EXPLICIT MULTIVARIABLE MEROMORPHIC SERIES

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ABSTRACT. In this paper, we present a constructive approach to multivariable meromorphic functions.

1. INTRODUCTION

For a field K , let $Y_1, \dots, Y_m$ be indeterminates over K where $m \in \mathbb{N}_+ =$ the set of all positive integers. Then for the polynomial and power series rings we have

$$K[Y_1, \dots, Y_m] \subset K[[Y_1, \dots, Y_m]]$$

and for their quotient fields we have

$$K[Y_1, \dots, Y_m] \subset K(Y_1, \dots, Y_m) = \text{the rational function field}$$

and

$$K[[Y_1, \dots, Y_m]] \subset K((Y_1, \dots, Y_m)) = \text{the meromorphic function field.}$$

Hence we get the subfield inclusion

$$K(Y_1, \dots, Y_m) \subset K((Y_1, \dots, Y_m)).$$

In case of $m = 1$, members of $K((Y_1))$ have concrete representations as meromorphic series, i.e., expressions of the form

$$g(Y_1) = \sum_{i \in \mathbb{Z}} b_i Y_1^i \quad \text{with } b_i \in K$$

such that the support $\text{Supp}(g) = \{i \in \mathbb{Z} : b_i \neq 0\}$ has only a finite number of negative integers; see page 69 of [Ab1] or page 39 of [Ab2]. We may ask the **explicitization question**: is there such an explicit representation of members of $K((Y_1, \dots, Y_m))$ for $m > 1$? To move towards an answer, consider the set $K\langle Y_1, \dots, Y_m \rangle$ consisting of all expressions of the form

$$f(Y_1, \dots, Y_m) = \sum_{(i_1, \dots, i_m) \in \mathbb{Z}^m} a_{i_1 \dots i_m} Y_1^{i_1} \dots Y_m^{i_m} \quad \text{with } a_{i_1 \dots i_m} \in K.$$

In an obvious manner $K\langle Y_1, \dots, Y_m \rangle$ can be identified with $K^{\mathbb{Z}^m} =$ the set of all maps $\mathbb{Z}^m \rightarrow K$, and we define the support of f by putting

$$\text{Supp}(f) = \{(i_1, \dots, i_m) \in \mathbb{Z}^m : a_{i_1 \dots i_m} \neq 0\}.$$

In a natural manner we convert $K\langle Y_1, \dots, Y_m \rangle$ into a K -vector-space; in view of the one complex variable theorem on laurent expansion of functions in a punctured

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disc, we call this vector space the **laurent vector space** (in $Y_1, \dots, Y_m$ over K). Identifying $K[[Y_1, \dots, Y_m]]$ with $K^{\mathbb{N}^m}$, where $\mathbb{N}$ = the set of all nonnegative integers, we have a natural monomorphism of K -vector-spaces

$$\phi : K[[Y_1, \dots, Y_m]] \rightarrow K\langle Y_1, \dots, Y_m \rangle$$

with

$$\text{im}(\phi) = \{f \in K\langle Y_1, \dots, Y_m \rangle : \text{Supp}(f) \in \mathbb{N}^m\}.$$

Remark 1.1. Let $\text{lex}(\mathbb{Z}^m)$ denote the ordered abelian group obtained by lexicographically ordering $\mathbb{Z}^m$ whereby: $(i_1, \dots, i_m) < (j_1, \dots, j_m)$ means for some $p \in \{1, \dots, m\}$ we have $i_1 = j_1, \dots, i_{p-1} = j_{p-1}, i_p < j_p$. We put

$$K(\langle Y_1, \dots, Y_m \rangle) = \{f \in K\langle Y_1, \dots, Y_m \rangle : \text{Supp}(f) \text{ is well ordered}\}$$

and we call $K(\langle Y_1, \dots, Y_m \rangle)$ the **lexomorphic field** (in $Y_1, \dots, Y_m$ over K). Now

$$\text{im}(\phi) = \phi(K[[Y_1, \dots, Y_m]]) \subset K(\langle Y_1, \dots, Y_m \rangle)$$

because, in the lexicographic ordering, $\mathbb{N}^m$ is a well ordered subset of $\mathbb{Z}^m$. To see this let T be any nonempty subset of $\mathbb{N}^m$. Let

$$\text{the projection } \pi_j : \mathbb{Z}^m \rightarrow \mathbb{Z} \text{ be given by } i = (i_1, \dots, i_m) \mapsto i_j.$$

Using the fact that $\mathbb{N}$ is well ordered we can inductively define

$$\begin{cases} t_1 = \min\{\pi_1(i) : i \in T\}, \dots, t_j = \min\{\pi_j(i) : i \in T \cap \bigcap_{1 \leq l \leq j-1} \pi_l^{-1}(t_l)\}, \\ \dots, t_m = \min\{\pi_m(i) : i \in T \cap \bigcap_{1 \leq l \leq m-1} \pi_l^{-1}(t_l)\} \end{cases}$$

and then clearly $(t_1, \dots, t_m) = \min(T)$.

Remark 1.2. In Section 2 we shall answer the explicitization question by showing that: $K(\langle Y_1, \dots, Y_m \rangle)$ can be made into an overfield of K and ϕ can be uniquely extended to a ring monomorphism

$$\Phi : K((Y_1, \dots, Y_m)) \rightarrow K(\langle Y_1, \dots, Y_m \rangle)$$

thus expanding all the members of $K((Y_1, \dots, Y_m))$ as multivariable meromorphic series.

Remark 1.3. In Section 3 we shall indicate how, by using Galois theory, it can be shown that, if $m > 1$, then the field $K(\langle Y_1, \dots, Y_m \rangle)$ is much larger than the subfield $\Phi(K((Y_1, \dots, Y_m)))$; the details of the Galois theory proof will be given in another paper. In Section 3 we shall also describe some other aspects of the extensions

$$\Phi(K((Y_1, \dots, Y_m))) \subset K(\langle Y_1, \dots, Y_m \rangle) \subset K\langle Y_1, \dots, Y_m \rangle.$$

In Section 5, by a combinatorial argument we shall show that if $m > 1$, then $K(\langle Y_1, \dots, Y_m \rangle)$ is an infinite dimensional vector space over $\Phi(K((Y_1, \dots, Y_m)))$. In Section 6, by extending this argument we shall show that if $m > 1$, then $K(\langle Y_1, \dots, Y_m \rangle)$ has infinite transcendence degree over $\Phi(K((Y_1, \dots, Y_m)))$.

Remark 1.4. To sketch a proof of the first half of Remark 1.2, given any ordered abelian group G , as on page 41 of [Ab2], for all $A \in K^G$ we define

$$\text{Supp}(A) = \{g \in G : A(g) \neq 0\}$$

and we put

$$K((X))_G = \{A \in K^G : \text{Supp}(A) \text{ is well ordered}\}.$$

We call $K((X))_G$ the **unomorphic field** (in X over K relative to G) and, moreover, for all $A \in K((X))_G$ we define

$$\text{ord}(A) = \begin{cases} \min \text{Supp}(A) & \text{if } A \neq 0, \\ \infty & \text{if } A = 0. \end{cases}$$

