F-STABILITY OF *f*-MINIMAL HYPERSURFACE

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ABSTRACT. In this paper we study the classification of the *f*-minimal hypersurface immersed in the manifold $M^n \times R$, where (M^n, g) is an Einstein manifold with positive Ricci curvature. By using the *F* functional and *F*-stability which were introduced by Huisken and Colding-Minicozzi respectively, we prove that among all complete *f*-minimal hypersurfaces with polynomial volume growth, only $M^n \times \{0\}$ is *F*-stable.

1. INTRODUCTION

A hypersurface Σ immersed into a Riemannian manifold $(\overline{M}, \overline{g})$ is called an *f*-minimal hypersurface if it satisfies that, for any point $p \in \Sigma$,

$$(1.1) \quad H = \langle \overline{\nabla} f, \nu \rangle,$$

where H is the mean curvature, f is a smooth function defined on $\overline{M}$, $\overline{\nabla} f$ is the gradient of f on $\overline{M}$, and ν is the outer unit normal. In fact, if a hypersurface Σ satisfies (1.1), then Σ is a minimal hypersurface of $\overline{M}$ with respect to the conformal metric $e^{-\frac{2f}{n}} \overline{g}$.

Let Σ be a hypersurface immersed into R^{n+1} . Given $x_0 \in R^{n+1}$ and $t_0 \geq 0$, Huisken [7] defined the functional F_{x_0, t_0} for a mean curvature flow $\Sigma_t, t < t_0$, by

$$(1.2) \quad F_{x_0, t_0} = (4\pi(t_0 - t))^{-\frac{n}{2}} \int_{\Sigma_t} e^{-\frac{|x-x_0|^2}{4(t_0-t)}} d\mu$$

and proved the functional is monotonically decreasing under the mean curvature flow. In [12], John Lott proved that Huisken's monotonicity holds when the ambient manifold is a gradient Ricci soliton. He also showed that the mean curvature soliton immersed into gradient Ricci solitons is just the *f*-minimal hypersurface, where f is the potential function. Recently, *f*-minimal hypersurfaces have been studied extensively; see [2–4, 10, 11] for instance.

A hypersurface immersed into R^{n+1} is called a self-shrinker if it satisfies

$$H = \langle x, \nu \rangle.$$

Self-shrinker is one of the most important *f*-minimal hypersurfaces by letting $\overline{M} = R^{n+1}$ and $f = \frac{|x|^2}{2}$ in (1.1), and plays a crucial role in the study of the singularity of the mean curvature flow. Using the functional (1.2), Colding and Minicozzi [6] classified the self-shrinkers with nonnegative mean curvature and polynomial

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volume growth. They showed that every self-shrinker is unstable in the usual sense, so they introduced the concept of F -stability in a natural way and classified the F -stable self-shrinkers with polynomial volume growth. In [9], Hussey classified the self-shrinkers with low index. For the harmonic heat flow, Y.-B. Zhang [14] considered the F -stability of self-similar solutions to the harmonic map flow.

In this paper, we consider the case that $\overline{M} = M^n \times R$, where (M^n, g) is an Einstein manifold with positive Ricci curvature κ , and $f = \frac{\kappa z^2}{2}$. By Myers' Theorem [13] we know (M^n, g) is compact. We denote a point $p \in \overline{M}$ by (p', z) , where $p' \in M^n$ and $z \in R$. Let $\overline{g} = g + dz^2$ be the product metric on $M^n \times R$. Thus, $(\overline{M}, \overline{g}, f)$ is a shrinking gradient Ricci soliton, and

$$\overline{Ric} + \overline{\nabla}^2 f = \kappa \overline{g}.$$

Now a hypersurface immersed in $M^n \times R$ is an f -minimal hypersurface if it satisfies

$$H = \kappa z \langle \partial z, \nu \rangle.$$

We have

Theorem 1.1. *Let $\Sigma \subset M^n \times R$ be a complete hypersurface with $H = \kappa z \langle \partial z, \nu \rangle$. Assume $\langle \partial z, \nu \rangle \geq 0$.*

- (1) *If Σ is compact, then $\langle \partial z, \nu \rangle \equiv 1$, and $\Sigma = M^n \times \{0\}$.*
- (2) *If Σ is complete noncompact and has polynomial volume growth, then $\langle \partial z, \nu \rangle \equiv 0$, and $\Sigma = \Sigma' \times R$. In this case, Σ' is a minimal hypersurface in M^n .*

