THE INDEPENDENCE OF THE NOTIONS OF HOPFIAN AND CO-HOPFIAN ABELIAN p -GROUPS

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ABSTRACT. Hopfian and co-Hopfian Abelian groups have recently become of great interest in the study of algebraic and adjoint entropy. Here we prove that the existence of the following three types of infinite abelian p -groups of size less than $2^{\aleph_0}$ is independent of ZFC: (a) both Hopfian and co-Hopfian, (b) Hopfian but not co-Hopfian, (c) co-Hopfian but not Hopfian. All three types of groups of size $2^{\aleph_0}$ exist in ZFC.

1. INTRODUCTION

A very simple result from linear algebra states that an endomorphism of a finite dimensional vector space is injective if and only if it is surjective and hence an isomorphism. However, an infinite dimensional vector space never has this property, as the standard shift endomorphisms show.

Reinhold Baer [B] was the first to investigate these properties for Abelian groups rather than vector spaces. While he used the terminology Q -group and S -group, researchers now use the notions of Hopfian group and co-Hopfian group. These groups have arisen recently in the study of algebraic entropy and its dual, adjoint entropy; see e.g., [DGSZ, GG], which are the motivations for our work.

Recall that an (Abelian) group G is *Hopfian* if every epimorphism $G \rightarrow G$ is an automorphism. Dually, G is said to be *co-Hopfian* if every monomorphism $G \rightarrow G$ is an automorphism.

Some obvious examples: finite groups are both Hopfian and co-Hopfian, the group of integers is Hopfian but not co-Hopfian, and the Prüfer group $\mathbb{Z}(p^\infty)$ is co-Hopfian but not Hopfian. Hence the notions are independent of each other. However, the existence of infinite co-Hopfian reduced p -groups was first established by Crawley [Cr].

As it is often the case in Abelian group theory, one attacks a problem by looking at three different cases, namely, the torsion case, the torsion-free case and the mixed case. Here we will concentrate on the p -groups, but for the convenience of

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the reader we list some easy observations about (co-)Hopfian groups in general (see also [GG2]).

- A torsion-free group is co-Hopfian if and only if it is divisible of finite rank; i.e., it is isomorphic to a finite dimensional $\mathbb{Q}$ -space.
- A torsion-free group of finite rank is Hopfian.
- Finitely generated groups are Hopfian, and dually, finitely co-generated groups (like $\mathbb{Z}(p^\infty)$) are co-Hopfian.
- Every group G with endomorphism ring $\text{End}(G) \cong \mathbb{Z}$ is Hopfian; thus arbitrarily large Hopfian groups exist.
- Reduced Hopfian or co-Hopfian p -groups are semi-standard and so have cardinality at most $2^{\aleph_0}$;
- Reduced countable Hopfian or co-Hopfian p -groups are finite.

Inspired by the last two statements, Goldsmith and Gong asked in [GG2] if the notions of Hopficity and co-Hopficity are independent at all for reduced p -groups and if there is a Hopfian and/or co-Hopfian group of size $\aleph_1$ if the continuum hypothesis fails. Note that there are Hopfian and co-Hopfian reduced p -groups of size the continuum. Here is our main theorem showing that the existence of such groups is independent of ZFC:

Theorem 1.1. *The following statements are consistent with ZFC even if the continuum $2^{\aleph_0}$ is arbitrarily large:*

- (1) *For every prime p there are reduced p -groups of size $\aleph_1$ of the following three types:*
 - (a) *both Hopfian and co-Hopfian,*
 - (b) *Hopfian but not co-Hopfian,*
 - (c) *co-Hopfian but not Hopfian.*
- (2) *For every prime p every reduced infinite p -group of size strictly less than the continuum is neither Hopfian nor co-Hopfian.*

Remark 1.2. There always exist continuum-sized reduced p -groups of types (1.a), (1.b) and (1.c) in ZFC.

The proof of this statement uses rather technical but old and classic ideas; cf. [Cr]. As usual one kills unwanted endomorphisms by adding new elements during the construction, establishing Lemmas 2.1 and 3.1 for the groups. One then finishes as below.

Actually, as we shall see, (1) holds in the forcing extension of the universe $\mathfrak{M}$ by $\text{Fin}(\omega_1, 2)$, i.e., the partial poset of partial maps $q: \omega_1 \rightarrow \{0, 1\}$ with finite domain $\text{dom}(q) \subseteq \omega_1$. Furthermore, (2) follows from Martin's axiom, even from its weak version $\text{MA}(\sigma\text{-centered})$.

