from (1.2) that $u' \rightarrow 0$ at infinity in M .

We claim that the only solution to (2.3) is the trivial solution $v = 0$. Thus, suppose first that $\max v > 0$ at some point in M . Then $\max_M v > 0$ occurs at a point $p \in \partial M$ and, by the Hopf maximum principle (cf. [8] for instance) $-N(v(p)) > 0$ where $-N$ is the outward unit normal to M . Hence, $\Delta v(p) \leq 0$ and $N(v(p)) < 0$, contradicting (2.3). Hence, $\max_M v = 0$. The same argument working at a point achieving $\min v$ shows that $\min_M v = 0$ so that $v = 0$, which proves the claim. It follows in particular that $u' = 0$ on M .

By the linearized static vacuum equations, one has then $u' Ric + u Ric' = (D^2)'u + D^2 u'$ which thus gives $Ric'_k = 0$, so that k is an infinitesimal Ricci flat deformation of $(\mathbb{R}^n \setminus B^n(1), g_{Eucl})$ with $k^T = 0$.

Now suppose $n = 3$ so that k is then an infinitesimal flat deformation of $(M, g_{Eucl}) = (\mathbb{R}^3 \setminus B^3(1), g_{Eucl})$. It is well known that the round sphere is infinitesimally isometrically rigid in \mathbb{R}^3 , i.e. the only infinitesimal isometric deformations of $S^2(1)$ in \mathbb{R}^3 are trivial, pure gauge deformations, so that $k = \mathcal{L}_Z g$ for some vector field Z , $Z = 0$ on ∂M . It follows that $K = Ker(D\Pi) = 0$, as claimed. (Note here that Killing field deformations X of $S^2(1)$ satisfy $k = \delta^* X = 0$). \square

To prove Corollary 1.2, the inverse function theorem in Banach spaces implies that Π is a local diffeomorphism near the standard exterior solution. Corollary 1.2 is just a restatement of this fact. \square

Remark 2.1. The proof of Theorem 1.1 and Corollary 1.2 generalizes to arbitrary $n \geq 3$, provided one knows that an infinitesimal Ricci-flat deformation of the standard flat exterior (M, g_{Eucl}) satisfying the boundary conditions $k^T = H'_k = 0$ is necessarily a flat deformation. This is in fact true, and essentially follows from the (exterior) isometry extension theorem in [2]; cf. [2, Propositions 5.2 and 5.3]. However, the proof of the isometry extension result is rather long and uses different methods from those discussed in this brief note. Hence, we leave it to the interested reader to verify this in detail.

Another important class of static vacuum solutions are the Schwarzschild metrics of the form

$$(2.5) \quad g_m = \left(1 - \frac{2m}{r^{n-2}}\right)^{-1} dr^2 + r^2 g_{S^{n-1}(1)},$$

with $r > (2m)^{1/(n-2)}$ for $m \geq 0$ and $r > 0$ for $m \leq 0$, and

$$(2.6) \quad u = \sqrt{1 - \frac{2m}{r^{n-2}}}.$$

Consider the exterior region $M = r^{-1}[1, \infty)$. One then has boundary data

$$\gamma = \gamma_{+1} = g_{S^{n-1}(1)}, \quad \text{and} \quad H = (n-1)\sqrt{1-2m}.$$

The fact that regular points of the smooth map Π_B are open implies that the exterior Schwarzschild domain (M, g_m, u) above is a regular point of Π_B , for m small. We conjecture that in fact all such round exterior Schwarzschild domains

are regular points of Π_B . The proof of Theorem 1.1, however, does not go through in this case, mainly since the relation (2.4) now becomes

$$\Delta u' = -\Delta' u = \langle D^2 u, k \rangle + \langle \beta(k), du \rangle,$$

where $\beta = \delta + \frac{1}{2}dtr$ is the Bianchi operator. It is not clear if the right side can be made to vanish, or have a particular sign, for u of the form (2.6).

It is also unknown if general flat exterior solutions are regular points of Π_B , i.e. $M = \mathbb{R}^n \setminus B$ with B a ball with smooth boundary $\partial B = \partial M$, $g = g_{Eucl}$, $u = 1$. The main difficulty with the method above is handling the term $\langle A'_k, A \rangle$. It would be very interesting to know this one way or the other.