Given $a \in R$, we define the functional F_a by

$$(1.3) \quad F_a(\Sigma) = \int_{\Sigma} e^{-\frac{\kappa(z^2 - 2az + a^2)}{2}} d\mu.$$

We will show that the f -minimal hypersurface in our case is the critical point of the F_a -functional defined by (1.3). Now we can introduce the concept of the stability of the F_a -functional for the f -minimal hypersurfaces. A critical point Σ for the F_a -functional is F -stable if for any compactly supported variation Σ_s with $\Sigma_0 = \Sigma$, there exists variation a_s of a that makes $F'' = (F_{a_s}(\Sigma_s))'' \geq 0$ at $s = 0$. We have that

Theorem 1.2. *$M^n \times \{0\}$ is the only compact F -stable hypersurface in $M^n \times R$.*

Theorem 1.3. *If $\Sigma \subset M^n \times R$ is a complete noncompact hypersurface with polynomial volume growth and satisfies $H = \kappa z \langle \partial z, \nu \rangle$, then Σ is F -unstable.*

In [4], by using the L_f -index, Cheng and Zhou studied the f -minimal hypersurfaces in a cylinder shrinker soliton $(S^n(\sqrt{2(n-1)}) \times R, \overline{g}, f)$ and showed that such hypersurface with L_f -index one must be either $S^n(\sqrt{2(n-1)}) \times \{0\}$ or $S^{n-1}(\sqrt{2(n-1)}) \times R$. By our Theorem 1.2, $S^n(\sqrt{2(n-1)}) \times \{0\}$ is the only compact F -stable hypersurface in $S^n(\sqrt{2(n-1)}) \times R$.

The rest of this paper is organized as follows. In Section 2, we compute the first variation formula of the F_a -functional. In Section 3, we compute the second variation formula of the F_a -functional at the critical point and we present the proof of Theorem 1.1. In Section 4, we consider the F -stable f -minimal hypersurface and give the proofs of Theorem 1.2 and Theorem 1.3.

2. FIRST VARIATION AND f -MINIMAL HYPERSURFACE

Lemma 2.1. *Let $\Sigma_s \subset \overline{M}$ be a variation of the hypersurface $\Sigma = \Sigma_0$ with variation vector field $\Sigma'_0 = \varphi\nu$, and let $a_s \in R$ be the variation of a with $a' = h$. Then*

$$(2.1) \quad \frac{\partial}{\partial s} \Big|_{s=0} F_{a_s}(\Sigma_s) = \int_{\Sigma} [\varphi(H - \kappa z \langle \nu, \partial z \rangle + a\kappa \langle \partial z, \nu \rangle) + \kappa h(z - a)] e^{-\frac{\kappa}{2}(z-a)^2} d\mu.$$

Proof. Using $\frac{\partial X_s}{\partial s} \Big|_{s=0} = \varphi\nu$ and $\frac{\partial a_s}{\partial s} \Big|_{s=0} = h$, we get

$$(2.2) \quad (d\mu)' = \varphi H d\mu.$$

The derivative of $e^{-\frac{\kappa}{2}(z-a_s)^2}$ at $s = 0$ is

$$(2.3) \quad -e^{-\frac{\kappa}{2}(z-a)^2} \kappa(z-a)(\varphi \langle \partial z, \nu \rangle - h).$$

Combining (2.2) and (2.3) gives the desired result. $\square$

Let $\Sigma \subset M^n \times R$ be a smooth immersed hypersurface, and let Δ and ∇ be the Laplacian and gradient on Σ , respectively. We define the linear elliptic operator as follows:

$$\ell v = \Delta v - \kappa z \langle \partial z, \nabla v \rangle,$$

where v is any smooth function on Σ . By Stokes' theorem, we have the following.

Lemma 2.2. *If $\Sigma \subset M^n \times R$ is a hypersurface, $u \in C_c^1(\Sigma)$, and $v \in C^2(\Sigma)$, then*

$$\int_{\Sigma} u(\ell v) e^{-\frac{\kappa z^2}{2}} = - \int_{\Sigma} \langle \nabla u, \nabla v \rangle e^{-\frac{\kappa z^2}{2}}.$$

By a similar argument as in Corollary 3.10 in [6], we can obtain

Corollary 2.1. *Suppose $\Sigma \subset M^n \times R$ is a complete hypersurface, and u and v are C^2 functions satisfying*

$$\int_{\Sigma} (|u \nabla v| + |\nabla u| |\nabla v| + |u \ell v|) e^{-\frac{\kappa z^2}{2}} < \infty.$$

Then

$$\int_{\Sigma} u(\ell v) e^{-\frac{\kappa z^2}{2}} = - \int_{\Sigma} \langle \nabla u, \nabla v \rangle e^{-\frac{\kappa z^2}{2}}.$$

Next we show the behaviour of ℓ on the coordinate function.