We prove the first part of the theorem in Sections 2 and 3. In the former section we will construct two Hopfian groups: a co-Hopfian one and a non-co-Hopfian one. In the latter section we carry out a similar construction of a co-Hopfian group.

Finally, in Section 4 we prove the second part of the theorem via constructing endomorphisms.

In the sequel the word group shall always mean an additively written Abelian group; the book [F] shall serve as a reference to ideas needed in Abelian group theory.

2. A HOPFIAN BUT NOT CO-HOPFIAN GROUP

In this section we construct Hopfian groups in the forcing extension of the universe $\mathfrak{M}$ by $\text{Fin}(\omega_1, 2)$. Actually, it will be more convenient to consider the equivalent forcing poset $P = \text{Fin}(\omega_1 \times \mathbb{Z}^+, 2)$, i.e., the partial poset of partial maps $q: \omega_1 \times \mathbb{Z}^+ \rightarrow \{0, 1\}$ with finite domain. Thus we work in $\mathfrak{M}[F]$, where F is a P -generic filter over $\mathfrak{M}$. Here we have new functions $\eta_\alpha: \mathbb{Z}^+ \rightarrow \{0, 1\}$ added to the universe for $\alpha < \omega_1$.

Let us construct some subgroups of $C = t(\prod_{n=1}^\infty \mathbb{Z}_{p^n} e_n)$ for some fixed prime p . Let x be the monomorphism defined informally by $e_k x := p e_{k+1}$ for every $k \in \mathbb{N}$, i.e., $(\sum_{k=1}^\infty n_k e_k) x := \sum_{k=1}^\infty p n_k e_{k+1}$ in general. Moreover, we put

$$(2.1) \quad a_{\alpha, n} := \sum_k^\infty \eta_\alpha(k) p^{k-n} e_k \quad \text{for } \alpha < \omega_1, n \in \mathbb{Z}^+,$$

hence

$$(2.2) \quad a_{\alpha, n} = \eta_\alpha(n) e_n + p a_{\alpha, n+1} \quad \text{for } \alpha < \omega_1, n \in \mathbb{Z}^+.$$

We define

$$(2.3) \quad G_{h,c} := \langle e_n, a_{\alpha, n} : n = 1, \dots, \infty, \alpha < \omega_1 \rangle,$$

$$(2.4) \quad G_h := \langle e_n, a_{\alpha, n} x^k : k = 0, \dots, \infty, n = 1, \dots, \infty, \alpha < \omega_1 \rangle.$$

A common basic subgroup of C , $G_{h,c}$ and G_h is

$$(2.5) \quad B = \bigoplus_{n=1}^\infty \mathbb{Z}_{p^n} e_n,$$

because it is a pure cyclic subgroup and factoring with it leads to divisible quotients. The divisibility of $G_{h,c}/B$ and G_h/B follows from (2.2).

We first determine the endomorphism rings of $G_{h,c}$ and G_h . Recall that a homomorphism $\phi: A \rightarrow X$ between two Abelian p -groups is called *small* if for all $k > 0$ there is an $n \geq 0$ such that $p^n A[p^k] \phi = 0$. By $\widehat{X}$ we denote the completion of X in p -adic topology.

Lemma 2.1.

$$(2.6) \quad \text{End } G_{h,c} = J_p \oplus \text{small}(G_{h,c}),$$

$$(2.7) \quad \text{End } G_h = \widehat{J_p[x]} \oplus \text{small}(G_h).$$

Proof. First note that the groups $G_{h,c}$ and G_h are pure p -torsion subgroups of C ; hence every homomorphism between these groups is uniquely determined on the basis elements e_n . Therefore a common generalization of the results stated in the lemma is

$$(2.8) \quad \text{Hom}(G_{h,c}, G_h) = \widehat{J_p[x]} \oplus \text{small}(G_{h,c}, G_h).$$

In other words, for every homomorphism f and every positive integer N , there is a polynomial $k' \in \mathbb{Z}[x]$ making $f - k'$ small on $G_{h,c}[p^N]$. We are now going to prove this.

For simplicity, we abuse notation, and let f also denote a name of the homomorphism $G_{h,c} \rightarrow G_h$. Let $N \in \mathbb{N}$.