We put

$$K[[X]]_G = \{A \in K((X))_G : \text{ord}(A) \geq 0\}$$

and we call $K[[X]]_G$ the **unomorphic ring** (in X over K relative to G). On page 42 of [Ab2] (with more details on pages 674-677 of [Ab2]) it is proved that $K((X))_G$ is indeed a field and $A \mapsto \text{ord}(A)$ gives a valuation $K \rightarrow G \cup \{\infty\}$ whose valuation ring is $K[[X]]_G$. Generalizing what we have said on page 42 of [Ab2], concretely speaking, a typical member of K^G can be written as

$$A(X) = \sum_{i \in G} A_i X^i$$

where we are writing A_i for the previous $A(i)$. The K -vector-space K^G may be denoted by $K\langle X \rangle_G$ and called the **laurent vector space** (in X over K relative to G). Clearly

$$f \mapsto A(X) = \sum_{i=(i_1, \dots, i_m) \in \mathbb{Z}^m} A_i X^i \quad \text{with } A_i = a_{i_1, \dots, i_m}$$

gives a K -vector-space bijection

$$\theta : K\langle Y_1, \dots, Y_m \rangle \rightarrow K\langle X \rangle_{\text{lex}(\mathbb{Z}^m)}$$

and remembering that $K(\langle Y_1, \dots, Y_m \rangle) \subset K\langle Y_1, \dots, Y_m \rangle$ we have

$$\theta(K(\langle Y_1, \dots, Y_m \rangle)) = K((X))_{\text{lex}(\mathbb{Z}^m)}.$$

This takes care of the first part of Remark 1.2 saying that $K(\langle Y_1, \dots, Y_m \rangle)$ can be made into an overfield of K . We put

$$K[\langle Y_1, \dots, Y_m \rangle] = \theta^{-1}(K[[X]]_{\text{lex}(\mathbb{Z}^m)})$$

and we call this the **lexomorphic ring** (in $Y_1, \dots, Y_m$ over K). In connection with Remarks 1.2 and 1.3, in Section 3 we shall show that

$$(\bullet) \quad \Phi(K[[Y_1, \dots, Y_m]]) \subset K[\langle Y_1, \dots, Y_m \rangle].$$

Remark 1.5. Here is a **paradox**. By a “crude” calculation based on the geometric series identity $(1 - Z)(\sum_{n=0}^{\infty} Z^n) = 1$ we surmise that

$$\frac{1}{Y_1 + Y_2} = \frac{1}{Y_1} \sum_{n=0}^{\infty} \left(\frac{-Y_2}{Y_1} \right)^n = \sum_{n=0}^{\infty} (-1)^n Y_1^{-n-1} Y_2^n$$

and

$$\frac{1}{Y_1 + Y_2} = \frac{1}{Y_2} \sum_{n=0}^{\infty} \left(\frac{-Y_1}{Y_2} \right)^n = \sum_{n=0}^{\infty} (-1)^n Y_1^n Y_2^{-n-1}.$$

Hence by equating the two extreme RHSs we obtain

$$\sum_{n=0}^{\infty} (-1)^n Y_1^{-n-1} Y_2^n = \sum_{n=0}^{\infty} (-1)^n Y_1^n Y_2^{-n-1}$$

which is absurd.

More generally, for any positive integer $q \leq m$, a “crude” calculation based on the geometric series indicates that

$$\frac{1}{Y_1 + \cdots + Y_m} = H_q(Y_1, \dots, Y_m) \in K\langle Y_1, \dots, Y_m \rangle$$

where

$$H_q(Y_1, \dots, Y_m) = \sum_{n=0}^{\infty} (-1)^n Y_q^{-n-1} (Y_1 + \cdots + Y_{q-1} + Y_{q+1} + \cdots + Y_m)^n.$$

Therefore for all $q \neq q'$ in $\{1, \dots, m\}$ we have $H_q(Y_1, \dots, Y_m) = H_{q'}(Y_1, \dots, Y_m)$ which (in case of $m > 1$) is absurd because for all $i = (i_1, \dots, i_m) \in \text{Supp}(H_q)$ we obviously have $i_j \in \mathbb{N} \Leftrightarrow j \neq q$.

In Section 4, resolution of this paradox will lead us to interesting automorphisms of $K\langle Y_1, \dots, Y_m \rangle$ as a K -vector-space.

2. EXPANSIONS OF MULTIVARIABLE MEROMORPHIC FUNCTIONS

At the end of Remark 1.4 we have shown that $\theta(K(\langle Y_1, \dots, Y_m \rangle)) = K((X))_{\text{lex}(\mathbb{Z}^m)}$ and hence $K(\langle Y_1, \dots, Y_m \rangle)$ can be made into an overfield of K . By Remark 1.1 we have $\phi(K[[Y_1, \dots, Y_m]]) \subset K(\langle Y_1, \dots, Y_m \rangle)$, and by page 42 of [Ab2] multiplication in $K((X))_{\text{lex}(\mathbb{Z}^m)}$ is defined by the “Cauchy rule”. Consequently, ϕ induces a ring monomorphism $K[[Y_1, \dots, Y_m]] \rightarrow K(\langle Y_1, \dots, Y_m \rangle)$. Since $K(\langle Y_1, \dots, Y_m \rangle)$ is field, it follows that ϕ can be uniquely extended to ring monomorphism

$$\Phi : K((Y_1, \dots, Y_m)) \rightarrow K(\langle Y_1, \dots, Y_m \rangle).$$

This completes the proof of Remark 1.2.

3. STRUCTURE OF VARIOUS RING AND FIELD EXTENSIONS

We start by converting the lexomorph field into the iterated meromorphic field.

Remark 3.1 (Iterated meromorphic field). Let us consider the **iterated meromorphic field**

$$L_{m,1} = K((Y_m))(Y_{m-1}) \cdots (Y_1)$$

defined, with decreasing induction, by putting

$$L_{m,m} = K((Y_m)), L_{m,m-1} = L_{m,m}((Y_{m-1})), \dots, L_{m,1} = L_{m,2}((Y_1))$$

where $L_{m,j}$ is **temporary notation**. Obviously $L_{m,j}$ may be regarded as a subset of $K\langle Y_1, \dots, Y_m \rangle$. Let us prove that then

$$(3.1.1) \quad K(\langle Y_1, \dots, Y_m \rangle) = K((Y_m))(Y_{m-1}) \cdots (Y_1).$$

For $1 \leq j \leq m$, we may identify $K(\langle Y_j, \dots, Y_m \rangle)$ with the set of those members of $K(\langle Y_1, \dots, Y_m \rangle)$ whose support belongs to $P_j = \pi_1^{-1}(0) \cap \cdots \cap \pi_{j-1}^{-1}(0)$ where $P_1 = \mathbb{Z}^m$; note that now the field $K(\langle Y_j, \dots, Y_m \rangle)$ equals the intersection of $K\langle Y_j, \dots, Y_m \rangle$ with the field $K(\langle Y_1, \dots, Y_m \rangle)$. To prove (3.1.1), by decreasing induction of j we shall show that $K(\langle Y_j, \dots, Y_m \rangle) = L_{m,j}$ for $1 \leq j \leq m$. For $j = m$ this is obvious. Moreover, for any positive integer $j < m$, the j case follows from the $j+1$ case by noting that a subset Q of P_j is well ordered iff $\pi_j(Q)$ is well ordered and $\pi_j^{-1}(R) \cap P_j$ is well ordered for all $R \in \pi_j(Q)$.

