A NOTE ON THE DZIOBEK CENTRAL CONFIGURATIONS

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ABSTRACT. For the Newtonian n -body problem in $\mathbb{R}^{n-2}$ with $n \geq 3$ we prove that the following two statements are equivalent.

- (a) Let x be a Dziobek central configuration having one mass located at the center of mass.
- (b) Let x be a central configuration formed by $n - 1$ equal masses located at the vertices of a regular $(n - 2)$ -simplex together with an arbitrary mass located at its barycenter.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

The main problem of the classical Celestial Mechanics is the *n-body problem*; i.e. the description of the motion of n particles of positive masses under their mutual Newtonian gravitational forces. This problem is completely solved only when $n = 2$, and for $n > 2$ there are only few partial results.

The equations of motion of the Newtonian n -body problem in the d -dimensional space $\mathbb{R}^d$ are

$$\ddot{x}_i = \sum_{j=1, j \neq i}^n \frac{m_j(x_j - x_i)}{r_{ij}^3}, \quad \text{for } i = 1, \dots, n,$$

where $x_i \in \mathbb{R}^d$ are the positions of the bodies, $r_{ij} = |x_i - x_j|$ are their mutual distances, and m_i are their masses.

The vector $x = (x_1, \dots, x_n) \in \mathbb{R}^{nd}$ denotes the *configuration* of the system formed by the n bodies. Clearly the differential equations of motion are well-defined when the configuration is of non-collision, that is when $r_{ij} \neq 0$ for $i \neq j$.

The *dimension* $\delta(x)$ of a configuration x is defined as the dimension of the smallest affine subspace of $\mathbb{R}^d$ which contains all of the points x_i . Of course, the configurations with $\delta(x) = 1, 2, 3$ are called *collinear*, *planar* and *spatial*, respectively. The dimension $\delta(x)$ of a non-collision configuration of $n \geq 2$ bodies satisfies $1 \leq \delta(x) \leq n - 1$.

The *total mass* and the *center of mass* of the n bodies are defined as

$$M = m_1 + \dots + m_n, \quad c = \frac{1}{M} (m_1 x_1 + \dots + m_n x_n),$$

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respectively. A configuration x is a *central configuration* if the acceleration vectors of the bodies satisfy

$$(1) \quad \sum_{j=1, j \neq i}^n \frac{m_j(x_j - x_i)}{r_{ij}^3} + \lambda(x_i - c) = 0, \quad \text{for } i = 1, \dots, n.$$

Central configurations started to be studied in the second part of the 18th century, and there is extensive literature concerning these solutions. For a classical background, see the sections on central configurations in the books of Wintner [18] and Hagiwara [7]. For a modern background see, for instance, the papers of Albouy and Chenciner [3], Albouy and Kaloshin [4], Hampton and Moeckel [8], Moeckel [10], Palmore [14], Saari [15], Schmidt [16], Xia [19], ... One of the reasons why central configurations are important is that they allow us to obtain the unique explicit solutions in function of the time of the n -body problem known until now, as the *homographic solutions* for which the ratios of the mutual distances between the bodies remain constant. They are also important because the total collision or the total parabolic escape at infinity in the n -body problem is asymptotic to central configurations, see for more details Dziobek [6] and [15]. Also if we fix the total energy h and the angular momentum c of the n -body problem, then some of the bifurcation points (h, c) for the topology of the level sets with energy h and angular momentum c are related with the central configurations, see Meyer [12] and Smale [17] for a full background on these topics.

Moulton [13] proved that for a fixed mass vector $m = (m_1, \dots, m_n)$ and a fixed ordering of the bodies along the line, there exists a unique collinear central configuration, up to translation and scaling.

At the other extreme of the dimension range, Lagrange [9] showed that for $n = 3$, the only central configuration x with $\delta(x) = n - 1 = 2$ is the equilateral triangle, and it is central for all choices of the masses. An analogous result of Lagrange's result holds for all n . Thus, it is well known that for $n \geq 3$ the only central configuration x with $\delta(x) = n - 1$ of the n -body problem is formed by the vertices of a regular $(n - 1)$ -simplex, which is central for all choices of the masses. Of course, a 1-simplex is a closed interval, a 2-simplex is a closed equilateral triangle, a 3-simplex is a closed regular tetrahedron, and so on.

For other values of the dimension $\delta(x)$ the problem of finding or even counting the central configurations x of the n -body problem is very difficult, see for instance Moeckel [11]. The dimension $\delta(x) = 2$ is of course the most interesting of all, because planar central configurations give rise to physically realistic periodic orbits. For $n = 4$, Dziobek [6] formulated the planar central configuration problem in terms of mutual distances r_{ij} and obtained algebraic equations characterizing the central configurations. His approach has been adopted and developed by Albouy [1, 2] in his study of the central configurations with four equal masses, see also Albouy and Llibre [5]. The natural generalization to higher n is the case $\delta(x) = n - 2$. Following [1] and [11] we call such central configurations *Dziobek configurations*.

The goal of this paper is to prove the following result.