Lemma 2.3. *If $\Sigma \subset M^n \times R$ is a hypersurface with $H = \kappa z \langle \partial z, \nu \rangle$, then*

$$\ell z = -\kappa z,$$

$$\ell z^2 = 2 - 2\kappa z^2 - 2\langle \partial z, \nu \rangle^2.$$

Proof. We denote by $\overline{\Delta}$ and $\overline{\nabla}$ the Laplacian and gradient on $\overline{M}$, respectively. Then

$$\Delta z = \overline{\Delta} z - (\overline{\nabla}^2 z)(\nu, \nu) - H \langle \overline{\nabla} z, \nu \rangle,$$

$$\overline{\Delta} z = 0, \quad \overline{\nabla}^2 z(\nu, \nu) = 0.$$

Using $H = \kappa z \langle \partial z, \nu \rangle$, we obtain

$$\begin{aligned} \ell z &= \Delta z - \kappa z \langle \partial z, \nabla z \rangle \\ &= -\kappa z \langle \partial z, \nu \rangle^2 - \kappa z \langle (\partial z)^T, (\partial z)^T \rangle \\ &= -\kappa z. \end{aligned}$$

By the same argument,

$$\begin{aligned}\Delta z^2 &= \overline{\Delta} z^2 - (\overline{\nabla}^2 z^2)(\nu, \nu) - H \langle \overline{\nabla} z^2, \nu \rangle, \\ \overline{\Delta} z^2 &= 2, \quad \overline{\nabla}^2 z^2(\nu, \nu) = 2 \langle \partial z, \nu \rangle^2,\end{aligned}$$

and we get

$$\begin{aligned}\ell z^2 &= \Delta z^2 - \kappa z \langle \nabla z^2, \partial z \rangle \\ &= 2 - 2 \langle \partial z, \nu \rangle^2 - 2 \kappa z^2 \langle \partial z, \nu \rangle^2 - 2 \kappa z^2 \langle (\partial z)^T, (\partial z)^T \rangle \\ &= 2 - 2 \kappa z^2 - 2 \langle \partial z, \nu \rangle^2.\end{aligned}\quad \square$$

Corollary 2.2. *If $\Sigma \subset M^n \times R$ is a hypersurface with $H = \kappa z \langle \partial z, \nu \rangle$ and polynomial volume growth, then*

$$(2.4) \quad \int_{\Sigma} z e^{-\frac{\kappa z^2}{2}} = 0,$$

$$(2.5) \quad \int_{\Sigma} \kappa z^2 e^{-\frac{\kappa z^2}{2}} = \int_{\Sigma} (1 - \langle \partial z, \nu \rangle^2) e^{-\frac{\kappa z^2}{2}}.$$

Proof. Using $|\nabla z| \leq 1$, $|\ell z| = \kappa|z|$ and the polynomial volume growth of Σ , we have

$$\int_{\Sigma} e^{-\frac{\kappa z^2}{2}} < \infty \quad \text{and} \quad \int_{\Sigma} \kappa |z| e^{-\frac{\kappa z^2}{2}} < \infty.$$

Letting $u = 1$ and $v = z$ in Corollary 2.1 gives (2.4).

Similarly, we can get $|\nabla z^2| \leq 2|z|$, $|\ell z^2| \leq 2 + 2\kappa z^2$, and

$$(2.6) \quad \int_{\Sigma} (2\kappa z^2 + 2) e^{-\frac{\kappa z^2}{2}} < \infty.$$

Now setting $u = 1$ and $v = z^2$ in Corollary 2.1 yields (2.5). $\square$

Remark 2.1. If the Ricci curvature of M^n is negative, i.e., $\kappa < 0$, and assuming Σ is a compact hypersurface, then $\langle \partial z, \nu \rangle \equiv 0$ and $z(x) \equiv 0$ for any point $x \in \Sigma$, so $\Sigma = M^n \times \{0\}$. In particular, if M^n is a complete noncompact Einstein manifold with negative Ricci curvature, then there exists no compact f -minimal hypersurface in $M^n \times R$.