In connection with Remark 1.4(•) we note that, for any $f \in K[[Y_1, \dots, Y_m]]$ we have $\text{Supp}(\phi(f)) \in \mathbb{N}^m$ and hence $\theta(\phi(f)) \in K[[X]]_{\text{lex}(\mathbb{Z}^m)}$. This proves Remark 1.4(•) which we restate as

$$(3.1.2) \quad \Phi(K[[Y_1, \dots, Y_m]]) \subset K[\langle Y_1, \dots, Y_m \rangle].$$

Remark 3.2 (Galois tower). For any field L , by $GT(L)$ we denote the set of all finite Galois groups $\text{Gal}(L^*, L)$ of (finite) Galois extensions L^* of L . We call $GT(L)$ the **Galois tower** of L . We say that $GT(L)$ is solvable (resp.: cyclic, abelian) to mean that every member of $GT(L)$ is solvable (resp.: cyclic, abelian). We say that $GT(L)$ is unsolvable to mean that some member of $GT(L)$ is unsolvable. We claim that

$$(3.2.1) \quad \begin{cases} \text{if } GT(K) \text{ is solvable, then so is } GT(K\langle Y_1, \dots, Y_m \rangle); \\ \text{hence, in particular, if } K \text{ is either algebraically closed or finite,} \\ \text{then } GT(K\langle Y_1, \dots, Y_m \rangle) \text{ is solvable.} \end{cases}$$

Recall that S_m is the permutation group on m letters with size $|S_m| = m!$. We claim that

$$(3.2.2) \quad S_m \in GT(K((Y_1, \dots, Y_m))).$$

We also claim that

$$(3.2.3) \quad \text{if } m > 1, \text{ then } GT(K((Y_1, \dots, Y_m))) \text{ is unsolvable.}$$

For $m \geq 5$, clearly $(3.2.2) \Rightarrow (3.2.3)$. Proofs of claims (3.2.1) to (3.2.3) will be given in another paper.

4. INTERESTING VECTOR SPACE AUTOMORPHISMS

Let S_m be the permutation group on $(1, \dots, m)$. For every $\sigma \in S_m$ we get a bijection $\hat{\sigma} : \mathbb{Z}^m \rightarrow \mathbb{Z}^m$ which sends $i = (i_1, \dots, i_m)$ to $\hat{\sigma}(i) = (i_{\sigma(1)}, \dots, i_{\sigma(m)})$. In turn we get a bijection

$$\tilde{\sigma} : K^{\mathbb{Z}^m} = K\langle Y_1, \dots, Y_m \rangle \rightarrow K\langle Y_1, \dots, Y_m \rangle = K^{\mathbb{Z}^m}$$

by sending every $A \in K^{\mathbb{Z}^m}$ to $\tilde{\sigma}(A) \in K^{\mathbb{Z}^m}$ such that $\tilde{\sigma}(A)(i) = A(\hat{\sigma}(i))$ for all $i \in \mathbb{Z}^m$. Clearly $\tilde{\sigma}$ is a K -linear automorphism of $K\langle Y_1, \dots, Y_m \rangle$. Upon letting

$$K(\langle Y_1, \dots, Y_m \rangle)_{\sigma} = \tilde{\sigma}(K(\langle Y_1, \dots, Y_m \rangle))$$

we get a field $K(\langle Y_1, \dots, Y_m \rangle)_{\sigma}$. Clearly $\tilde{\sigma}(\phi(K[[Y_1, \dots, Y_m]])) = \phi(K[[Y_1, \dots, Y_m]])$ and hence, interpreting $\subset$ to mean subring, we get

$$\phi(K[[Y_1, \dots, Y_m]]) \subset K(\langle Y_1, \dots, Y_m \rangle)_{\sigma}.$$

Therefore ϕ can be uniquely extended to a ring monomorphism

$$\Phi_{\sigma} : K((Y_1, \dots, Y_m)) \rightarrow K(\langle Y_1, \dots, Y_m \rangle)_{\sigma}.$$

The paradox of Remark 1.5 is resolved by saying that for $\sigma \neq \tau$ in S_m we need not have $\text{im}(\Phi_{\sigma}) = \text{im}(\Phi_{\tau})$. This inspires

Question 4.1. Do we have $\bigcap_{\sigma \in S_m} K(\langle Y_1, \dots, Y_m \rangle)_{\sigma} = \phi(K[[Y_1, \dots, Y_m]])$?

5. EXAMPLES OF NONMEROMORPHIC LEXOMORPHIC FUNCTIONS

For $m > 1$ we shall write down explicit examples of lexomorphic functions which are not meromorphic. Briefly speaking, we shall consider lots of lexomorphic functions whose supports are suitable disjoint infinite subsets of

$$\Delta = \{(-n, n, 0, \dots, 0) : n \in \mathbb{N}_+\} \subset \mathbb{Z}^m.$$

These will give you examples of infinitely many lexomorphic functions which are linearly independent over the meromorphic series field and, as we shall see in the next section, they can even be chosen to be algebraically independent over that field. Let us put

$$C = \Phi(K[[Y_1, \dots, Y_m]])$$

and

$$E = \Phi(K(\langle Y_1, \dots, Y_m \rangle)) \subset K(\langle Y_1, \dots, Y_m \rangle) = F.$$

Now explicitly here are the:

Examples 5.1. Let $m > 1$ and consider

$$f = \sum_{1 \leq n < \infty} Y_1^{n^2} Y_2^{-n^2} \in F.$$

We shall show that then $f \notin E$. Moreover, we shall show that f gives rise to infinitely many members of F which are linearly independent over E . Namely, partition the support of f into infinitely many infinite disjoint subsets and let $f_1, f_2, \dots$ be the members of F having these supports. We shall show that then the elements $f_1, f_2, \dots$ are linearly independent over E .

Definition 5.2. To establish the claims made in the above examples, we shall start by developing some sufficient conditions of linear independence. Let p denote a positive integer.

Given any $\beta^{(j)} = (\beta_1^{(j)}, \dots, \beta_m^{(j)}) \in \mathbb{Z}^m$ for $1 \leq j \leq p$, we put

$$\beta^{(1)} + \dots + \beta^{(p)} = (\beta_1^{(1)} + \dots + \beta_1^{(p)}, \dots, \beta_m^{(1)} + \dots + \beta_m^{(p)}) \in \mathbb{Z}^m.$$

Given any $D_j \subset \mathbb{Z}^m$ for $1 \leq j \leq p$, we put

$$D_1 + \dots + D_p = \{\beta^{(1)} + \dots + \beta^{(p)} : \beta^{(1)} \in D_1, \dots, \beta^{(p)} \in D_p\} \subset \mathbb{Z}^m.$$

Given any $\beta \in \mathbb{Z}^m$ and $D \subset \mathbb{Z}^m$ we put

$$\beta + D = \{\beta + \gamma : \gamma \in D\} \subset \mathbb{Z}^m.$$

Given any subset D of $\mathbb{Z}^m$ we say that D is **unclosed** to mean that for every $\alpha \in \mathbb{N}^m$ there is a $\beta \in D$ such that $\beta + \alpha \notin \mathbb{N}^m$ and for all $\gamma \in D \setminus \{\beta\}$ we have $\beta + \alpha \notin \gamma + \mathbb{N}^m$.

Given any indexed family $D = (D_w)_{w \in W}$ of subsets D_w of $\mathbb{Z}^m$ we say that the family is **multiclosed** to mean that: for all $w \neq w'$ in W we have $D_w \cap D_{w'} = \emptyset$, and for every $(\alpha, w) \in \mathbb{N}^m \times W$ there is a $\beta \in D_w$ such that $\beta + \alpha \notin \mathbb{N}^m$ and for all $\gamma \in (\bigcup_{w' \in W} D_{w'}) \setminus \{\beta\}$ we have $\beta + \alpha \notin \gamma + \mathbb{N}^m$.

Given any overring $\overline{B}$ of a ring $\overline{C}$, an indexed family $(f_w)_{w \in W}$ of elements f_w in $\overline{B}$ is said to be **linearly overindependent** over $\overline{C}$ to mean that: $\sum_{v \in V} h_v f_v \in \overline{C}$ with $h_v \in \overline{C}$ for all $v \in V =$ a finite subset of $W \Rightarrow h_v = 0$ for all $v \in V$. Recall that the family $(f_w)_{w \in W}$ is said to be **linearly independent** over $\overline{C}$ to mean that: $\sum_{v \in V} h_v f_v = 0$ with $h_v \in \overline{C}$ for all $v \in V =$ a finite subset $W \Rightarrow h_v = 0$ for all

$v \in V$. Consequently: linearly overindependent $\Rightarrow$ linearly independent. Also note that if $\overline{B}$ is a domain, then: linearly independent over $\overline{C} \Rightarrow$ linearly independent over the quotient field of $\overline{C}$.

