ON NONLINEAR INTERPOLATION

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ABSTRACT. In a case study on asymptotics of spectral quantities of Schrödinger operators in fractional Sobolev spaces on the circle we show how a nonlinear version of the Riesz-Thorin theorem on the interpolation of linear operators can be applied.

1. INTRODUCTION

The aim of this paper is to derive asymptotics of spectral quantities of Schrödinger operators on the circle for potentials in fractional Sobolev spaces by a suitable theorem on nonlinear interpolation which we now state in the set-up used for our applications.

For any $0 \leq s < \infty$ denote by $H_{\mathbb{C}}^s \equiv H^s(\mathbb{T}, \mathbb{C})$ the complex Sobolev space of order s of the $1d$ torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Its elements can be represented by a Fourier series $\varphi(x) = \sum_{k \in \mathbb{Z}} \hat{\varphi}(k) e^{2\pi i k x}$, where $\hat{\varphi}(k) := \int_0^1 \varphi(x) e^{-2\pi i k x} dx \in \mathbb{C}$, and

$$\|\varphi\|_s := \left(\sum_{k \in \mathbb{Z}} (1 + |k|)^{2s} |\hat{\varphi}(k)|^2 \right)^{1/2} < \infty.$$

Note that $H_{\mathbb{C}}^{s'} \hookrightarrow H_{\mathbb{C}}^s$ for any $0 \leq s \leq s'$. Denote by $B_{\mathbb{C}}^s(R) \subseteq H_{\mathbb{C}}^s$ the open ball of radius R , centered at 0. Furthermore, for any $1 \leq p < \infty$ and $t \in \mathbb{R}$ denote by $\ell_{\mathbb{C}}^{p,t} \equiv \ell^{p,t}(\mathbb{Z}, \mathbb{C})$ the complex sequence space

$$\ell_{\mathbb{C}}^{p,t} = \{ \xi = (\xi_j)_{j \in \mathbb{Z}} \subseteq \mathbb{C} \mid \|\xi\|_{p,t} < \infty \}$$

where

$$\|\xi\|_{p,t} := \left(\sum_{j \in \mathbb{Z}} ((1 + |j|)^t |\xi_j|)^p \right)^{1/p}.$$

Note that $\ell_{\mathbb{C}}^{p,t'} \hookrightarrow \ell_{\mathbb{C}}^{p,t}$ for any $0 \leq t \leq t'$. Finally, for $b > a \geq 0$ and $\alpha \geq 0$, $\beta > 0$, assume that $F : B_{\mathbb{C}}^a(R) \rightarrow \ell_{\mathbb{C}}^{p,\alpha+\beta a}$ is an analytic map, bounded by $M_a > 0$, so that for some $M_b > 0$,

$$(1.1) \quad \sup_{\varphi \in B_{\mathbb{C}}^b(R)} \|F(\varphi)\|_{p,\alpha+\beta b} \leq M_b.$$

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By the characterization of analytic maps with sequence spaces as their range (cf. e.g. [5, Appendix A]) and the analyticity of F it follows from (1.1) that $F|_{B_{\mathbb{C}}^b(R)} : B_{\mathbb{C}}^b(R) \rightarrow \ell_{\mathbb{C}}^{p,\alpha+\beta b}$ is analytic as well. Hence, equivalently, we are given the following commutative diagram:

$$\begin{array}{ccc} B_{\mathbb{C}}^a(R) & \xrightarrow{F} & \ell_{\mathbb{C}}^{p,\alpha+\beta a} \\ \uparrow & & \uparrow \\ B_{\mathbb{C}}^b(R) & \xrightarrow{F|_{B_{\mathbb{C}}^b(R)}} & \ell_{\mathbb{C}}^{p,\alpha+\beta b} \end{array}$$

where the horizontal maps F and $F|_{B_{\mathbb{C}}^b(R)}$ are analytic,

$$\sup_{\varphi \in B_{\mathbb{C}}^a(R)} \|F(\varphi)\|_{p,\alpha+\beta a} \leq M_a, \quad \sup_{\varphi \in B_{\mathbb{C}}^b(R)} \|F(\varphi)\|_{p,\alpha+\beta b} \leq M_b,$$

and the vertical arrows denote the standard inclusions of the corresponding spaces.

