ISOMORPHIC SUBGROUPS OF SOLVABLE GROUPS

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ABSTRACT. Let H and K be isomorphic subgroups of a solvable group G , and suppose that H is maximal in G . We show that if either H is supersolvable, or a Sylow 2-subgroup of H is abelian, then K is also maximal in G .

Assume that H and K are isomorphic subgroups of some finite group G , and suppose that H is maximal in G . In this paper we investigate the question of whether or not K must also be maximal in G . In general, the answer is “no” since, for example, we can choose K to be an arbitrary nonabelian simple group and $G = K \times K$. Then K is clearly not maximal in G , but the diagonal subgroup H is isomorphic to K , and it is not hard to see that H must be maximal in G .

The situation is quite different, however, if G is assumed to be solvable. In a posting on the MathOverflow web site, Moshe Newman asked if it can ever happen that a *solvable* group G has isomorphic subgroups H and K , where H is maximal and K is not. Although we have not been able to answer Newman’s question fully, we do have some partial results.

Theorem A. *Let H be a maximal subgroup of a solvable group G , and suppose that $K \subseteq G$ and $K \cong H$. If H has a Sylow tower, and, in particular, if H is supersolvable, then K is maximal in G .*

Theorem B. *Let H be a maximal subgroup of a solvable group G , and suppose that $K \subseteq G$ and $K \cong H$. If a Sylow 2-subgroup of H is abelian, then K is maximal in G .*

Although our proof of Theorem A is relatively elementary, Theorem B seems deeper, and our proof relies on Glauberman’s ZJ-Theorem. We begin with some easy general observations.

Lemma 1. *Let $G = HU$, where $U \triangleleft G$ and $H \cap U = 1$, and suppose that U is solvable. Then H is a maximal subgroup of G if and only if U is a minimal normal subgroup of G .*

Proof. First suppose that H is maximal, and let $V \triangleleft G$ with $1 < V \subseteq U$. To show that $V = U$, note that $V \not\subseteq H$, and thus $HV = G$. We have $V \subseteq U \subseteq HV$, and thus by Dedekind’s lemma, $U = (H \cap U)V = V$, as wanted.

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Conversely, assume that U is minimal normal in G , and note that U is abelian since we are assuming that it is solvable. Let $H \subseteq X < G$. To show that $X = H$, observe that $X = H(X \cap U)$ by Dedekind's lemma, so it suffices to show that $X \cap U = 1$. Now $X \cap U < U$ since $H(X \cap U) = X < G = HU$. Also $XU \supseteq HU = G$ and both X and U normalize $X \cap U$, so $X \cap U \triangleleft G$. Since U is minimal normal, it follows that $X \cap U = 1$. $\square$

Since maximal subgroups of solvable groups have prime-power index, it follows in the situation of Newman's question that $|G : H|$ is a power of some prime p . Also $|H| = |K|$, so $|G : H| = |G : K|$, and thus $|G : K|$ is also a power of p . It should be clear, therefore, why the following lemma is relevant.

Lemma 2. *Let $H \subseteq G$, where $|G : H|$ is a power of a prime p . Then $\mathbf{O}_p(G) \cap H = \mathbf{O}_p(H)$.*

Proof. Since $\mathbf{O}_p(G) \cap H$ is a normal p -subgroup of H , we have $\mathbf{O}_p(G) \cap H \subseteq \mathbf{O}_p(H)$. To prove the reverse containment, consider a Sylow p -subgroup P of G , and note that $HP = G$. Then $\mathbf{O}_p(H)^G = \mathbf{O}_p(H)^{HP} = \mathbf{O}_p(H)^P \subseteq P$, and we see that $\mathbf{O}_p(H)^G$ is a normal p -subgroup of G . Then $\mathbf{O}_p(H) \subseteq \mathbf{O}_p(H)^G \subseteq \mathbf{O}_p(G)$, and we have $\mathbf{O}_p(H) \subseteq \mathbf{O}_p(G) \cap H$, as required. $\square$

We can now begin to address the Newman problem.

Theorem 3. *Let H be a maximal subgroup of a solvable group G with index a power of the prime p , and suppose that $K \subseteq G$ and $K \cong H$. If $\mathbf{O}_p(G) \not\subseteq H$, then K is maximal in G .*

Proof. We proceed by induction on $|G|$. Let $U = \mathbf{O}_p(G)$, and suppose $U \not\subseteq H$, so that $HU > H$, and thus $HU = G$. Write $V = H \cap U$, and observe that $V \triangleleft H$, so $H \subseteq \mathbf{N}_G(V)$. Now $V < U$, so $\mathbf{N}_G(V) \cap U > V$. But $H \cap U = V$, and we deduce that $\mathbf{N}_G(V) > H$. By the maximality of H , therefore, $\mathbf{N}_G(V) = G$, and we have $V \triangleleft G$. By Lemma 1 applied in the group G/V , we conclude that U/V is minimal normal in G/V , and in particular, U/V is abelian.