We shall now prove the following theorem and then use it to establish the above explicit examples.

Theorem 5.3. *As sufficient conditions of linear independence we have the following:*

(5.3.1) *If $f \in F$ is such that $\text{Supp}(f)$ is unclosed, then $f \notin E$.*

(5.3.2) *If $(f_w)_{w \in W}$ is a family in F such that $\text{Supp}(f_w)_{w \in W}$ is multiclosed, then the elements $(f_w)_{w \in W}$ are linearly overindependent over C .*

Proof of (5.3.1). Let $f \in F$ be such that upon letting $D = \text{Supp}(f)$ we have that D is unclosed. We want to prove that $f \notin E$. Assume the contrary. Then for some $0 \neq h \in C$ we have $hf \in C$. Since $0 \neq h \in C$, we can take $\alpha \in \mathbb{N}^m$ such that (i) $\alpha \in \text{Supp}(h)$. Since D is unclosed, there is (ii) $\beta \in D$ such that (iii) $\beta + \alpha \notin \mathbb{N}^m$. Regarding hf as a map $\mathbb{Z}^m \rightarrow K$ and calculating its value at $\beta + \alpha \in \mathbb{Z}^m$, by (i) and (ii) we get $(hf)(\beta + \alpha) \neq 0$. This is a contradiction because $hf \in C$ but $\beta + \alpha \notin \mathbb{N}^m$.

Proof of (5.3.2). Let $(f_w)_{w \in W}$ be a family of elements $f_w \in F$ such that, letting $D_w = \text{Supp}(f_w)$ for all $w \in W$, the family $D = (D_w)_{w \in W}$ is multiclosed. We want to prove that the elements $(f_w)_{w \in W}$ are linearly overindependent over C .

Clearly it suffices to show that, given any finite subset V of W with $D_w \neq \emptyset$ for all $w \in V$ and given any $0 \neq h_w \in C$ for all $w \in V$ such that $\sum_{w \in V} h_w f_w \in C$, we must have $V = \emptyset$. Assume the contrary and fix $w \in V$. We can then find some $\alpha \in \text{Supp}(h_w)$. Since D is multiclosed, there is a $\beta \in D_w$ such that $\beta + \alpha \notin \mathbb{N}^m$ and for all $\gamma \in (\bigcup_{w' \in W} D_{w'}) \setminus \{\beta\}$ we have $\beta + \alpha \notin \gamma + \mathbb{N}^m$. It obviously follows that: (1 $\bullet$) for all $\gamma \in (\bigcup_{w' \in V} D_{w'}) \setminus \{\beta\}$ we have $\beta + \alpha \notin \gamma + \mathbb{N}^m$. By definition of multiclosed we also have: (2 $\bullet$) $D_{w'} \cap D_{w''} = \emptyset$ for all $w' \neq w''$ in V . Regarding $h_{w'} f_{w'}$ as a map $\mathbb{Z}^m \rightarrow K$ and calculating its value at $\beta + \alpha \in \mathbb{Z}^m$, by (1 $\bullet$) and (2 $\bullet$) we see that: (3 $\bullet$) $(h_{w'} f_{w'})(\beta + \alpha) = 0$ for all $w' \in V \setminus \{w\}$. Since $\alpha \in \text{Supp}(h_w)$ and $\beta \in D_w$, by (1 $\bullet$) we get: (4 $\bullet$) $(h_w f_w)(\beta + \alpha) \neq 0$. By (3 $\bullet$) and (4 $\bullet$) we conclude that: (5 $\bullet$) $(\sum_{w' \in V} h_{w'} f_{w'})(\beta + \alpha) \neq 0$. This is a contradiction because $\sum_{w' \in V} h_{w'} f_{w'} \in C$ but $\beta + \alpha \notin \mathbb{N}^m$.

Lemma 5.4. *Let $a : \mathbb{N}_+ \rightarrow \mathbb{N}_+$ be such that:*

$$(\dagger) \quad \begin{cases} \text{for all } n \in \mathbb{N}_+ \text{ we have } a(n) < a(n+1), \\ \text{and for every } \nu \in \mathbb{N}_+ \text{ there exists } \mu(\nu) \in \mathbb{N}_+ \\ \text{such that } a(n) + \nu < a(n+1) \text{ for all } n \geq \mu(\nu). \end{cases}$$

Then, assuming $m > 1$, and letting

$$D = \{(a(n), -a(n), 0, \dots, 0) \in \mathbb{Z}^m : n \in \mathbb{N}_+\}$$

we have (5.4.1) to (5.4.3) stated below.

(5.4.1) *Given any $\alpha \in \mathbb{N}^m$ and any infinite subset $\widehat{D}$ of D , there exists $\beta \in \widehat{D}$ such that $\beta + \alpha \notin \mathbb{N}^m$ and for all $\gamma \in D \setminus \{\beta\}$ we have $\beta + \alpha \notin \gamma + \mathbb{N}^m$.*

(5.4.2) *D is unclosed.*

(5.4.3) For all $i \in \mathbb{N}_+$ let D_i be an infinite subset of D such that for all $i \neq j$ in $\mathbb{N}_+$ we have $D_i \cap D_j = \emptyset$. Then $(D_i)_{i \in \mathbb{N}_+}$ is a multiclosed indexed family of subsets of $\mathbb{Z}^m$.

(5.4.4) Without assuming $m > 1$, if for some integer $k \geq 2$ we have $a(n) = k^n$ for all $n \in \mathbb{N}_+$, then a satisfies condition $(\dagger)$.

(5.4.5) Without assuming $m > 1$, if for some integer $k \geq 2$ we have $a(n) = n^k$ for all $n \in \mathbb{N}_+$, then a satisfies condition $(\dagger)$.

Proof of (5.4.1). Now $\alpha = (\alpha_1, \dots, \alpha_m)$ with $\alpha_1, \dots, \alpha_m$ in $\mathbb{N}$. Letting $\nu = \alpha_1 + \alpha_2 + 1$ we get $\nu \in \mathbb{N}_+$. Since $\widehat{D}$ is assumed to be infinite, we can find $n \in \mathbb{N}_+$ such that $n > \mu(\nu)$ and upon letting $\beta = (a(n), -a(n), 0, \dots, 0)$ we have $\beta \in \widehat{D}$. Now $\mu(\nu) \leq n - 1 \in \mathbb{N}_+$ and hence $a(n - 1) + \nu < a(n)$ and therefore $\alpha_2 \leq \nu \leq a(n - 1) + \nu < a(n)$. Hence $-a(n) + \alpha_2 < 0$ and therefore $\beta + \alpha \notin \mathbb{N}^m$. Given any $\gamma \in D \setminus \{\beta\}$, we have $\gamma = (a(n'), -a(n'), 0, \dots, 0)$ with $n \neq n' \in \mathbb{N}_+$. Clearly,

$$n' < n \Rightarrow \beta_2 + \alpha_2 \leq -a(n) + \nu < -a(n - 1) \leq -a(n') = \gamma_2 \Rightarrow \beta + \alpha \notin \gamma + \mathbb{N}^m$$

and

$$n' > n \Rightarrow \beta_1 + \alpha_1 \leq a(n) + \nu < a(n + 1) \leq \gamma_1 \Rightarrow \beta + \alpha \notin \gamma + \mathbb{N}^m.$$

Proof of (5.4.2). Take $D = \widehat{D}$ in (5.4.1).