By Lemma 2.1 and Corollary 2.2, we can get the following.

Proposition 2.1. *Σ is a critical point for $F_a(\Sigma)$ if and only if $H = \kappa z \langle \partial z, \nu \rangle - a\kappa \langle \partial z, \nu \rangle$.*

Proof. For any point $x = (x', z) \in \overline{M}$, let $\tilde{x} = (x', z - a)$ and $\tilde{\Sigma} = \{\tilde{x} = (x', z - a) | x \in \overline{M}\}$. If Σ satisfies

$$H = \kappa z \langle \partial z, \nu \rangle - a\kappa \langle \partial z, \nu \rangle,$$

then

$$H(\tilde{x}) = \kappa z(\tilde{x}) \langle \partial z, \nu \rangle$$

for any point $\tilde{x} \in \tilde{\Sigma}$, and

$$\int_{\Sigma} \kappa h(z - a) e^{-\frac{\kappa(z-a)^2}{2}} = \int_{\tilde{\Sigma}} z e^{-\frac{\kappa z^2}{2}} = 0,$$

where the second equality follows from Corollary 2.2. $\square$

3. SECOND VARIATION OF $F_a(\Sigma)$

Lemma 3.1. *If Σ_s is a normal variation of Σ , and a_s is a variation of a_0 with*

$$\partial_s|_{s=0} \Sigma_s = \varphi\nu, \partial_s|_{s=0} a_s = h,$$

$$\partial_s|_{s=0} \varphi = \varphi', \partial_{ss}|_{s=0} a_s = h',$$

then, setting $F'' = \partial_{ss}|_{s=0} (F_{a_s}(\Sigma_s))$, we have

$$\begin{aligned} F'' &= \int_{\Sigma} \{(-\varphi L_{a_0} \varphi + 2\kappa h \varphi \langle \partial z, \nu \rangle - \kappa h^2) \\ &\quad + (\varphi(H - \kappa z \langle \partial z, \nu \rangle + a_0 \kappa \langle \partial z, \nu \rangle) + \kappa h(z - a_0))^2 \\ &\quad + \varphi'(H - \kappa z \langle \partial z, \nu \rangle + a_0 \kappa \langle \partial z, \nu \rangle) + \kappa h'(z - a_0)\} e^{-\frac{\kappa}{2}(z-a_0)^2} d\mu. \end{aligned} \quad (3.1)$$

Here $L_{a_0} = \Delta - \kappa(z - a_0) \langle \nabla, \partial z \rangle + |A|^2 + \kappa$.

Proof. We use primes to denote the derivatives with respect to s at $s = 0$. For any smooth function ψ defined on Σ , we use $[\cdot]$ to denote the weighted integral, i.e.,

$$[\psi] = \int_{\Sigma} \psi e^{-\frac{\kappa z^2}{2}} d\mu.$$

Then

$$\begin{aligned} F'' &= [\varphi(H - \kappa z \langle \nu, \partial z \rangle + a_s \kappa \langle \partial z, \nu \rangle)]' + \kappa(hz - a_s h)' \\ &\quad + (\varphi(H - \kappa z \langle \nu, \partial z \rangle + a_0 \kappa \langle \partial z, \nu \rangle) + \kappa(hz - a_0 h))^2 \\ &\quad + \varphi'(H - \kappa z \langle \nu, \partial z \rangle + a_0 \kappa \langle \partial z, \nu \rangle)]. \end{aligned} \quad (3.2)$$

By standard calculations (see [8]), we can get

$$\begin{aligned} H' &= -\Delta \varphi - |A|^2 \varphi - \overline{Ric}(\nu, \nu) \varphi, \\ \nu' &= -\nabla \varphi. \end{aligned} \quad (3.3)$$

Note that

$$\overline{Ric} + \overline{\nabla}^2 \left(\frac{\kappa z^2}{2} \right) = \kappa \bar{g},$$

so

$$\overline{Ric}(\nu, \nu) + \overline{\nabla}^2 \frac{\kappa z^2}{2}(\nu, \nu) = \overline{Ric}(\nu, \nu) + \kappa \langle \partial z, \nu \rangle^2 = \kappa. \quad (3.4)$$