Theorem 1. *The following two statements are equivalent for the n -body problem with $n \geq 3$.*

- (a) *Let x be a central configuration with $\delta(x) = n - 2$ (i.e. a Dziobek central configuration) having one mass located at the center of mass.*

- (b) *Let x be a central configuration formed by $n - 1$ equal masses located at the vertices of a regular $(n - 2)$ -simplex together with an arbitrary mass located at its barycenter.*

Of course, statement (b) of Theorem 1 implies immediately statement (a). The converse implication is proved in section 3.

If the central configuration is not Dziobek, then the equivalence of Theorem 1 does not hold. Thus, for instance, consider a regular polygon with $n - 1 > 3$ equal masses located at its vertices and an arbitrary mass located at its barycenter. This is a non-Dziobek central configuration having one mass located at the center of mass, and since $n > 4$ it is different from the configuration formed by the vertices of a regular $(n - 2)$ -simplex together with its barycenter.

2. EQUATIONS FOR THE DZIOBEK CENTRAL CONFIGURATIONS

Since we want to study the Dziobek central configurations, we consider the n -body problem in $\mathbb{R}^{n-2}$. To each configuration $x = (x_1, \dots, x_n) \in \mathbb{R}^{n(n-2)}$ we associate the $n \times n$ matrix

$$X = \begin{pmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_n \\ 0 & \cdots & 0 \end{pmatrix}.$$

Let X_k be the $(n - 1) \times (n - 1)$ matrix obtained deleting from the matrix X its k -th column and its last row. Then define $\Delta_k = (-1)^{k+1} \det(X_k)$ for $k = 1, \dots, n$.

Dziobek [6] (see also equations (8) and (16) of Moeckel [11]) reduces the equations for the central configurations (1) of the n -body problem to the following system of

$$N = \frac{n(n-1)}{2} + n + 2$$

equations and N unknowns:

$$(2) \quad \begin{aligned} \frac{1}{r_{ij}^3} &= c_1 + c_2 \frac{\Delta_i \Delta_j}{m_i m_j}, \\ t_i - t_j &= 0, \end{aligned}$$

for $1 \leq i < j \leq n$, with

$$t_i = \sum_{j=1, j \neq i}^n \Delta_j r_{ij}^2.$$

The N unknowns in equations (2) are the $n(n - 1)/2$ mutual distances r_{ij} , the n variables Δ_i , and the two constants c_k .

3. PROOF OF THEOREM 1

In order to complete the proof of Theorem 1, we must prove that statement (a) of that theorem implies statement (b).

We assume that $x = (x_1, \dots, x_n) \in \mathbb{R}^{n(n-2)}$ is a Dziobek central configuration having one mass located at the center of masses. Without loss of generality we can suppose:

- (i) the center of mass is at the origin of coordinates;
- (ii) the mass m_n is located at the center of mass, i.e. $x_n = 0$;

(iii) the unit of mass is taken in such a way that $m_{n-1} = 1$. Then

$$x_{n-1} = - \sum_{i=1}^{n-2} m_i x_i.$$

Since x is a Dziobek central configuration it satisfies (2). Easy computations using the properties of the determinants show that

$$(3) \quad \begin{aligned} \frac{\Delta_i \Delta_j}{m_i m_j} &= \det(x_1 \cdots x_{n-2})^2 \quad \text{for } 1 \leq i < j \leq n-1, \\ \frac{\Delta_i \Delta_n}{m_i m_n} &= \frac{M - m_n}{m_n} \det(x_1 \cdots x_{n-2})^2 \quad \text{for } i = 1, \dots, n-1. \end{aligned}$$

Then, the first equations of (2) become

$$(4) \quad \begin{aligned} \frac{1}{r_{ij}^3} &= c_1 + c_2 \det(x_1 \cdots x_{n-2})^2 \quad \text{for } 1 \leq i < j \leq n-1, \\ \frac{1}{r_{in}^3} &= c_1 + c_2 \frac{M - m_n}{m_n} \det(x_1 \cdots x_{n-2})^2 \quad \text{for } i = 1, \dots, n-1. \end{aligned}$$

The equations $t_i = t_j$ of (2) are trivially satisfied by direct computations, but they are not relevant in this proof.

From equations (4) we obtain that

$$(5) \quad \begin{aligned} r_{ij} &= k_1 \quad \text{for } 1 \leq i < j \leq n-1, \\ r_{in} &= k_2 \quad \text{for } i = 1, \dots, n-1, \end{aligned}$$

where k_1 and k_2 are constants. Therefore, from the first equations of (5) it follows that the masses m_k for $k = 1, \dots, n-1$ are at the vertices of a regular $(n-2)$ -simplex, and from the second equations of (5) we obtain that the mass m_n is at the barycenter of this regular $(n-2)$ -simplex. Moreover, since the barycenter must be the center of mass of the $n-1$ masses located at the vertices of a regular $(n-2)$ -simplex, this forces that these $n-1$ masses must be equal. Of course, the mass m_n located at the barycenter is arbitrary. This completes the proof of the theorem.

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