Proof of (5.4.3). Given any $(\alpha, i) \in \mathbb{N}^m \times \mathbb{N}_+$, by taking $\widehat{D} = D_i$ in (5.4.1) we find $\beta \in D_i$ such that $\beta + \alpha \notin \mathbb{N}^m$ and for all $\gamma \in \mathbb{N}_+ \setminus \{i\}$ we have $\beta + \alpha \notin \gamma + \mathbb{N}^m$. Therefore the family $(D_i)_{i \in \mathbb{N}_+}$ is multiclosed.

Proof of (5.4.4) and (5.4.5). Take $\mu(\nu) = \nu$.

Examples 5.5. Let $a : \mathbb{N}_+ \rightarrow \mathbb{N}_+$ satisfy Lemma 5.4($\dagger$). Let $m > 1$ and consider

$$f = \sum_{1 \leq n < \infty} Y_1^{a(n)} Y_2^{-a(n)} \in F.$$

By (5.3.1) and (5.4.2) we see that $f \notin E$. Now let us partition the support of f into infinitely many infinite disjoint subsets and let $f_1, f_2, \dots$ be the members of F having these supports; then by (5.3.2) and (5.4.3) we see that the elements $f_1, f_2, \dots$ are linearly independent over E .

By (5.4.4) we see that, fixing any integer $k \geq 2$, we may take $a(n) = k^n$ for all $n \in \mathbb{N}_+$.

By (5.4.5) we see that, fixing any integer $k \geq 2$, we may take $a(n) = n^k$ for all $n \in \mathbb{N}_+$. The $k = 2$ case of this corresponds to Examples 5.1.

Examples 5.6. To show that the converse of (5.3.1) is **not true**, let $a' : \mathbb{N}_+ \rightarrow \mathbb{N}_+$ be such that for all $n \in \mathbb{N}_+$ we have $a'(n) + 2 < a'(n + 1)$. Assuming $m > 1$ let

$$D' = \{\beta'_n : n \in \mathbb{N}_+\} \text{ where } \beta'_n = (a'(n), -a'(n), 0, \dots, 0) \in \mathbb{Z}^m$$

and

$$D'' = \{\beta''_n : n \in \mathbb{N}_+\} \text{ where } \beta''_n = (a'(n) + 1, -a'(n) - 1, 0, \dots, 0) \in \mathbb{Z}^m.$$

Clearly, $D' \cap D'' = \emptyset$. Taking $\alpha = (1, 1, 0, \dots, 0) \in \mathbb{N}^m$ we see that for all $n \in \mathbb{N}_+$ we have

$$\beta' + \alpha = \beta''_n + (0, 2, 0, \dots, 0) \quad \text{and} \quad \beta'' + \alpha = \beta'_n + (2, 0, 0, \dots, 0)$$

and hence $D' \cup D''$ is NOT unclosed.

Let $a : \mathbb{N}_+ \rightarrow \mathbb{N}_+$ satisfy Lemma 5.4(†) and take $a'(n) = a(\mu(2) + n)$ for all $n \in \mathbb{N}_+$. Then clearly $a'(n) + 2 < a'(n+1)$ for all $n \in \mathbb{N}_+$ and a' satisfies Lemma 5.4(†). Therefore upon letting

$$f' = \sum_{1 \leq n < \infty} Y_1^{a'(n)} Y_2^{-a'(n)} \in F$$

we have $\text{Supp}(f') = D'$; by (5.4.2) we see that D' is unclosed and hence by (5.2.1) we conclude that $f' \notin E$. Clearly $0 \neq 1 + Y_1 Y_2^{-1} \in E$ and hence upon letting $f'' = (1 + Y_1 Y_2^{-1})f'$ we get $f'' \in F \setminus E$. But obviously

$$f'' = f' + Y_1 Y_2^{-1} f' = \sum_{1 \leq n < \infty} Y_1^{a'(n)} Y_2^{-a'(n)} + \sum_{1 \leq n < \infty} Y_1^{a'(n)+1} Y_2^{-a'(n)-1}$$

and hence $\text{Supp}(f'') = D' \cup D''$ which shows that the converse of (5.3.1) is not true.

6. EXAMPLES OF TRANSCENDENTAL LEXOMORPHIC FUNCTIONS

The examples of the previous section can be modified to write down transcendental lexomorphic functions. Let us put

$$C = \Phi(K[[Y_1, \dots, Y_m]])$$

and

$$E = \Phi(K(\langle Y_1, \dots, Y_m \rangle)) \subset K(\langle Y_1, \dots, Y_m \rangle) = F.$$

Now explicitly here are the:

Examples 6.1. Let $m > 1$ and consider

$$f = \sum_{1 \leq n < \infty} Y_1^{n^n} Y_2^{-n^n} \in F.$$

We shall show that then f is transcendental over E . Moreover, we shall show that f gives rise to infinitely many members of F which are algebraically independent over E . Namely, partition the support of f into infinitely many infinite disjoint subsets and let $f_1, f_2, \dots$ be the members of F having these supports. We shall show that then the elements $f_1, f_2, \dots$ are algebraically independent over E .

Definition 6.2. To establish the claims made in the above examples, we start by developing sufficient conditions of linear independence and algebraic independence.

For all p and j in $\mathbb{N}_+$ we put

$$N(p, j) = \{i \in \mathbb{N}^p : i_1 + \dots + i_p = j\}.$$

Given any $n \in \mathbb{N}$ and $D \subset \mathbb{Z}^m$ we define $[n]D \subset \mathbb{Z}^m$ by putting

$$[n]D = \begin{cases} D_1 + \dots + D_n \text{ with } D_1 = \dots = D_n = D & \text{if } n > 0, \\ \{(0, \dots, 0)\} & \text{if } n = 0. \end{cases}$$

Given any $i = (i_1, \dots, i_p) \in \mathbb{N}^p$ with $p \in \mathbb{N}_+$ and subsets $D_1, \dots, D_p$ of $\mathbb{Z}^m$, we put

$$[i](D_1, \dots, D_p) = [i_1]D_1 + \dots + [i_p]D_p.$$

Given any set W and any $p \in \mathbb{N}_+$, by $W^{(p)}$ we denote the set of all sequences $v = (v_1, \dots, v_p)$ of distinct elements $v_1, \dots, v_p$ in W . For any such p and v , and any indexed family $(D_w)_{w \in W}$ of subsets D_w of $\mathbb{Z}^m$ and $i \in \mathbb{N}^p$ we put

$$D_i^{(v)} = [i](D_{v_1}, \dots, D_{v_p}).$$

By a **boxing** of a set U we mean a family $(U_j)_{j \in \mathbb{N}_+}$ of subsets U_j of U such that $U = \bigcup_{j \in \mathbb{N}_+} U_j$ and $U_j \cap U_{j'} = \emptyset$ for all $j \neq j'$ in $\mathbb{N}_+$.

To sharpen Theorem 5.3, we first sharpen the concept of multiclosed thus. By a **polyfamily** we mean an indexed family $D = (\widehat{D}_i, D_i)_{i \in U}$ of subsets $\widehat{D}_i$ and D_i of $\mathbb{Z}^m$ together with a boxing $(U_j)_{j \in \mathbb{N}_+}$ of U . We say that such a polyfamily D is **polyclosed** to mean that: for all $i \in U$ we have $\widehat{D}_i \subset D_i$, and for every $(\alpha, j, i) \in \mathbb{N}^m \times \mathbb{N}_+ \times U_j$ there is a $\beta \in \widehat{D}_i$ such that $\beta + \alpha \notin \mathbb{N}^m$ and for all $\gamma \in (\bigcup_{i' \in (U_1 \cup \dots \cup U_j) \setminus \{i\}} D_{i'}) \cup (D_i \setminus \{\beta\})$ we have $\beta + \alpha \notin \gamma + \mathbb{N}^m$.