Now we have

$$\begin{aligned} &(H - \kappa z \langle \nu, \partial z \rangle + a_s \kappa \langle \partial z, \nu \rangle)' \\ &= -\Delta \varphi - |A|^2 \varphi - \overline{Ric}(\nu, \nu) \varphi - \kappa \varphi \langle \partial z, \nu \rangle^2 + \kappa z \langle \partial z, \nabla \varphi \rangle \\ &\quad - a_0 \kappa \langle \partial z, \nabla \varphi \rangle + h \kappa \langle \partial z, \nu \rangle \\ &= -L_{a_0} \varphi + h \kappa \langle \partial z, \nu \rangle. \end{aligned} \quad (3.5)$$

The second term is given by

$$\kappa(hz - a_s h)' = \kappa(h'z + h\varphi \langle \partial z, \nu \rangle - h^2 - a_0 h'). \quad (3.6)$$

Combining (3.2), (3.5), and (3.6) gives the lemma. $\square$

Theorem 3.1. Suppose Σ is a complete hypersurface, has polynomial volume growth, and satisfies $H = \kappa z \langle \partial z, \nu \rangle$. If Σ_s is the normal variation of Σ , $a_s \in R$ is the variation with $a_0 = 0$, and

$$\partial_s|_{s=0} \Sigma_s = \varphi \nu, \partial_s|_{s=0} a_s = h,$$

then

$$F'' = \int_{\Sigma} (-\varphi L\varphi + 2\kappa h \varphi \langle \nu, \partial z \rangle - \kappa h^2 \langle \partial z, \nu \rangle^2) e^{-\frac{\kappa z^2}{2}} d\mu.$$

Here $L = \Delta - \kappa z \langle \partial z, \nabla \cdot \rangle + |A|^2 + \kappa$.

Proof. Since $H = \kappa z \langle \partial z, \nu \rangle$ and $a_0 = 0$, by Lemma 3.1,

$$F'' = \int_{\Sigma} (-\varphi L\varphi + 2\kappa h \varphi \langle \partial z, \nu \rangle - \kappa h^2 + \kappa^2 h^2 z^2) e^{-\frac{\kappa z^2}{2}},$$

where we have used (2.4). By (2.5), it follows that

$$F'' = \int_{\Sigma} (-\varphi L\varphi + 2\kappa h \varphi \langle \nu, \partial z \rangle - \kappa h^2 + \kappa h^2 (1 - \langle \partial z, \nu \rangle^2) e^{-\frac{\kappa z^2}{2}} d\mu,$$

which completes the proof. $\square$

A hypersurface Σ with $H = \kappa z \langle \partial z, \nu \rangle$ is a minimal hypersurface in $(\overline{M}, e^{-\frac{\kappa z^2}{n}} \overline{g})$, and Σ is called stable [5] in the usual sense if, for any function φ with compact support,

$$\int_{\Sigma} -\varphi L\varphi e^{-\frac{\kappa z^2}{2}} d\mu \geq 0.$$

In [1], Cheng-Mejia-Zhou proved that if $(\overline{M}, \overline{g}, f)$ is a complete manifold with $\overline{Ric}_f \geq c$, where c is a positive constant, then there is no two-sided stable f -minimal hypersurface with finite weighted volume in $(\overline{M}, \overline{g}, f)$. In our case,

$$\overline{Ric}_f = \overline{Ric} + \overline{\nabla}^2 f = \kappa \overline{g}.$$

Since κ is a positive constant, there exists no stable complete f -minimal hypersurface with finite weighted volume. Now we introduce the following.

Definition 3.1. A hypersurface Σ with $H = \kappa z \langle \partial z, \nu \rangle$ is called F -stable if, for any compactly supported function φ , there exists some constant h , such that $F'' \geq 0$.

By definition, a stable f -minimal hypersurface must be F -stable. By Corollary 4 from [3], we have the following.

Lemma 3.2. $L \langle \partial z, \nu \rangle = \kappa \langle \partial z, \nu \rangle$.

We also need the following.