The following theorem is an instance of an extension of the Riesz-Thorin theorem (cf. e.g. [1]) to nonlinear maps and is inspired by a special case treated in [6, Appendix 2].

Theorem 1.1. *Under the assumptions stated above, F interpolates between $B_{\mathbb{C}}^a(R)$ and $B_{\mathbb{C}}^b(R)$. More precisely, for any $s = (1 - \lambda)a + \lambda b$ with $0 \leq \lambda \leq 1$, $F(B_{\mathbb{C}}^s(R)) \subseteq \ell_{\mathbb{C}}^{p,\alpha+\beta s}$ and $F|_{B_{\mathbb{C}}^s(R)} : B_{\mathbb{C}}^s(R) \rightarrow \ell_{\mathbb{C}}^{p,\alpha+\beta s}$ is an analytic map, bounded by $M_a^{1-\lambda} M_b^{\lambda}$.*

The proof of Theorem 1.1, presented in Section 2, uses in a crucial way the assumption that F is analytic on a ball in a complex (Hilbert) space. The proof does not apply to nonlinear maps

$$F : B_{\mathbb{C}}(R) \cap H^s(\mathbb{T}, \mathbb{R}) \rightarrow \ell^{p,\alpha+\beta s}(\mathbb{Z}, \mathbb{R})$$

which are merely real analytic. In the sequel we would like to discuss a type of problem where nevertheless Theorem 1.1 can be applied to real analytic nonlinear maps. For q in $L_0^2(\mathbb{T}, \mathbb{R}) = \{q \in L^2(\mathbb{T}, \mathbb{R}) \mid \int_0^1 q(x) dx = 0\}$ let $L(q)$ be the Schrödinger operator $L(q) := -d_x^2 + q$. The Dirichlet spectrum of $L(q)$, considered on the interval $[0, 1]$, is real and consists of simple eigenvalues. We list them in increasing order $\mu_1 < \mu_2 < \dots$. Furthermore denote by $M(x, \lambda)$ the fundamental solution of $L(q)$, i.e. the 2×2 matrix valued function satisfying $L(q)M = \lambda M$, $\lambda \in \mathbb{C}$, and $M(0, \lambda) = Id_{2 \times 2}$,

$$M(x, \lambda) = \begin{pmatrix} y_1(x, \lambda) & y_2(x, \lambda) \\ y_1'(x, \lambda) & y_2'(x, \lambda) \end{pmatrix}.$$

When evaluated at $\lambda = \mu_n$, the Floquet matrix $M(1, \lambda)$ is lower triangular, hence its eigenvalues are given by $y_1(1, \mu_n)$ and $y_2'(1, \mu_n)$. By the Wronskian identity, they satisfy $y_1(1, \mu_n)y_2'(1, \mu_n) = 1$. By deforming the potential q to the zero potential along the straight line tq , $0 \leq t \leq 1$, one sees that $(-1)^n y_1(1, \mu_n) > 0$. Hence the Floquet exponents are given by $\pm \kappa_n$ where

$$\kappa_n := -\log((-1)^n y_1(1, \mu_n))$$

and $\log$ denotes the principal branch of the logarithm. It turns out that together the μ_n 's and κ_n 's form a system of canonical coordinates for L_0^2 (see [8]). The κ_n 's also play an important role in proving the property of 1-smoothing of the periodic KdV equation [4]. We want to determine the asymptotics of the κ_n 's as $n \rightarrow \infty$. To state them, introduce for any $s \in \mathbb{R}_{\geq 0}$ the Sobolev spaces

$$H_0^s \equiv H_0^s(\mathbb{T}, \mathbb{R}) = H^s(\mathbb{T}, \mathbb{C}) \cap L_0^2(\mathbb{T}, \mathbb{R})$$

and denote by $\langle \cdot, \cdot \rangle$ the standard inner product in $L_0^2(\mathbb{T}, \mathbb{R})$,

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx.$$

In [3], the following theorem was proved.