There are conditions $r_{i,j}$ in the generic filter F forcing the e_j -coordinate of $e_i f$ to be some $k_{i,j} \in \mathbb{Z}_{p^j}$. The $r_{i,j}$ altogether have countable domain, which is therefore disjoint from $\{\alpha\} \times \mathbb{Z}^+$ for some $\alpha \in \omega_1$.

There is an r in the generic filter F forcing that f is a homomorphism $G_{h,c} \rightarrow G_h$, and

$$(2.9) \quad a_{\alpha,N} f = g + k a_{\alpha,M} + \sum_{i=1}^l k_i a_{\alpha_i, M_i}$$

for some $g \in B$, polynomials $k, k_i \in \mathbb{Z}[x]$ and pairwise distinct ordinals $\alpha, \alpha_i < \omega_1$.

For all positive integers $n \geq N$ where (α, n) is not in the domain of r , let us consider the automorphism φ of the forcing notion switching the value of η_α at n ; i.e., η_α^φ coincides with η_α except at the place n where $\eta_\alpha^\varphi(n) = 1 - \eta_\alpha(n)$. For $\beta \neq \alpha$, the function η_β remains unchanged: $\eta_\beta^\varphi = \eta_\beta$.

On the level of partial functions, for every partial function q , we have $\text{dom } q^\varphi = \text{dom } q$ and $(\beta, m)q^\varphi = (\beta, m)q$ for $(\beta, m) \in \text{dom } q$ and $(\beta, m) \neq (\alpha, n)$. The crucial change is $(\alpha, n)q^\varphi = 1 - (\alpha, n)q$ whenever $(\alpha, n) \in \text{dom } q$. In particular, φ leaves r and the $r_{i,j}$ invariant.

Now applying the automorphism φ , we obtain, since r forces it, that f^φ is again a homomorphism $G_{h,c} \rightarrow G_h$ satisfying

$$(2.10) \quad (a_{\alpha,N} \pm p^{n-N} e_n) f^\varphi = g + k(a_{\alpha,M} \pm p^{n-M} e_n) + \sum_{i=1}^l k_i a_{\alpha_i, M_i}.$$

Also, the e_j -coordinate of $e_i f^\varphi$ is $k_{i,j}$, because $r_{i,j}$ forces it, so $f^\varphi = f$ since the two maps agree on the e_n . Hence the difference of the two equations (2.9) and (2.10) simplifies to

$$(2.11) \quad p^{n-N} e_n f = p^{n-M} k e_n$$

(we repeat) for all but finitely many $n \geq N$ (those for which (α, n) is not in support of p).

It follows that $0 = p^n e_n f = p^{n+N-M} k e_n$ for almost all n . Therefore $p^{n+N-M} k$ must be divisible by p^n , which means that the polynomial $k' := p^{N-M} k$ has integral coefficients; thus (2.11) beautifies to

$$(2.12) \quad p^{n-N} e_n f = p^{n-N} k' e_n$$

for all but finitely many n .

Finally, we can conclude that $f - k'$ is zero on all but finitely many of the $p^{n-N} e_n$, hence $f - k'$ is small on $G_{h,c}[p^N]$. $\square$

It remains to show that the discovered endomorphism rings ensure that the groups $G_{h,c}$ and G_h are Hopfian. We thank the referee for suggesting the Theorem of Irwin and Walker to shorten our original proof.

Proposition 2.2. *Let G be a semi-standard reduced abelian p -group with a strictly height-increasing monomorphism x , i.e., $h(gx) > h(g)$ for all $g \in G \setminus \{0\}$. Assume $\text{End } G = \widehat{J_p[x]} \oplus \text{small}(G)$. Then G is Hopfian but not co-Hopfian.*

Proof. Obviously, G is not co-Hopfian, as x is a monomorphism which is not an isomorphism. To prove G is Hopfian, let $f: G \rightarrow G$ be an epimorphism. We will show that it is an isomorphism adapting [P, 16.14].