Lemma 3.3. Suppose ψ is a C^2 function satisfying $L\psi = -\mu\psi$, for some $\mu \in R$. If $\psi > 0$, then for any function φ in the weighted $W^{1,2}$ space, we have

$$(3.7) \quad [\varphi^2 |A|^2] \leq [|\nabla \varphi|^2 - \varphi^2 (\mu + \kappa)].$$

Proof. Since $\psi > 0$ and $\ln \psi$ is well defined,

$$\ell \ln \psi = -|\nabla \ln \psi|^2 + \frac{\Delta \psi - \kappa z \langle \partial z, \nabla \psi \rangle}{\psi} = -\mu - |A|^2 - \kappa - |\nabla \ln \psi|^2.$$

Suppose η is a function with compact support. By Lemma 2.2,

$$[(\nabla \eta^2, \nabla \ln \psi)] = -[\eta^2 \ell \ln \psi] = [\eta^2 (\mu + |A|^2 + \kappa + |\nabla \ln \psi|^2)].$$

Using

$$\langle \nabla \eta^2, \nabla \ln \psi \rangle \leq |\nabla \eta|^2 + \eta^2 |\nabla \ln \psi|^2,$$

we get

$$[\eta^2(\mu + |A|^2 + \kappa + |\nabla \ln \psi|^2)] \leq [|\nabla \eta|^2 + \eta^2 |\nabla \ln \psi|^2],$$

which yields

$$(3.8) \quad [\eta^2 |A|^2] \leq [|\nabla \eta|^2 - \eta^2(\mu + \kappa)].$$

Let η_j be 1 in $M^n \times (-j, j)$, and cut off linearly to zero in $M^n \times (-2j, 2j) \setminus M^n \times (-j, j)$. Since φ is in the weighted $W^{1,2}$ space, applying (3.8) with $\eta = \eta_j \varphi$, letting $j \rightarrow \infty$, and using the monotone convergence theorem gives (3.7). $\square$

Using Lemma 3.3, we may prove Theorem 1.1.

Proof of Theorem 1.1. (1) Consider the coordinate function z on Σ . Since Σ is compact, there exists a point $p \in \Sigma$ such that $z(p) = \max_{\Sigma} z$. So

$$|\nabla z|^2(p) = 1 - \langle \partial z, \nu \rangle^2(p) = 0.$$

Assume $\langle \partial z, \nu \rangle(p) = 1$; then the Harnack inequality gives

$$\langle \partial z, \nu \rangle(q) > 0, \forall q \in \Sigma.$$

By Lemma 3.2,

$$\begin{aligned} \ell \langle \partial z, \nu \rangle &= L \langle \partial z, \nu \rangle - (|A|^2 + \kappa) \langle \partial z, \nu \rangle \\ &= -|A|^2 \langle \partial z, \nu \rangle. \end{aligned}$$

Setting $u \equiv 1$ and $v = \langle \partial z, \nu \rangle$ in Lemma 2.2 gives

$$\int_{\Sigma} |A|^2 \langle \partial z, \nu \rangle e^{-\frac{\kappa z^2}{2}} = 0.$$

Thus $|A|^2 \equiv 0$, and $H = \kappa z \langle \partial z, \nu \rangle \equiv 0$. Since $\langle \partial z, \nu \rangle > 0$, then $z \equiv 0$. Σ is compact, so $\Sigma = M^n \times \{0\}$ and $\langle \partial z, \nu \rangle \equiv 1$.

(2) Suppose not. Then there exists some point $p \in \Sigma$ such that $\langle \partial z, \nu \rangle(p) > 0$. Applying the Harnack inequality, we get $\langle \partial z, \nu \rangle > 0$ in Σ . Obviously $\varphi \equiv 1$ is in the weighted $W^{1,2}$ space, so by Lemma 3.3, we know

$$\int_{\Sigma} |A|^2 e^{-\frac{\kappa z^2}{2}} < \infty.$$

Setting $u = 1$ and $v = \langle \partial z, \nu \rangle$ in Corollary 2.1, we obtain

$$\int_{\Sigma} |A|^2 \langle \partial z, \nu \rangle e^{-\frac{\kappa z^2}{2}} = 0.$$

By the same argument as before, we find $\Sigma = M^n \times \{0\}$, which contradicts our assumption that Σ is noncompact. Now $\langle \partial z, \nu \rangle \equiv 0$ and let

$$\Sigma' = \Sigma \cap (M^n \times \{0\});$$

then $\Sigma = \Sigma' \times R$. Since

$$H = \kappa z \langle \partial z, \nu \rangle \equiv 0,$$

Σ' is a minimal hypersurface in M^n . $\square$

4. F -STABLE f -MINIMAL HYPERSURFACE IN $M^n \times R$

In this section, we give the proofs of Theorem 1.2 and Theorem 1.3.