Theorem 1.2 ([3]). *For any $N \in \mathbb{Z}_{\geq 0}$,*

$$\kappa_n = \frac{1}{2\pi n} \left(\langle q, \sin 2\pi nx \rangle + \frac{1}{n^{N+1}} \ell_n^2 \right)$$

uniformly on bounded subsets of H_0^N . Here ℓ_n^2 denotes the n 'th component of a sequence in $\ell^2 \equiv \ell^2(\mathbb{N}, \mathbb{R})$.

Our aim is to extend Theorem 1.2 to any fractional order Sobolev space H_0^s with $s \in \mathbb{R}_{\geq 0}$. To this end we need to extend various quantities to the complex Hilbert space $H_{0,\mathbb{C}}^s := \{q \in H_{\mathbb{C}}^s \mid \int_0^1 q(x) dx = 0\}$. Let $L_{0,\mathbb{C}}^2 \equiv H_{0,\mathbb{C}}^0$ and denote by $B_{0,\mathbb{C}}^s(R)$ the complex ball $B_{0,\mathbb{C}}^s(R) := \{q \in H_{0,\mathbb{C}}^s \mid \|q\| < R\}$. In Section 3 we prove the following

Theorem 1.3. *For any $R > 0$ there exists $n_R > 0$ so that for any $n > n_R$, κ_n can be extended analytically to $B_{0,\mathbb{C}}^0(R)$. Moreover, for any $N \in \mathbb{Z}_{\geq 0}$ the sequence $(\kappa_n)_{n > n_R}$ satisfies the estimate*

$$\kappa_n = \frac{1}{2\pi n} \left(\langle q, \sin 2n\pi x \rangle + \frac{1}{n^{N+1}} \ell_n^2 \right)$$

uniformly on $B_{0,\mathbb{C}}^N(R)$.

At the end of Section 3 we show that Theorem 1.3 allows us to apply Theorem 1.1 to generalize Theorem 1.2 to any fractional order Sobolev space.

Theorem 1.4. *For any $s \in \mathbb{R}_{\geq 0}$*

$$\kappa_n = \frac{1}{2\pi n} \left(\langle q, \sin 2n\pi x \rangle + \frac{1}{n^{s+1}} \ell_n^2 \right)$$

uniformly on bounded subsets of H_0^s .

By the same method one can show that similar results hold for many other spectral quantities. We state without proof another such result. For $q \in L_0^2$, the periodic spectrum of $L(q)$ on the interval $[0, 2]$ is real and discrete. When listed in increasing order and with their multiplicities the eigenvalues satisfy

$$\lambda_0 < \lambda_1^- \leq \lambda_1^+ < \lambda_2^- \leq \lambda_2^+ < \dots$$

Using [3, Theorem 1.3, Theorem 1.4], one then shows with the help of Theorem 1.1 that the following result holds.

Theorem 1.5. *For any $s \in \mathbb{R}_{\geq 0}$*

$$(\lambda_n^+ + \lambda_n^-) / 2 - \mu_n = \langle q, \cos 2\pi nx \rangle + \frac{1}{n^{s+1}} \ell_n^2$$

uniformly on bounded subsets of H_0^s .

Related work. In the context of nonlinear PDEs, Tartar obtained a nonlinear interpolation theorem in [12] which later was slightly improved in [2, Theorem 1]. However, for nonlinear maps such as the ones encountered in the analysis of Schrödinger operators, the assumptions of these theorems are not satisfied and hence they cannot be applied. See also [7] and the references therein.

2. PROOF OF THEOREM 1.1

The proof of Theorem 1.1 uses arguments of the proof of the Riesz-Thorin theorem on the interpolation of linear operators (see e.g. [1]). The key ingredient is Hadamard's three line theorem.