If f is small, then $G = f(G) = f(B)$, where B is basic in G . Since G is semi-standard, B is countable, and so too is G . As a consequence of Zippins Theorem (see [F, Proposition 77.5]) one can write $G = \bigoplus_{i < \omega} G_i$ where G_i is non-trivial. It follows that the canonical image of $\text{End}(G)$ in $\text{Hom}(G[p], G)$ has size the continuum: if $\delta_n : G_n[p] \rightarrow G$ is either the zero or the identity map on $G_n[p]$, then the vector $\underline{\delta} = (\delta_0, \delta_1, \dots)$ represents a homomorphism: $G[p] \rightarrow G$. Observe that there are continuum many such homomorphisms. Now if $\epsilon_n : G_n \rightarrow G$ is the corresponding zero or identity map on G_n , then the vector $\underline{\epsilon} = (\epsilon_0, \epsilon_1, \dots)$ represents a homomorphism: $G \rightarrow G$ such that $\underline{\epsilon}$ restricts to $\underline{\delta}$ on $G[p]$.

However, the images of both $\widehat{J_p[x]}$ and $\text{small}(G)$ are only countable: First, $\widehat{J_p[x]} = \mathbb{Z}[x] + pJ_p[x]$ with the first summand countable, and the image of the second summand is clearly 0. Second, every small endomorphism φ restricted to $G[p]$ should be 0 on the cofinite subgroup $p^n G[p]$ for some n . This leaves only countably many possibilities for the restriction of φ to $G[p]$. All in all, the images of $\widehat{J_p[x]}$ and $\text{small}(G)$ in $\text{Hom}(G[p], G)$ are countable, but the image of $\text{End } G$ is continuum, a contradiction.

Therefore f is not small, hence it is of the form $f = \alpha + \phi$ where $0 \neq \alpha \in \widehat{J_p[x]}$ and $\phi \in \text{small}(G)$.

Second, we want to show that α has an invertible constant term in several steps.

As the first step, we decompose α : let $\alpha = p^k \alpha'$, where $p \nmid \alpha$. Treating α' as a power series in x , let the x^j -term be the first one whose coefficient is not divisible by p ; i.e., $\alpha' = \beta x^j + p\gamma$, where βx^j is the sum of terms of degree at least j with β having an invertible constant term, and $p\gamma$ is the sum of terms of x -degree less than j . In particular, β is height-preserving, i.e., $h(g\beta) = h(g)$ for all $g \in G$, hence it is a monomorphism. It follows that

$$(2.13) \quad \alpha = p^k x^j \beta + p^{k+1} \gamma, \quad \beta, \gamma \in \widehat{J_p[x]}, \quad \deg \gamma < j.$$

The second step is determining the kernel of α :

$$(2.14) \quad \ker \alpha[p^{k+1}] = \ker p^k x^j \beta[p^{k+1}] = \ker p^k 1_G[p^{k+1}] = G[p^k]$$

using that x and β are monomorphisms, and $p^{k+1}\gamma$ is 0 on $G[p^{k+1}]$. We conclude that

$$(2.15) \quad \ker \alpha = G[p^k].$$

Since ϕ is small, there is a natural number N such that

$$(2.16) \quad p^N G[p^{k+1}] \subseteq \ker \phi.$$

Let $n > N$ be arbitrary. As

$$(2.17) \quad p^n (G[p^n] f^{-1})[p^{k+1}] \subseteq \ker f \cap \ker \phi \subseteq \ker \alpha = G[p^k],$$

we have $p^{n-1} (G[p^n] f^{-1})[p^{k+2}] \subseteq G[p^{k+1}]$ (cf. [P, 16.1(d)]), hence

$$(2.18) \quad p^{n-1} (G[p^n] f^{-1}) = p^{n-1} (G[p^n] f^{-1})[p^{k+1}].$$

Consequently (cf. [P, 16.1(b)])

$$(2.19) \quad \begin{aligned} p^{n-1} G[p] &= p^{n-1} (G[p^n]) = p^{n-1} (G[p^n] f^{-1}) f = p^{n-1} (G[p^n] f^{-1})[p^{k+1}] f \\ &= p^{n-1} (G[p^n] f^{-1})[p^{k+1}] \alpha = p^{n-1} (G[p^n] f^{-1})[p^{k+1}] p^k x^j \beta \subseteq p^{n-1+k+j} G[p], \end{aligned}$$

using that f is epic, ϕ and $p^{k+1}\gamma$ are 0 on $p^{n-1}(G[p^n]f^{-1})[p^{k+1}]$, and that x is strictly height-increasing.

So we have $p^{n-1}G[p] \subseteq p^{n-1+k+j}G[p]$ for all $n > N$. As G is unbounded, this is only possible if $k = j = 0$, i.e., $\alpha = \beta$. Therefore α is also height-preserving.