Proof of Theorem 1.2. Firstly, we show that $\Sigma = M^n \times \{0\}$ is F -stable. Now $H = 0$, $|A|^2 = 0$, and $\nu = \partial z$. Then $L = \Delta + \kappa$ and

$$F'' = \int_{\Sigma} -\varphi(\Delta\varphi + \kappa\varphi) + 2\kappa h\varphi - \kappa h^2.$$

We use λ_1 to denote the first nonzero eigenvalue of the Laplacian on M , then by Lichnerowicz's theorem, $\lambda_1 \geq \frac{n\kappa}{n-1}$. The eigenfunctions corresponding to $\lambda_0 = 0$ are constant functions; let E be the space of functions which are orthogonal to the constants, i.e.,

$$E = \{\psi \in W^{1,2}(\Sigma) : \int_{M^n} \psi = 0\}.$$

For any smooth function φ , choose $b \in R$ such that

$$\varphi_0 \equiv \varphi - b \in E.$$

Now we get that

$$F'' = \int_{M^n} [-(\varphi_0 + b)\Delta\varphi_0 - \kappa(\varphi_0 + b)^2 + 2\kappa h(\varphi_0 + b) - \kappa h^2].$$

By the orthogonality of the different eigenspaces, it follows that

$$F'' \geq \int_{M^n} \varphi_0^2(\lambda_1 - \kappa) + \int_{M^n} -\kappa(b^2 - 2bh + h^2).$$

Since $\lambda_1 = \frac{n\kappa}{n-1} > \kappa$, then

$$F'' \geq \int_{M^n} -\kappa(b - h)^2.$$

So F'' can be made nonnegative by choosing $h = b$.

We will show that if the compact hypersurface Σ satisfies $H = \kappa z \langle \partial z, \nu \rangle$ and $\Sigma \neq M^n \times \{0\}$, then Σ is F -unstable, i.e., there exists some function φ , such that for any constant h , we have $F'' < 0$.

Since $\Sigma \neq M^n \times \{0\}$, we know that μ_1 , the first eigenvalue of L , satisfies

$$\mu_1 < -\kappa.$$

Let φ be the eigenfunction corresponding to μ_1 with $\int_{\Sigma} \varphi^2 e^{-\frac{\kappa z^2}{2}} = 1$. Since

$$L\langle \partial z, \nu \rangle = \kappa \langle \partial z, \nu \rangle,$$

φ and $\langle \partial z, \nu \rangle$ are orthogonal in the weighted L^2 space. For any constant h , it follows that

$$F'' = \int_{\Sigma} (\mu_1 \varphi^2 - \kappa h^2 \langle \partial z, \nu \rangle) e^{-\frac{\kappa z^2}{2}} \leq \int_{\Sigma} \mu_1 \varphi^2 e^{-\frac{\kappa z^2}{2}} < 0. \quad \square$$

Now we consider the complete noncompact f -minimal hypersurface. Suppose $\Sigma \subset M^n \times R$ is a hypersurface with polynomial volume growth and satisfies $H = \kappa z \langle \partial z, \nu \rangle$.

Proof of Theorem 1.3. We will consider two separate cases.

Case 1: $\langle \partial z, \nu \rangle$ doesn't change sign, so we can assume $\langle \partial z, \nu \rangle \geq 0$. By Theorem 1.1, $\langle \partial z, \nu \rangle \equiv 0$, $\Sigma = \Sigma' \times R$, and

$$L = \Delta - \kappa z \langle \partial z, \nabla \cdot \rangle + |A|^2 + \kappa.$$

We use Δ' to denote the Laplacian on Σ' . Then

$$L = \frac{\partial^2}{\partial z^2} + \Delta' - \kappa z \frac{\partial}{\partial z} + |A|^2 + \kappa.$$