Proof of Theorem 1.1. As the case $p = 1$ is similar we restrict our attention to the case when $1 < p < \infty$. For any $k \in \mathbb{Z}$, denote by F_k the k 'th component of F . As $F : B_{\mathbb{C}}^a(R) \rightarrow \ell_{\mathbb{C}}^{p, \alpha + \beta b}$ is analytic, $F_k : B_{\mathbb{C}}^a(R) \rightarrow \mathbb{C}$ is analytic. In particular, the restriction $F_k|_{B_{\mathbb{C}}^s(R)}$ of F_k to $B_{\mathbb{C}}^s(R) \subseteq B_{\mathbb{C}}^a(R)$ is analytic for any $a \leq s \leq b$. Hence once the claimed bound for $F|_{B_{\mathbb{C}}^s(R)}$ is established, the analyticity of $F|_{B_{\mathbb{C}}^s(R)} : B_{\mathbb{C}}^s(R) \rightarrow \ell_{\mathbb{C}}^{p, \alpha + \beta s}$ will follow (cf. e.g. [5, Appendix A]). To prove the claimed bound consider $\varphi \in B_{\mathbb{C}}^s(R)$ and define the complex family

$$\varphi_z(x) := \sum_{k \in \mathbb{Z}} \langle k \rangle^{s-z} \hat{\varphi}(k) e^{2\pi i k x}$$

where $z = u + iv \in \mathbb{C}$, $u, v \in \mathbb{R}$, and $\langle k \rangle := 1 + |k|$. As u and v are real we see that for any $\varphi \in B_{\mathbb{C}}^s(R)$,

$$\|\varphi_{u+iv}\|_u^2 = \sum_{k \in \mathbb{Z}} \langle k \rangle^{2u} |\langle k \rangle^{-u-iv+s} \hat{\varphi}(k)|^2 = \sum_{k \in \mathbb{Z}} \langle k \rangle^{2s} |\hat{\varphi}(k)|^2 = \|\varphi\|_s^2 < R^2.$$

In particular, for any $v \in \mathbb{R}$,

$$(2.1) \quad \|F(\varphi_{a+iv})\|_{p, \alpha + \beta a} \leq M_a \quad \text{and} \quad \|F(\varphi_{b+iv})\|_{p, \alpha + \beta b} \leq M_b.$$

Similarly, for any $a \leq u \leq b$ and $v \in \mathbb{R}$,

$$\|\varphi_{u+iv}\|_a \leq \|\varphi_{u+iv}\|_u = \|\varphi\|_s < R.$$

Hence $\varphi_{u+iv} \in B_{\mathbb{C}}^a(R)$ for any $a \leq u \leq b$ and any $v \in \mathbb{R}$. In particular, $F(\varphi_{u+iv}) \in \ell_{\mathbb{C}}^{p, \alpha + \beta a}$ and

$$(2.2) \quad \|F(\varphi_{u+iv})\|_{p, \alpha + \beta a} \leq M_a.$$

As $\varphi_s = \varphi$ for any $\varphi \in B_{\mathbb{C}}^s(R)$, we need to prove the claimed estimate for $z = s$. Denote by $1 < q < \infty$ the number conjugate to p , i.e. $\frac{1}{p} + \frac{1}{q} = 1$, and consider an arbitrary sequence $\xi = (\xi_k)_{k \in \mathbb{Z}} \subseteq \mathbb{C}$ with *finite support* so that $\|\xi\|_{q, -(\alpha + \beta s)} \leq 1$. Similarly as in the case of $H_{\mathbb{C}}^s$, define for $z = u + iv \in \mathbb{C}$, $u, v \in \mathbb{R}$,

$$\xi_z := \left(\langle k \rangle^{-\beta(s-z)} \xi_k \right)_{k \in \mathbb{Z}}.$$

Then

$$(2.3) \quad \begin{aligned} \|\xi_{u+iv}\|_{q, -(\alpha + \beta u)}^q &= \sum_{k \in \mathbb{Z}} \left(\langle k \rangle^{-(\alpha + \beta u)} |\langle k \rangle^{-\beta(s-z)} \xi_k| \right)^q \\ &\leq \sum_{k \in \mathbb{Z}} \left(\langle k \rangle^{-(\alpha + \beta s)} |\xi_k| \right)^q \leq 1. \end{aligned}$$

Similarly as above, $\xi_s = \xi$, and for any $a \leq u \leq b$ and $v \in \mathbb{R}$,

$$(2.4) \quad \|\xi_{u+iv}\|_{q, -(\alpha+\beta a)} \leq \|\xi\|_{q, -\alpha+\beta b}.$$