Finally, let $B = \bigoplus_{n=1}^{\infty} B_n$ be a basic subgroup of G with B_n a direct sum of cyclic groups of order p^n . Taking the N from above (chosen to satisfy (2.16)), the canonical direct complement of $B_1 \oplus \cdots \oplus B_N$ is $H_N := \langle p^N G, B_{N+1}, B_{N+2}, \dots \rangle$. Now on $H_N[p] = p^N G[p]$, the small map ϕ is 0 by (2.16), so f coincides with α here. Hence f is height-preserving on $H_N[p] = p^N G[p]$. It follows that f is injective on H_N and $H_N f$ is a direct complement of $B_1 \oplus \cdots \oplus B_N$ by [IW, Theorem 16]. This leads to the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_N & \longrightarrow & G & \longrightarrow & B_1 \oplus \cdots \oplus B_N \longrightarrow 0 \\ & & \cong \downarrow & & f \downarrow & & \downarrow \\ 0 & \longrightarrow & H_N f & \longrightarrow & G & \longrightarrow & B_1 \oplus \cdots \oplus B_N \longrightarrow 0 \end{array}$$

Since f is epic, the right-most vertical arrow is epic. As the B_n are finite, this arrow must be an isomorphism. By the Five Lemma, therefore, f is an isomorphism. $\square$

Finally, we conclude from Lemma 2.1 and Proposition 2.2 that G_h is Hopfian, while it is obviously not co-Hopfian. It also follows from Lemma 2.1 that $G_{h,c}$ is both Hopfian and co-Hopfian. Obviously, both groups have size $\aleph_1$, establishing the Hopfian parts (1.a) and (1.b) of Theorem 1.1 (1).

3. A CO-HOPFIAN BUT NOT HOPFIAN GROUP

We follow an analogous strategy as in the previous section to construct a co-Hopfian group which is not Hopfian. Therefore we outline only the differences.

This time we use the equivalent forcing poset $\text{Fin}(\omega_1 \times \mathbb{Z}^+ \times \mathbb{Z}^+, 2)$, so the newly added functions are $\eta_\alpha: \mathbb{Z}^+ \times \mathbb{Z}^+ \rightarrow \{0, 1\}$. Let $C = t(\prod_{n=1}^{\infty} \bigoplus_{m=1}^n \mathbb{Z}_{p^n} e_{n,m})$ and put

$$(3.1) \quad a_{\alpha,n}^{(i)} := \sum_{k=n+i}^{\infty} \sum_{m=1}^k \eta_\alpha(k-i, m) p^{k-n} e_{k,m} \quad \text{for } \alpha < \omega_1, i \geq 0, n \geq 1.$$

We define

$$(3.2) \quad G_c := \langle e_{n,m}, a_{\alpha,n}^{(i)} x^k : n \geq m \geq 1, k, i \geq 0 \rangle \subseteq C,$$

where $x: C \rightarrow C$ is defined by

$$(3.3) \quad e_{n,n} x := 0$$

$$(3.4) \quad e_{n+1,m} x := e_{n,m} \quad m \leq n.$$

Hence

$$(3.5) \quad a_{\alpha,n+1}^{(i+1)} x = a_{\alpha,n}^{(i)} + \sum_{m=1}^{n+i} \eta_\alpha(n, m) p^i e_{n+i,m},$$

$$(3.6) \quad a_{\alpha,n}^{(i)} = \sum_{m=1}^{n+i} \eta_\alpha(n, m) p^i e_{n+i,m} + p a_{\alpha,n+1}^{(i)}.$$

Note that a common basic subgroup of C and G_c is

$$(3.7) \quad B := \bigoplus_{n=1}^{\infty} \bigoplus_{m=1}^n \mathbb{Z}_{p^n} e_{n,m}.$$

It is clear from the above that x is an epimorphism of C and G_c .

As in the last section we determine the endomorphism ring of G_c . Note that $C[p]x = 0$, and hence the (p, x) -adic completion $J_p[[x]]$ of $\mathbb{Z}[x]$ acts on C .

Lemma 3.1.

$$(3.8) \quad \text{End } G_c = J_p[[x]] \oplus \text{small}(G_c).$$

Proof. The proof is analogous to that of Lemma 2.1, so we point out only the differences.