We also conjecture that the standard flat exterior boundary data $(\gamma_{+1}, n-1)$, and similarly standard Schwarzschild exterior boundary data $(\gamma_{+1}, (n-1)\sqrt{1-2m})$ are uniquely realized by the flat and Schwarzschild metrics respectively. This is a global uniqueness conjecture, somewhat analogous to the well-known black hole uniqueness theorem for the Schwarzschild metric, cf. [7]. It does not appear that the elementary methods used above can be adapted to settle this issue.

Remark 2.2. It is of interest to compare the proof of Corollary 1.2 with the main result of Miao in [9] which gives the same result in dimension 3 but with a $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry condition imposed. Loosely speaking, and in physics terminology, Miao works off-shell and with a fixed gauge and boundary conditions, while the approach in this work is on-shell (i.e. on the space of solutions), without any gauge and studying the behavior of the gauge-invariant map to the boundary conditions. For this to work, it is important to know that the on-shell moduli space $\mathcal{E}^{m,\alpha}$ is itself a smooth Banach manifold.

In a little more detail, Miao proves that the linearized Einstein operator in a Bianchi gauge $D\Phi$, together with the boundary conditions (γ, H) imposed, has a surjective linearization at the standard round exterior in \mathbb{R}^3 , when all data satisfy the symmetry condition $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ above.

Note that the operator $T \sim D\Phi$ in [9] is not an isomorphism; the L^2 orthogonal complement of ImT is described in [9, Prop. 2]. In particular, it has non-zero gauge components $\eta, \hat{\eta}$. The proof of Theorem 1.1 above implies that $KerT = 0$ so that the Fredholm operator T has negative Fredholm index. An examination of the form of $(ImT)^\perp$ given in [9, Prop. 2] also shows that $D\Pi_B$ in (1.5) is surjective in dimension 3, [10] so that the two methods of proof are consistent.

ACKNOWLEDGMENT

The author would like to thank Pengzi Miao for interesting discussions related to the topic of this paper.

REFERENCES

- [1] Michael T. Anderson, *On boundary value problems for Einstein metrics*, *Geom. Topol.* **12** (2008), no. 4, 2009–2045, DOI 10.2140/gt.2008.12.2009. MR2431014 (2009f:58035)
- [2] Michael T. Anderson, *Extension of symmetries on Einstein manifolds with boundary*, *Selecta Math. (N.S.)* **16** (2010), no. 3, 343–375, DOI 10.1007/s00029-010-0028-9. MR2734335 (2012a:53073)
- [3] Michael T. Anderson and Marcus A. Khuri, *On the Bartnik extension problem for the static vacuum Einstein equations*, *Classical Quantum Gravity* **30** (2013), no. 12, 125005, 33, DOI 10.1088/0264-9381/30/12/125005. MR3064190

- [4] Robert Bartnik, *New definition of quasilocal mass*, Phys. Rev. Lett. **62** (1989), no. 20, 2346–2348, DOI 10.1103/PhysRevLett.62.2346. MR996396 (90e:83041)
- [5] Robert Bartnik, *Mass and 3-metrics of non-negative scalar curvature*, (Beijing, 2002), Higher Ed. Press, Beijing, 2002, pp. 231–240. MR1957036 (2003k:53034)
- [6] R. Beig and W. Simon, *Proof of a multipole conjecture due to Geroch*, Comm. Math. Phys. **78** (1980/81), no. 1, 75–82. MR597031 (81m:83016)
- [7] Piotr T. Chruściel, Gregory J. Galloway, and Daniel Pollack, *Mathematical general relativity: A sampler*, Bull. Amer. Math. Soc. (N.S.) **47** (2010), no. 4, 567–638, DOI 10.1090/S0273-0979-2010-01304-5. MR2721040 (2011j:53152)
- [8] David Gilbarg and Neil S. Trudinger, *Elliptic partial differential equations of second order*, 2nd ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 224, Springer-Verlag, Berlin, 1983. MR737190 (86c:35035)
- [9] Pengzi Miao, *On existence of static metric extensions in general relativity*, Comm. Math. Phys. **241** (2003), no. 1, 27–46. MR2013750 (2004j:83043)
- [10] P. Miao, *personal communication*.

DEPARTMENT OF MATHEMATICS, STONY BROOK UNIVERSITY, STONY BROOK, NEW YORK 11794-3651

E-mail address: anderson@math.sunysb.edu

URL: <http://www.math.sunysb.edu/~anderson>