Denote by $\langle \cdot, \cdot \rangle$ the ℓ^2 dual pairing as well as its extension to $\ell_{\mathbb{C}}^{p, \alpha+\beta a} \times \ell_{\mathbb{C}}^{q, -(\alpha+\beta a)}$ and introduce the vertical strip

$$(2.5) \quad S_{a,b} := \{z = u + iv \mid v \in \mathbb{R}, a < u < b\}$$

and its closure $\overline{S}_{a,b}$. To obtain the claimed estimate we want to apply Hadamard's three line theorem to the following function:

$$f : \overline{S}_{a,b} \rightarrow \mathbb{C}, \quad z \mapsto \langle F(\varphi_z), \xi_z \rangle$$

where

$$\langle F(\varphi_z), \xi_z \rangle = \sum_{k \in \mathbb{Z}} F_k(\varphi_z) \langle k \rangle^{-\beta(s-z)} \xi_k.$$

As the support of ξ is finite, the latter sum is finite, and hence the function f is well defined and $\|\xi\|_{q, -\alpha+\beta b} < \infty$. In view of (2.2) and (2.4), $f : \overline{S}_{a,b} \rightarrow \mathbb{C}$ is *bounded* as for any $z \in \overline{S}_{a,b}$,

$$|f(z)| = |\langle F(\varphi_z), \xi_z \rangle| \leq \|F(\varphi_z)\|_{p, \alpha+\beta a} \|\xi_z\|_{q, -(\alpha+\beta a)} \leq M_a \|\xi\|_{q, -\alpha+\beta b}.$$

Note that on the strip $S_{a,b}$, the curves $z \mapsto \xi_z \in \ell_{\mathbb{C}}^{q, -(\alpha+\beta a)}$ and $z \mapsto \varphi_z \in B_{\mathbb{C}}^a(R) \subseteq H_{\mathbb{C}}^a$ are analytic. As $F_k : B_{\mathbb{C}}^a(R) \rightarrow \mathbb{C}$ is analytic for any $k \in \mathbb{Z}$ it then follows that f , being a finite sum of analytic functions, is analytic on $S_{a,b}$ and continuous on the closure $\overline{S}_{a,b}$. Moreover, in view of (2.1) and (2.3), the following estimates hold for any $v \in \mathbb{R}$:

$$|f(a + iv)| \leq \|F(\varphi_{a+iv})\|_{p, \alpha+\beta a} \|\xi_{a+iv}\|_{q, -(\alpha+\beta a)} \leq M_a$$

and

$$|f(b + iv)| \leq \|F(\varphi_{b+iv})\|_{p, \alpha+\beta b} \|\xi_{b+iv}\|_{q, -(\alpha+\beta b)} \leq M_b.$$

Hence we can apply Hadamard's three line theorem to f (cf. e.g. [9, Appendix to IX,.4]) to conclude that, for $s = (1 - \lambda)a + \lambda b$ and any $v \in \mathbb{R}$

$$|f(s + iv)| \leq M_a^{1-\lambda} M_b^{\lambda}.$$

In particular, for $z = s$ one has

$$|f(s)| = |\langle F(\varphi), \xi \rangle| \leq M_a^{1-\lambda} M_b^{\lambda}$$

where we took into account that $\varphi_s = \varphi$ and $\xi_s = \xi$. The claimed estimate $\|F(\varphi)\|_{p, \alpha+\beta s} \leq M_a^{1-\lambda} M_b^{\lambda}$ then follows from the Hahn-Banach theorem and the fact that $\ell_{\mathbb{C}}^{q, -(\alpha+\beta s)}$ is the dual space of $\ell_{\mathbb{C}}^{p, \alpha+\beta s}$. $\square$