Now the reformulation of the statement reads: for every homomorphism f and every positive integer N , there is a polynomial $k' \in \mathbb{Z}[x]$ making $f - k'$ small on $G_c[p^N]$.

This time the forcing argument provides an α such that

$$(3.9) \quad a_{\alpha,N}^{(0)} f = g + k a_{\alpha,M}^{(i)} + \sum_{j=1}^l k_j a_{\alpha_j, M_j}^{(i_j)}$$

for some $g \in B$, polynomials $k, k_i \in \mathbb{Z}[x]$, non-negative integer i and pairwise distinct ordinals α, α_j . We may assume that the constant term of k is not divisible by p^M when $i > 0$. Moreover (still by the forcing argument via switching $\eta_\alpha(n, m)$), for all but finitely many pairs n, m of positive integers with $n \geq N$,

$$(3.10) \quad \left(a_{\alpha,N}^{(0)} \pm p^{n-N} e_{n,m} \right) f = g + k \left(a_{\alpha,M}^{(i)} \pm p^{n+i-M} e_{n+i,m} \right) + \sum_{j=1}^l k_j a_{\alpha_j, M_j}^{(i_j)},$$

with the convention that $e_{p,q} = 0$ for $p < q$.

The difference of the two equations (3.9) and (3.10) reduces to

$$(3.11) \quad p^{n-N} e_{n,m} f = k p^{n+i-M} e_{n+i,m}$$

(we repeat) for all but finitely many pairs n, m with $n \geq N$. If $i > 0$, then choosing $m = n + i$ and n large enough, we obtain $0 = p^{n-N} e_{n,n+i} f = k p^{n+i-M} e_{n+i,n+i}$, contradicting that the constant term of k is not divisible by p^M .

Hence $i = 0$, and we finish as in Lemma 2.1, deriving that $f - k'$ is small on $G_c[p^N]$, where $k' := p^{N-M} k$ is an integral polynomial. $\square$

Finally, we have to show that the above group is co-Hopfian.

Proposition 3.2. *Let G be a semi-standard reduced abelian p -group with an endomorphism x , which is 0 on $G[p]$. Assume $\text{End } G = J_p[[x]] \oplus \text{small}(G_h)$. Then G is co-Hopfian (but not Hopfian if x is epic).*

Proof. To prove G is co-Hopfian, let $f: G \rightarrow G$ be a monomorphism. We will show that it is an isomorphism.

First, f is of the form $f = \alpha + \phi$, where $\alpha \in J_p[[x]]$ and $\phi \in \text{small}(G)$.

Let n be a natural number. Since ϕ is small, there is a natural number m such that

$$(3.12) \quad p^m G[p^n] \subseteq \ker \phi,$$

so f coincides with α on $p^m G[p^n]$.

Consider the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & p^m G[p^n] & \longrightarrow & G[p^n] & \longrightarrow & \frac{G[p^n]}{p^m G[p^n]} \longrightarrow 0 \\ & & \alpha \downarrow & & f \downarrow & & \downarrow \\ 0 & \longrightarrow & p^m G[p^n] & \longrightarrow & G[p^n] & \longrightarrow & \frac{G[p^n]}{p^m G[p^n]} \longrightarrow 0 \end{array}$$

Since f is monic, the left-most vertical arrow α is monic. Hence α must have a constant term not divisible by p , so α is invertible and therefore an isomorphism. It follows that the right-most vertical arrow is a monomorphism. As $G[p^n]/p^m G[p^n]$ is finite, this arrow must be an isomorphism. By the Five Lemma, therefore f is an isomorphism on $G[p^n]$ for all n , and hence f is also an isomorphism of G . $\square$

It is easy to see that G_c is co-Hopfian by Lemma 3.1 and Proposition 3.2. As x is an epimorphism but not a monomorphism, G_c is not Hopfian, proving Theorem 1.1(1.c).

4. MARTIN'S AXIOM AND HOPFIAN GROUPS

In this section we show that $\text{MA}(\sigma\text{-centered})$ implies that every reduced infinite p -group G of size less than $2^{\aleph_0}$ is neither Hopfian nor co-Hopfian, i.e., Theorem 1.1(2).

Let G be a reduced p -group. We may assume that G is semi-standard, as otherwise it is neither Hopfian nor co-Hopfian. Let B be a basic subgroup of G .