Use $\varphi(x) = \varphi(x', z)$ as a variation, where

$$\varphi(x', z) = \begin{cases} 1, & |z| \leq R, \\ 2 - \frac{|z|}{R}, & R \leq |z| \leq 2R, \\ 0, & |z| \geq 2R, \end{cases}$$

where R is a constant large enough to be determined. Then

$$\begin{aligned} F'' &= \int_{\Sigma} -\varphi L \varphi e^{-\frac{\kappa z^2}{2}} \\ &= \int_{\Sigma} -\varphi \left(\frac{\partial^2 \varphi}{\partial z^2} + \Delta' \varphi - \kappa z \frac{\partial \varphi}{\partial z} + |A|^2 \varphi + \kappa \varphi \right) e^{-\frac{\kappa z^2}{2}} \\ &= \int_{\Sigma} -(|A|^2 \varphi^2 + \kappa \varphi^2) e^{-\frac{\kappa z^2}{2}} + \int_{\Sigma \cap \{R \leq |z| \leq 2R\}} \kappa z \varphi \frac{\partial \varphi}{\partial z} e^{-\frac{\kappa z^2}{2}} \\ &\leq \int_{\Sigma} -(|A|^2 \varphi^2 + \kappa \varphi^2) e^{-\frac{\kappa z^2}{2}} + \int_{\Sigma \cap \{R \leq |z| \leq 2R\}} \kappa \left(\frac{2|z|}{R} + \frac{z^2}{R^2} \right) e^{-\frac{\kappa z^2}{2}}, \end{aligned}$$

where we have used

$$\frac{\partial^2 \varphi}{\partial z^2} = \Delta' \varphi = 0.$$

By the monotone convergence theorem,

$$\lim_{R \rightarrow \infty} \int_{\Sigma} -(|A|^2 \varphi^2 + \kappa \varphi^2) e^{-\frac{\kappa z^2}{2}} = \int_{\Sigma} -(|A|^2 + \kappa) e^{-\frac{\kappa z^2}{2}}.$$

On the other hand,

$$\lim_{R \rightarrow \infty} \int_{\Sigma \cap \{R \leq |z| \leq 2R\}} \left(\frac{2|z|}{R} + \frac{z^2}{R^2} \right) e^{-\frac{\kappa z^2}{2}} = 0,$$

then $F'' < 0$ if we choose R large enough, thus Σ is F -unstable.

Case 2: $\langle \partial z, \nu \rangle$ changes sign. We denote by μ_1 the first eigenvalue of the operator L .

Claim. $\mu_1 < -\kappa$.

Observe that

$$L \langle \partial z, \nu \rangle = \kappa \langle \partial z, \nu \rangle,$$

and that $\langle \partial z, \nu \rangle$ does not vanish identically. Then by the Lemma 9.25 in [6], we only need to show $\langle \partial z, \nu \rangle$ is in the weighted $W^{1,2}$ space. Since 1 and $|A|$ are in the weighted L^2 space, and

$$|\langle \partial z, \nu \rangle| \leq 1, |\nabla \langle \partial z, \nu \rangle| \leq |A|,$$

then $\langle \partial z, \nu \rangle$ is in the weighted $W^{1,2}$ space.

For a fixed point $p \in \Sigma$, let $B_r(p)$ denote the intrinsic ball of radius r and centered at p . Since $\mu_1 < -\kappa$, then we can choose some r large enough, denoted by

$\bar{r}$, such that the lowest eigenvalue $\mu_1(B_{\bar{r}}) < -\kappa$. Let $u_{\bar{r}}$ be the nonzero solution to the Dirichlet equation:

$$Lu_{\bar{r}} = -\mu_1(B_{\bar{r}})u_{\bar{r}}.$$

Now let $\varphi = u_{\bar{r}}$; then

$$\begin{aligned} F'' &= \int_{\Sigma} (-\varphi L\varphi + 2\kappa h\varphi\langle\partial z, \nu\rangle - \kappa h^2\langle\partial z, \nu\rangle^2) e^{-\frac{\kappa z^2}{2}} \\ &= \int_{\Sigma} (\mu_1(B_{\bar{r}})u_{\bar{r}}^2 + 2\kappa h u_{\bar{r}}\langle\partial z, \nu\rangle - \kappa h^2\langle\partial z, \nu\rangle^2) e^{-\frac{\kappa z^2}{2}} \\ &= \int_{\Sigma} [(\mu_1(B_{\bar{r}}) + \kappa)u_{\bar{r}}^2 - \kappa(u_{\bar{r}} - h\langle\partial z, \nu\rangle)^2] e^{-\frac{\kappa z^2}{2}} \\ &\leq \int_{\Sigma} (\mu_1(B_{\bar{r}}) + \kappa)u_{\bar{r}}^2 e^{-\frac{\kappa z^2}{2}} \\ &< 0. \end{aligned}$$

□

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