3. PROOFS OF THEOREM 1.3 AND THEOREM 1.4

Before proving Theorem 1.3 we need to make some preparatory considerations. Note that for q in $L_{0,\mathbb{C}}^2$ the operator $L(q)$ is no longer symmetric with respect to the L^2 -inner product $\langle f, g \rangle = \int_0^1 f(x) \overline{g(x)} dx$. The Dirichlet spectrum is still discrete, but the eigenvalues might be complex valued and multiple. We list them according to their algebraic multiplicities and in lexicographic ordering

$$\mu_1 \preccurlyeq \mu_2 \preccurlyeq \mu_3 \preccurlyeq \dots$$

where for any complex numbers a, b

$$a \preccurlyeq b \text{ iff } [\operatorname{Re} a < \operatorname{Re} b] \text{ or } [\operatorname{Re} a = \operatorname{Re} b \text{ and } \operatorname{Im} a \leq \operatorname{Im} b].$$

By [8] there exists for any $R > 0$ an integer $m_R > 0$ so that for any $q \in L^2_{0,\mathbb{C}}$ with $\|q\| := \langle q, q \rangle^{1/2} < R$ the Dirichlet eigenvalues $\mu_n \equiv \mu_n(q), n \geq 1$, satisfy the following estimates:

$$|\mu_n - n^2\pi^2| < \pi/4 \quad \forall n > m_R$$

and

$$|\mu_n| < m_R^2\pi^2 + \pi/4 \quad \forall 1 \leq n \leq m_R.$$

In particular, for any $n > m_R$, μ_n is a simple Dirichlet eigenvalue of $L(q)$ and hence analytic on the complex ball $B^0_{0,\mathbb{C}}(R) = \{q \in L^2_{0,\mathbb{C}} \mid \|q\| < R\}$. To see that κ_n can be analytically extended to $B^0_{0,\mathbb{C}}(R)$ for n sufficiently large we first note that by [8], the Floquet matrix $M(1, \lambda, q)$ is analytic on $\mathbb{C} \times L^2_{0,\mathbb{C}}$ and, with $\lambda = \nu^2$, $y_1(1, \nu^2, q)$ satisfies the following estimate:

$$(3.1) \quad |y_2(1, \nu^2, q) - \cos \nu| \leq \frac{1}{|\nu|} \exp(|\operatorname{Im} \nu| + \|q\|).$$

As for $n > m_R$, one has $|\mu_n - n^2\pi^2| < \pi/4$. It follows that $\nu_n \equiv \nu_n(q) = \sqrt[4]{\mu_n(q)}$ is well defined. Here $\sqrt[4]{z}$ denotes the principal branch of the square root. Then for any $n > m_R$ and $q \in B^0_{0,\mathbb{C}}(R)$, $\nu_n = n\pi \sqrt[4]{1 + z_n}$, where $z_n = \frac{\mu_n - n^2\pi^2}{n^2\pi^2}$ satisfies $|z_n| \leq \frac{1}{4\pi n^2} \leq \frac{1}{10}$ and thus

$$|\sqrt[4]{1 + z_n} - 1| = \left| \frac{(\sqrt[4]{1 + z_n} - 1)(\sqrt[4]{1 + z_n} + 1)}{\sqrt[4]{1 + z_n} + 1} \right| \leq |z_n|$$

or

$$|\nu_n - n\pi| \leq \frac{\pi}{4} \frac{1}{4n} \leq \frac{1}{4n}; \quad |\operatorname{Im} \nu_n| \leq \frac{1}{4n}.$$

These estimates are used in the asymptotics of $y_1(1, \nu_n^2, q)$. As $(-1)^n \cos \nu_n = 1 + (\nu_n - n\pi) \int_0^1 -\sin(t(\nu_n - n\pi)) dt$ one concludes that $|(-1)^n \cos \nu_n - 1| \leq \frac{1}{4n}$ for any $n > m_R$ and as $|\nu_n| \geq n\pi - \frac{1}{4n} \geq 2n$ it follows from (3.1) that

$$|(-1)^n y_1(1, \mu_n, q) - 1| \leq \frac{1}{4n} + \frac{1}{2n} \exp\left(\frac{1}{4n} + R\right)$$

for any $n > m_R$ and $\|q\| < R$. Now choose $n_R \geq m_R$ so large that

$$\frac{1}{4n} + \frac{1}{2n} \exp\left(\frac{1}{4n} + R\right) \leq \frac{1}{2} \quad \forall n > n_R.$$

As a consequence, for any $q \in L^2_{0,\mathbb{C}}$ with $\|q\| < R$ and any $n > n_R$,

$$\kappa_n = -\log((-1)^n y_1(1, \mu_n, q))$$

is well defined. We thus have proved.