We shall build endomorphisms of G by gluing together partial endomorphisms defined on finite subgroups. The partial endomorphisms will be selected via Martin's Axiom. To ensure the countable chain condition, we voluntarily restrict to endomorphisms f satisfying $G(f - 1) \subseteq B$ so that every element of G can be mapped to only countably many others. As a by-product, this also simplifies the presentation by allowing us to restrict to separable G , which we explain in the next two paragraphs.

The first Ulm factor $G/G^{(1)}$ is separable with a basic group isomorphic to B via the natural map $B \rightarrow G/G^{(1)}$. By abuse of notation, let B also denote this basic subgroup.

Now every endomorphism f of $G/G^{(1)}$ with the image of $f - 1_{G/G^{(1)}}$ in B lifts uniquely to an endomorphism $\tilde{f}$ of G which is the identity on $G^{(1)}$ and satisfies $G(\tilde{f} - 1_G) \subseteq B$. This is easily seen by considering $g = f - 1_{G/G^{(1)}}$ and $\tilde{g} = \tilde{f} - 1_G$ instead of f and $\tilde{f}$ respectively, with the co-domain of g and $\tilde{g}$ restricted to B . Then the condition is that $\tilde{g}$ is the composition of the natural projection $G \rightarrow G/G^{(1)}$ and g .

In particular, $\tilde{f}$ is a monomorphism or an epimorphism if and only if f is so, respectively. This reduces the problem to separable groups G .

Also note that because f induces the identity on G/B , we derive analogously that f is monic or epic, respectively, if so is the restriction $f|_B: B \rightarrow B$. In particular, f is epic if its image contains B .

From now on, let G be an infinite semi-standard separable p -group of cardinality less than the continuum. In particular G is unbounded. Let $B = \bigoplus_{n=1}^{\infty} B_n$ be a decomposition of the basic subgroup with B_n being a direct sum of cyclic subgroups $\mathbb{Z}_{p^n}$ of order p^n . Let $H_N := \langle p^N G, B_{N+1}, B_{N+2}, \dots \rangle$ be the canonical direct complement of $\bigoplus_{n=1}^N B_n$ in G .

In the following subsections we construct a non-monic epic endomorphism and a non-epic monic endomorphism of G , respectively. Therefore G is neither Hopfian nor co-Hopfian, as claimed.

4.1. Group is not Hopfian. To show that G is not Hopfian, we apply Martin's Axiom to the following poset:

Elements: partial homomorphisms $g: H \rightarrow G$, where H is a finite pure subgroup of G with $H(g-1) \subseteq B$.

Order: $h \leq g$ if and only if h is an extension of g .

Dense subsets:

- $\mathbb{D}_{x \in \text{dom}} := \{g \mid x \in \text{dom } g\}$ for $x \in G$,
- $\mathbb{D}_{b \in \text{im}} := \{g \mid b \in \text{im } g\}$ for $b \in B$,
- $\mathbb{D}_{\text{non-monic}} := \{g \mid g \text{ not monic}\}$.

Filters: $\mathcal{F}_\varphi := \{g \mid g \subseteq \varphi\}$ for φ an endomorphism of G , which is the identity on some H_n and $G(\varphi-1) \subseteq B$.

This is obviously a poset satisfying the conditions of MA(σ -centered). E.g., there are only countably many filters $\mathcal{F}_\varphi$, as there are only countably many choices of φ thanks to the restrictions $G(\varphi-1) \subseteq B$ and $\varphi|_{H_n} = 1$. These filters cover the poset, and as for any partial homomorphism $g: H \rightarrow G$, we can find a direct complement C of H in G containing some H_n . Defining φ to be g on H and the identity on C , we obtain $g \in \mathcal{F}_\varphi$. A similar but easier argument proves that the set $\mathbb{D}_{b \in \text{im}}$ is dense for $b \in B$. Now for $g: H \rightarrow G$, we choose a pure extension $K = H \oplus L$ with L a cyclic direct summand of B with sufficiently large order; thus g extends to an $h: K \rightarrow G$ with $b \in Lh \subseteq B$. It is even easier to show that the remaining sets claimed to be dense are really dense.

Applying MA(σ -centered), there is a filter intersecting the dense subsets above. Taking the union of the homomorphisms in the filter, we obtain a non-monic endomorphism f of G with $G(f-1) \subseteq B$ and $B \subseteq Gf$. It follows that f is epic but not monic, indicating that G is not Hopfian.