Sharpening Theorem 5.3 we get

Theorem 6.3. *As sufficient conditions of linear independence and algebraic independence we have the following:*

(6.3.1) *Let $(f_i)_{i \in U}$ be a family of elements f_i in F . Assume that there exists a polyclosed polyfamily D as above such that for all $i \in U$ we have $\text{Supp}(f_i) = \widehat{D}_i$. Then the elements $(f_i)_{i \in U}$ are linearly overindependent over C .*

(6.3.2) *Given any $f \in F$, let $(U_j)_{j \in \mathbb{N}_+}$ be the boxing of $U = \mathbb{N}_+$ given by putting $U_j = \{j\}$ for all $j \in \mathbb{N}_+$, consider the polyfamily $D^* = (\widehat{D}_n^*, D_n^*)_{n \in U}$ obtained by putting $\widehat{D}_n^* = \text{Supp}(f^n)$ and $D_n^* = [n]\text{Supp}(f)$ for all $n \in \mathbb{N}_+$, and assume that the polyfamily D^* is polyclosed. Then f is transcendental over E .*

*Note**. In the proof we shall tacitly use the following facts: Let $\overline{C}$ be any domain with quotient field $\overline{E}$, let $\overline{F}$ be any overfield of $\overline{E}$, and let f be any element of $\overline{F}$. Then f is transcendental over $E \Leftrightarrow$ the elements $(f^n)_{n \in \mathbb{N}_+}$ are linearly overindependent over $\overline{C}$. Moreover, in (6.3.2) we have $\widehat{D}_n^* \subset D_n^*$ for all $n \in \mathbb{N}_+$.

(6.3.3) Given any indexed family $(f_w)_{w \in W}$ of elements f_w in F , let $D = (D_w)_{w \in W}$ be the family of subsets of $\mathbb{Z}^m$ obtained by putting $D_w = \text{Supp}(f_w)$ for all $w \in W$. For any $p \in \mathbb{N}_+$ and $v \in W^{(p)}$, let $(U_j)_{j \in \mathbb{N}_+}$ be the boxing of $U = \mathbb{N}^p \setminus \{(0, \dots, 0)\}$ given by putting $U_j = N(p, j)$ for all $j \in \mathbb{N}_+$, and consider the polyfamily $D^{(v)} = (\widehat{D}_i^{(v)}, D_i^{(v)})_{i \in U}$ obtained by putting $\widehat{D}_i^{(v)} = \text{Supp}(f_{v_1}^{i_1} \dots f_{v_p}^{i_p})$ for all $i \in U$. Assume that for all $p \in \mathbb{N}_+$ and $v \in W^{(p)}$, the polyfamily $D^{(v)}$ is polyclosed. Then the elements $(f_w)_{w \in W}$ are algebraically independent over E .

*Note***. In the proof we shall tacitly use the following facts: Let $\overline{C}$ be any domain with quotient field $\overline{E}$, let $\overline{F}$ be any overfield of $\overline{E}$, and let $(f_w)_{w \in W}$ be any indexed family of elements f_w in $\overline{F}$. Then the elements $(f_w)_{w \in W}$ are algebraically independent over $\overline{E} \Leftrightarrow$ for every $j \in \mathbb{N}_+$ and every finite sequence $v_1, \dots, v_p$ of distinct elements in W , the monomials $(f_{v_1}^{i_1} \dots f_{v_p}^{i_p})_{1 \leq i_1 + \dots + i_p \leq j}$ are linearly overindependent over $\overline{C}$. Moreover, in (6.3.3) we have $\widehat{D}_i^{(v)} \subset D_i^{(v)}$ for all $v \in W^{(p)}$ and $i \in \mathbb{N}^p$ with $p \in \mathbb{N}_+$.

Proof of (6.3.1). Clearly it suffices to show that, given any finite subset V of U with $\widehat{D}_i \neq \emptyset$ for all $i \in V$ and given any $0 \neq h_i \in C$ for all $i \in V$ such that $\sum_{i \in V} h_i f_i \in C$, we must have $V = \emptyset$. Assuming the contrary, let j be the largest positive integer such that $V \cap U_j \neq \emptyset$ and fix $i \in V \cap U_j$. We can then find some $\alpha \in \text{Supp}(h_i)$. Since D is polyclosed, there is a $\beta \in \widehat{D}_i$ such that $\beta + \alpha \notin \mathbb{N}^m$ and: (1 $\bullet$) for all $\gamma \in (\bigcup_{i' \in (U_1 \cup \dots \cup U_j) \setminus \{i\}} D_{i'}) \cup (D_i \setminus \{\beta\})$ we have $\beta + \alpha \notin \gamma + \mathbb{N}^m$. Regarding $h_{i'} f_{i'}$ as a map $\mathbb{Z}^m \rightarrow K$ and calculating its value at $\beta + \alpha \in \mathbb{Z}^m$, by (1 $\bullet$) we see that: (2 $\bullet$) $(h_{i'} f_{i'})(\beta + \alpha) = 0$ for all $i' \in V \setminus \{i\}$. Since $\alpha \in \text{Supp}(h_i)$ and

$\beta \in \widehat{D}_i$, by (1 $\bullet$) we get: (3 $\bullet$) $(h_i f_i)(\beta + \alpha) \neq 0$. By (2 $\bullet$) and (3 $\bullet$) we conclude that: (4 $\bullet$) $(\sum_{i' \in V} h_{i'} f_{i'}) (\beta + \alpha) \neq 0$. This is a contradiction because $\sum_{i' \in V} h_{i'} f_{i'} \in C$ but $\beta + \alpha \notin \mathbb{N}^m$.

Proof of (6.3.2). In view of Note*, it suffices to show that the elements $(f^n)_{n \in \mathbb{N}_+}$ are linearly overindependent over C . This follows by taking D^* for D in (6.3.1).

Proof of (6.3.3). In view of Note**, it suffices to show that for every $j \in \mathbb{N}_+$ and every finite sequence $v_1, \dots, v_p$ of distinct elements in W , the monomials $(f_{v_1}^{i_1} \cdots f_{v_p}^{i_p})_{1 \leq i_1 + \dots + i_p \leq j}$ are linearly overindependent over C . This follows by taking $D^{(v)}$ for D in (6.3.1).

Lemma 6.4. *Let $b : \mathbb{N}_+ \rightarrow \mathbb{N}_+$ be such that*

$$(\sharp) \quad \begin{cases} \text{for all } n \in \mathbb{N}_+ \text{ we have } b(n) < b(n+1), \\ \text{and } b(n+1)/b(n) \text{ tends to } \infty \text{ as } n \text{ tends to } \infty. \end{cases}$$

Then, given any $\nu \in \mathbb{N}_+$ and $\nu' \in \mathbb{N}_+$, there exists $\mu(\nu, \nu') \in \mathbb{N}_+$ such that for all $n \geq \mu(\nu, \nu')$ we have $b(n+1)/b(n) > \nu(\nu'+1)$. Moreover, for every such $\mu(\nu, \nu')$ we have that: if $\mu(\nu, \nu') < n_1 \leq n_2 \leq \dots \leq n_r$ and $1 \leq m_1 \leq m_2 \leq \dots \leq m_s$ are any integers with $1 \leq s \leq r \leq \nu$ and $(n_1, \dots, n_r) \neq (m_1, \dots, m_s)$, then $|n' - m'| > \nu'$ where $n' = b(n_1) + \dots + b(n_r)$ and $m' = b(m_1) + \dots + b(m_s)$.

(Here $\|$ denotes norm, and $(n_1, \dots, n_r) = (m_1, \dots, m_s)$ means $r = s$ and $n_i = m_i$ for $1 \leq i \leq r$.)