Lemma 3.1. *For any $R > 0$ there exists $n_R > 0$ so that $\forall n > n_R$*

$$\kappa_n : B^0_{0,\mathbb{C}}(R) \rightarrow \mathbb{C}, \quad q \mapsto -\log((-1)^n y_1(1, \mu_n, q))$$

is well defined and analytic.

Proof of Theorem 1.3. In view of Lemma 3.1 it remains to prove for any $R > 0$, and any $N \in \mathbb{Z}_{\geq 0}$,

$$\sum_{n > n_R} n^{2N+2} |2\pi n \kappa_n - \langle q, \sin 2n\pi x \rangle|^2$$

is bounded on $B_{0,\mathbb{C}}^N(R)$. Going through the arguments of the proof of Theorem 1.1 in [3], one sees that this is indeed the case. $\square$

It remains to show Theorem 1.4. As already mentioned in the introduction, we will use the result of Theorem 1.1 on nonlinear interpolation.

Proof of Theorem 1.4. By Theorem 1.2, Theorem 1.4 holds in the case where s is an integer. Let $N \in \mathbb{Z}_{\geq 0}$ and $N < s < N + 1$. For any given $R > 0$ choose $n_R > 0$ as in Theorem 1.3. It follows from Theorem 1.2 that there exists $M_R > 0$ such that for any $q \in L_0^2$ with $\|q\| < R$ and any $1 \leq n \leq n_R$

$$|2\pi n\kappa_n - \langle q, \sin 2\pi nx \rangle| \leq M_R$$

and therefore

$$(3.2) \quad \left(\sum_{n \leq n_R} n^{2(s+1)} |2\pi n\kappa_n - \langle q, \sin 2\pi nx \rangle|^2 \right)^{1/2} \leq n_R^{s+3/2} M_R.$$

On the other hand, by Theorem 1.3, for any $n > n_R$, κ_n extends analytically to $B_{0,\mathbb{C}}^0(R)$ and there exist constants $M_{N,R}, M_{N+1,R}$ so that

$$\left(\sum_{n > n_R} n^{2(N+1)} |2\pi n\kappa_n - \langle q, \sin 2\pi nx \rangle|^2 \right)^{1/2} \leq M_{N,R} \quad \forall q \in B_{0,\mathbb{C}}^N(R)$$

as well as

$$\left(\sum_{n > n_R} n^{2(N+2)} |2\pi n\kappa_n - \langle q, \sin 2\pi nx \rangle|^2 \right)^{1/2} \leq M_{N+1,R} \quad \forall q \in B_{0,\mathbb{C}}^{N+1}(R).$$

Hence Theorem 1.1 applies and implies that with $0 \leq \lambda \leq 1$ chosen so that $s = (1 - \lambda)N + \lambda(N + 1)$,

$$\left(\sum_{n > n_R} n^{2(s+1)} |2\pi n\kappa_n - \langle q, \sin 2\pi nx \rangle|^2 \right)^{1/2} \leq M_{s,R} \quad \forall q \in B_{0,\mathbb{C}}^s(R)$$

where $M_{s,R} = (M_{N,R})^{(1-\lambda)}(M_{N+1,R})^\lambda$. From this together with (3.2), the claimed result follows. $\square$

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The motivation of this paper originated from our observation that in the proof of Theorem 6.1 in the paper [10] of A. Savchuk and A. Shkalikov, the theorem of Tartar on nonlinear interpolation [12] cannot be applied as stated by the authors. After having pointed this out to A. Shkalikov, he sent us the statement of an unpublished interpolation theorem obtained by them earlier, which can be used to prove their results. Due to unfortunate personal circumstances their paper with a proof of this interpolation theorem was delayed. In order to be able to further progress on our projects, we came up with our own version in the set-up needed for our applications and with an optimal estimate on the bounds of the interpolating maps. After having sent our preprint to A. Shkalikov, he sent us a preprint with the proof of their theorem [11]. As the two papers turned out to be quite different, we agreed to publish them independently.

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