4.2. Group is not co-Hopfian. The construction is similar to the previous case with a suitable adjustment of the poset.

Let $0 \neq b_0 \in B$ be a fixed non-zero element of B . Our aim is to construct a monic endomorphism of G whose image does not contain b_0 .

Elements: partial monomorphisms $g: H \rightarrow G$, where H and the image Hg are finite pure subgroups of G with $b_0 \notin Hg$ and $H(g-1) \subseteq B$.

Order: $h \leq g$ if and only if h is an extension of g .

Dense subsets: $\mathbb{D}_{x \in \text{dom}} := \{g \mid x \in \text{dom } g\}$ for $x \in G$.

Filters: $\mathcal{F}_\varphi := \{g \mid g \subseteq \varphi\}$ for φ a monomorphism to G defined on a subgroup of G and extending the identity on some H_n . Moreover, the image of $\varphi-1$ lies in B and b_0 is outside the image of φ .

(For the filters, we allow φ to be defined only on a subgroup, because a priori we cannot come up with a monic endomorphism of G which is not epic. Actually, all the φ we use will be restrictions of some isomorphism of G .)

Since H_n is of finite index, there are only finitely many subgroups containing it, and hence there are altogether only countably many filters $\mathcal{F}_\varphi$.

To show that every partial monomorphism $g: H \rightarrow G$ is contained in an $\mathcal{F}_\varphi$, we proceed in several steps.

We choose decompositions $G = H \oplus C_1 \oplus H_n$ and $G = Hg \oplus C_2 \oplus H_n$ for a suitable n with $C_1, C_2 \subseteq B$. By factoring with H_n , it follows that $H \oplus C_1 \cong Hg \oplus C_2$. As the groups involved here are finite, and $H \cong Hg$ as g is a monomorphism, by the fundamental theorem of abelian groups, the primary decomposition of C_1 and C_2 should coincide, hence $C_1 \cong C_2$. We define φ to be g on H , an isomorphism to C_2 on C_1 and the identity on H_n . Hence φ is an automorphism of G extending g and satisfying $G(\varphi - 1) \subseteq B$. By restricting to a complement of an appropriate cyclic direct summand of $C_1 \oplus H_n$, we can omit b_0 from the image of φ with the domain still containing an H_m for some $m \geq n$. Hence $g \in \mathcal{F}_\varphi$.

An easier argument shows that the $\mathbb{D}_{x \in \text{dom}}$ are dense, but care must be taken to omit b_0 from the image of the extension. For example, given $g: H \rightarrow G$, we can find decompositions $G = H \oplus C_1 \oplus H_n$ with $x \in H \oplus C_1$, and $G = Hg \oplus C_2 \oplus H_n$ with $C_2 \subseteq B$ for a suitable n . It follows that $H \oplus C_1 \cong Hg \oplus C_2$, and hence $C_1 \cong C_2$.

We define the extension $h: H \oplus C_1 \rightarrow G$ of g . Obviously, it should coincide with g on H . On the other part, C_1 , let h be a modification of the inclusion of C_1 into $G = Hg \oplus C_2 \oplus H_n$ by replacing the C_2 -component with an isomorphism $C_1 \cong C_2$. Thus, $G = Hg \oplus C_1 h \oplus H_n$, hence h is an extension of g to an embedding onto a finite pure subgroup. The definition was carefully made to keep the image of $h - 1$ inside B .

However, there is no guarantee that b_0 hasn't smuggled itself into the image. As a remedy, let's consider a variant h' of h : since $H_n \cap B$ is reduced and unbounded, there is an embedding $\alpha: C_1 \rightarrow H_n \cap B$. Now $h' := h + \alpha$ has the same properties as h : it extends g , it embeds its domain into a finite pure subgroup, and the image of $h' - 1$ is in B . Moreover, $\text{im } h \cap \text{im } h' = Hg$, and therefore b_0 cannot be contained in the image of both h and h' , so at least one of them is an extension of g in $\mathbb{D}_{x \in \text{dom}}$.

We conclude that the poset satisfies the conditions of MA(σ -centered). So there is a filter intersecting all the dense subsets above. The union of the monomorphisms of the filter produces a monic endomorphism f of G with $G(f - 1) \subseteq B$ and $b_0 \notin Gf$. Hence f is monic but not epic, showing that G is not co-Hopfian.

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