Proof. We are assuming that $b(n+1)/b(n) \rightarrow \infty$ as $n \rightarrow \infty$, and

(1) $b(n) < b(n+1)$ for all $n \in \mathbb{N}_+$.

These assumptions tell us that, given any $\nu \in \mathbb{N}_+$ and $\nu' \in \mathbb{N}_+$, there exists $\mu(\nu, \nu') \in \mathbb{N}_+$ such that

(2) for all $n \in \mathbb{N}_+$ with $n \geq \mu(\nu, \nu')$ we have $b(n+1) > \nu(\nu'+1)b(n)$.

Now given any integers

(3) $\mu(\nu, \nu') < n_1 \leq n_2 \leq \dots \leq n_r$, and

(4) $1 \leq m_1 \leq m_2 \leq \dots \leq m_s$ with $1 \leq s \leq r \leq \nu$,

we want to show that

$$(\bullet) \quad (n_1, \dots, n_r) \neq (m_1, \dots, m_s) \Rightarrow |n' - m'| > \nu'.$$

We shall prove $(\bullet)$ by induction on r . By (1) to (3) we see that

(5) for all $n \in \mathbb{N}_+$ with $n > n_r$ we have $b(n) > \nu(\nu'+1)b(n_r)$, and

(6) for all $n \in \mathbb{N}_+$ with $n < n_r$ we have $\nu(\nu'+1)b(n) < b(n_r)$.

In view of (1) to (4), respectively by (5) and (6) we see that

(7) $m_s > n_r \Rightarrow m' \geq b(m_s) > \nu(\nu'+1)b(n_r) \geq (\nu'+1)n' \geq n' + \nu'$, and

(8) $m_s < n_r \Rightarrow m' + \nu' \leq (\nu'+1)m' \leq \nu(\nu'+1)b(m_s) < b(n_r) \leq n'$.

Now if $(n_1, \dots, n_r) \neq (m_1, \dots, m_s)$ and $r = 1 = s$, then clearly $m_s \neq n_r$; consequently, by (7) and (8) we see that

(9) $(n_1, \dots, n_r) \neq (m_1, \dots, m_s)$ with $r = 1 = s \Rightarrow |n' - m'| > \nu'$.

By (2) we see that if $r > 1$, then $b(n_{r-1}) > \nu'$, and hence

(10) if $r > 1 = s$ with $m_s = n_r$, then $n' - m' = b(n_1) + \dots + b(n_{r-1}) > \nu'$.

Obviously,

$$(11) \quad \begin{cases} \text{if } r > 1 < s \text{ with } m_s = n_r, \text{ then:} \\ (n_1, \dots, n_r) \neq (m_1, \dots, m_s) \Leftrightarrow (n_1, \dots, n_{r-1}) \neq (m_1, \dots, m_{s-1}) \\ \text{and } n' - m' = b(n_1) + \dots + b(n_{r-1}) - b(m_1) - \dots - b(m_{s-1}). \end{cases}$$

In case of $r = 1$, $(\bullet)$ follows from (9).

In case of $r > 1$, by (7), (8), (10), (11) we see that: $(\bullet)$ for $r - 1 \Rightarrow (\bullet)$ for r .

Lemma 6.5. *Let $b : \mathbb{N}_+ \rightarrow \mathbb{N}_+$ satisfy Lemma 6.4($\sharp$). In case of $m > 1$, for every $n \in \mathbb{N}_+$ we put $\delta(n) = (n, -n, 0, \dots, 0) \in \mathbb{Z}^m$. Then, assuming $m > 1$, and letting $\tilde{D} = \{\delta(n) : n \in \mathbb{N}_+\} \subset \mathbb{Z}^m$, we have (6.5.1) and (6.5.2) stated below.*

(6.5.1) *Let $f \in F$ be such that $\text{Supp}(f) = \tilde{D}$. Then f is transcendental over E .*

(6.5.2) *Let $W = \mathbb{N}_+$. For all $w \in W$ let D_i be an infinite subset of $\tilde{D}$ such that for all $w \neq w'$ in W we have $D_w \cap D_{w'} = \emptyset$. For every $w \in W$ let $f_w \in F$ be such that $\text{Supp}(f_w) = D_w$. Then the elements $f_1, f_2, \dots$ are algebraically independent over E .*

(6.5.3) *Without assuming $m > 1$, if $b(n) = n!$ for all $n \in \mathbb{N}_+$, then b satisfies Lemma 6.4($\sharp$).*

(6.5.4) *Without assuming $m > 1$, if $b(n) = n^n$ for all $n \in \mathbb{N}_+$, then b satisfies Lemma 6.4($\sharp$).*

Proof of (6.5.1). Let $(U_j)_{j \in \mathbb{N}_+}$ be the boxing of $U = \mathbb{N}_+$ given by putting $U_j = \{j\}$ for all $j \in \mathbb{N}_+$, and consider the polyfamily $D^* = (\hat{D}_i^*, D_i^*)_{i \in U}$ obtained by putting $\hat{D}_i^* = \text{Supp}(f^i)$ and $D_i^* = [i]\text{Supp}(f)$ for all $i \in \mathbb{N}_+$. To prove f is transcendental over E , in view of (6.3.2) it suffices to show that D^* is polyclosed. By Note* we have $\hat{D}_i^* \subset D_i^*$ for all $i \in \mathbb{N}_+$. It only remains to show that given any $(\alpha, i) \in \mathbb{N}^m \times \mathbb{N}_+$ there is a $\beta \in \text{Supp}(f^i)$ such that $\beta + \alpha \notin \mathbb{N}^m$ and for all $\gamma \in D_1^* \cup \dots \cup D_{i-1}^* \cup (D_i^* \setminus \{\beta\})$ we have $\beta + \alpha \notin \gamma + \mathbb{N}^m$. Taking

$$(1) \quad \nu = \nu' = \alpha_1 + \alpha_2 + i$$

in Lemma 6.4 we find $\mu(\nu, \nu) \in \mathbb{N}_+$ such that

$$(2) \quad \text{for all } n \geq \mu(\nu, \nu) \text{ we have } b(n+1)/b(n) > \nu(\nu+1).$$

By Lemma 6.4 we know that

$$(\sharp) \quad \begin{cases} \text{if } \mu(\nu, \nu) < n_1 \leq n_2 \leq \dots \leq n_r \text{ and } 1 \leq m_1 \leq m_2 \leq \dots \leq m_s \\ \text{are any integers with } 1 \leq s \leq r \leq \nu \\ \text{and } (n_1, \dots, n_r) \neq (m_1, \dots, m_s), \text{ then } |n' - m'| > \nu \\ \text{where } n' = b(n_1) + \dots + b(n_r) \text{ and } m' = b(m_1) + \dots + b(m_s). \end{cases}$$

Fixing any integer

$$(3) \quad n > \mu(\nu, \nu)$$

and letting

$$(4) \quad n_1 = \dots = n_i = n \quad \text{and} \quad n' = ib(n)$$

by $(\ddagger)$ we see that

$$(5) \quad \begin{cases} \text{if } 1 \leq m_1 \leq \cdots \leq m_s \text{ are any integers with } 1 \leq s \leq i \\ \text{and } (n_1, \dots, n_i) \neq (m_1, \dots, m_s), \text{ then } |n' - m'| > \nu \\ \text{where } m' = b(m_1) + \cdots + b(m_s). \end{cases}$$

Since $\text{Supp}(f) = \tilde{D}$, upon letting $\beta = \delta(n')$, by (4) and the $s = i$ case of (5) we conclude that $\beta \in \text{Supp}(f^i)$. Mimicking the proof of (5.4.1) we shall show that $\beta + \alpha \notin \mathbb{N}^m$ and for all $\gamma \in D_1^* \cup \cdots \cup D_{i-1}^* \cup (D_i^* \setminus \{\beta\})$ we have $\beta + \alpha \notin \gamma + \mathbb{N}^m$. This will complete the proof.

By (2) and (3) we have $n' > \nu$, and by (1) we have $\nu > \alpha_2$; consequently, $n' > \alpha_2$; but by definition $\beta_2 = -n'$ and hence $\beta_2 + \alpha_2 < 0$ and therefore $\beta + \alpha \notin \mathbb{N}^m$. Given any $\gamma \in D_1^* \cup \cdots \cup D_{i-1}^* \cup (D_i^* \setminus \{\beta\})$, we may assume that in the situation of (5) we have $\gamma = \delta(m')$; now it suffices to note that

$$m' < n' \Rightarrow \beta_2 + \alpha_2 = -n' + \alpha_2 \leq -n' + \nu < -m' = \gamma_2 \Rightarrow \beta + \alpha \notin \gamma + \mathbb{N}^m$$

and

$$m' > n' \Rightarrow \beta_1 + \alpha_1 = n' + \alpha_1 \leq n' + \nu < m' = \gamma_1 \Rightarrow \beta + \alpha \notin \gamma + \mathbb{N}^m.$$

Proof of (6.5.2). For any $p \in \mathbb{N}_+$ and any sequence $v = (v_1, \dots, v_p)$ of distinct positive integers, let $(U_j)_{j \in \mathbb{N}_+}$ be the boxing of $U = \mathbb{N}^p \setminus \{(0, \dots, 0)\}$ given by putting $U_j = N(p, j)$ for all $j \in \mathbb{N}_+$, and consider the polyfamily $D^{(v)} = (\widehat{D}_i^{(v)}, D_i^{(v)})_{i \in U}$ obtained by putting $\widehat{D}_i^{(v)} = \text{Supp}(f_{v_1}^{i_1} \cdots f_{v_p}^{i_p})$ for all $i \in U$. To prove that the elements $f_1, f_2, \dots$ are algebraically independent over E , in view of (6.3.3) it suffices to show that $D^{(v)}$ is polyclosed. By Note** we have $\widehat{D}_i^{(v)} \subset D_i^{(v)}$ for all $i \in U$. It only remains to show that given any $(\alpha, j, i) \in \mathbb{N}^m \times \mathbb{N}_+ \times N(p, j)$ there is a $\beta \in \widehat{D}_i^{(v)}$ such that $\beta + \alpha \notin \mathbb{N}^m$ and for all $\gamma \in (\bigcup_{i' \in (U_1 \cup \cdots \cup U_j) \setminus \{i\}} D_{i'}) \cup (D_i \setminus \{\beta\})$ we have $\beta + \alpha \notin \gamma + \mathbb{N}^m$. Taking

$$(1) \quad \nu = \nu' = \alpha_1 + \alpha_2 + j$$

in Lemma 6.4 we find $\mu(\nu, \nu) \in \mathbb{N}_+$ such that

$$(2) \quad \text{for all } n \geq \mu(\nu, \nu) \text{ we have } b(n+1)/b(n) > \nu(\nu+1).$$

By Lemma 6.4 we know that

$$(\ddagger) \quad \begin{cases} \text{if } \mu(\nu, \nu) < n_1 \leq n_2 \leq \cdots \leq n_r \text{ and } 1 \leq m_1 \leq m_2 \leq \cdots \leq m_s \\ \text{are any integers with } 1 \leq s \leq r \leq \nu \\ \text{and } (n_1, \dots, n_r) \neq (m_1, \dots, m_s), \text{ then } |n' - m'| > \nu \\ \text{where } n' = b(n_1) + \cdots + b(n_r) \text{ and } m' = b(m_1) + \cdots + b(m_s). \end{cases}$$

Clearly, we can find integers $n'_1, \dots, n'_p$ such that

$$(3) \quad \mu(\nu, \nu) < n'_1 < \cdots < n'_p \text{ with } \delta(n'_q) \in D_{v_q} \text{ for } 1 \leq q \leq p$$

and now letting

$$(4) \quad n_l = n'_q \text{ for } 1 \leq q \leq p \text{ and } \sum_{1 \leq t \leq q-1} i_t < l \leq \sum_{1 \leq t \leq q} i_t \text{ with } n' = \sum_{1 \leq q \leq p} i_q b(n'_q)$$

by $(\ddagger)$ we see that

$$(5) \quad \begin{cases} \text{if } 1 \leq m_1 \leq \cdots \leq m_s \text{ are any integers with } 1 \leq s \leq j \\ \text{and } (n_1, \dots, n_j) \neq (m_1, \dots, m_s), \text{ then } |n' - m'| > \nu \\ \text{where } m' = b(m_1) + \cdots + b(m_s). \end{cases}$$

Since $\text{Supp}(f_{v_q}) = D_{v_q}$ for $1 \leq q \leq p$, upon letting $\beta = \delta(n')$, by (4) and the $s = j$ case of (5) we conclude that $\beta \in \widehat{D}_i^{(v)}$. Mimicking the proof of (5.4.1) we shall show that $\beta + \alpha \notin \mathbb{N}^m$ and for all $\gamma \in (\bigcup_{i' \in (U_1 \cup \cdots \cup U_j) \setminus \{i\}} D_{i'}) \cup (D_i \setminus \{\beta\})$ we have $\beta + \alpha \notin \gamma + \mathbb{N}^m$. This will complete the proof.

By (2) to (4) we have $n' > \nu$, and by (1) we have $\nu > \alpha_2$; consequently, $n' > \alpha_2$; but by definition $\beta_2 = -n'$. Hence $\beta_2 + \alpha_2 < 0$ and therefore $\beta + \alpha \notin \mathbb{N}^m$. Given any $\gamma \in (\bigcup_{i' \in (U_1 \cup \cdots \cup U_j) \setminus \{i\}} D_{i'}) \cup (D_i \setminus \{\beta\})$ we may assume that in the situation of (5) we have $\gamma = \delta(m')$; now it suffices to note that

$$m' < n' \Rightarrow \beta_2 + \alpha_2 = -n' + \alpha_2 \leq -n' + \nu < -m' = \gamma_2 \Rightarrow \beta + \alpha \notin \gamma + \mathbb{N}^m$$

and

$$m' > n' \Rightarrow \beta_1 + \alpha_1 = n' + \alpha_1 \leq n' + \nu < m' = \gamma_1 \Rightarrow \beta + \alpha \notin \gamma + \mathbb{N}^m.$$

Proof of (6.5.3) and (6.5.4). Obvious.

Examples 6.6. Let $b : \mathbb{N}_+ \rightarrow \mathbb{N}_+$ satisfy Lemma 6.5($\ddagger$). Let $m > 1$ and consider

$$f = \sum_{1 \leq n < \infty} Y_1^{b(n)} Y_2^{-b(n)} \in F.$$

By (6.5.1) we see that f is transcendental over E . Now partition the support of f into infinitely many infinite disjoint subsets and let $f_1, f_2, \dots$ be the members of F having these supports; then by (6.5.2) we see that the elements $f_1, f_2, \dots$ are algebraically independent over E .

By (6.5.3) we see that we may take $b(n) = n!$ for all $n \in \mathbb{N}_+$.

By (6.5.4) we see that we may take $b(n) = n^n$ for all $n \in \mathbb{N}_+$; this corresponds to Examples 6.1.

Question 6.7. In the situation of Examples 6.6 we may ask whether E is relatively algebraically closed in F . As a partial answer let $f = \sum_{1 \leq n < \infty} Y_1^{2^n} Y_2^{-2^n}$ and assume K is of characteristic 2. By Examples 5.5 we see that $f \in F \setminus E$, but clearly $f^2 + f + Y_1^2 Y_2^{-2} = 0$ and hence $[E(f) : E] = 2$.

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