Now write $W = K \cap U$, and observe that $V = \mathbf{O}_p(H)$ and $W = \mathbf{O}_p(K)$ by Lemma 2, and thus $|V| = |W|$ since $H \cong K$. Since $G = HU$, we have

$$|G| = \frac{|H||U|}{|V|} = \frac{|K||U|}{|W|} = |KU|,$$

and we conclude that $KU = G$.

Now K normalizes W , and thus K normalizes WV . Also, U normalizes WV since U/V is abelian, and thus WV is normal in $KU = G$. Since U/V is a chief factor of G , it follows that either $WV = V$ or $WV = U$, and we suppose first that $WV = V$. Then $W \subseteq V$, and since we know that $|V| = |W|$, we conclude that $V = W$. Now $KU = G$ and $K \cap U = V$, and we know that U/V is minimal normal in G/V , so it follows by Lemma 1 applied in the group G/V that K is maximal in G , as required.

We can now assume that $WV = U$. Write $Y = K \cap H$, and let $X = \theta(Y)$, where $\theta : K \rightarrow H$ is an isomorphism. Then X and Y are isomorphic subgroups of H , and we wish to apply the inductive hypothesis in H , so we work to show that X is maximal in H and $\mathbf{O}_p(H) \not\subseteq X$.

First, we show that X is maximal in H , or, equivalently, that Y is maximal in K . We have $G = KU = K WV = KV$, so there exists an isomorphism $\varphi : G/V \rightarrow K/(K \cap V)$, and we see that $\varphi(H/V) = (K \cap H)/(K \cap V) = Y/(K \cap V)$. Since H/V is maximal in G/V , it follows that $Y/(K \cap V)$ is maximal in $K/(K \cap V)$, so Y is maximal in K , as wanted.

Next, to show that $\mathbf{O}_p(H) \not\subseteq X$, it suffices to show that $\mathbf{O}_p(K) \not\subseteq Y$. Recall that $W = \mathbf{O}_p(K)$, and if $W \subseteq Y$, then $W \subseteq H$, and thus $U = WV \subseteq H$, which is not the case. The inductive hypothesis now implies that Y is maximal in H . We have $KH \supseteq KV = G$, so by Dedekind's lemma, intersection with H is an injective map from the set of subgroups S with $K \subseteq S \subseteq G$ into the set of subgroups T with $K \cap H \subseteq T \subseteq H$. Since $K \cap H = Y$ is maximal in H , it follows that K is maximal in G , as wanted. $\square$

Theorem A now follows fairly easily.

Proof of Theorem A. We are given a maximal subgroup $H \subseteq G$, where G is solvable, and we assume that H has a Sylow tower, which means that every nontrivial homomorphic image of H has a nontrivial normal Sylow subgroup. Suppose that $K \subseteq G$ and $\theta : H \rightarrow K$ is an isomorphism. We proceed by induction on $|G|$ to show that K is maximal in G .

First, suppose that there is a nontrivial subgroup $N \triangleleft G$, where $N \subseteq H$ and $\theta(N) = N$. Then $N \subseteq K$, and θ induces an isomorphism from H/N to K/N . Now H/N and K/N are isomorphic subgroups of G/N , and H/N is maximal in G/N . Since H/N has a Sylow tower, the inductive hypothesis applies in G/N , and we conclude that K/N is maximal in G/N , and so K is maximal in G .

We can suppose, therefore, that there is no nontrivial subgroup N as in the previous paragraph. If p is the unique prime divisor of $|G : H|$, we can assume by Theorem 3 that $\mathbf{O}_p(G) \subseteq H$. By Lemma 2, we see that $\mathbf{O}_p(G) = \mathbf{O}_p(H)$, and thus $\theta(\mathbf{O}_p(G)) = \theta(\mathbf{O}_p(H)) = \mathbf{O}_p(K) \subseteq \mathbf{O}_p(G)$, where the containment holds by Lemma 2. It follows that $\theta(\mathbf{O}_p(G)) = \mathbf{O}_p(G)$, and since $\mathbf{O}_p(G) \triangleleft G$, we have $\mathbf{O}_p(G) = 1$, and thus also $\mathbf{O}_p(H) = 1$.

We can certainly assume that $H > 1$, and thus by hypothesis, H has a nontrivial normal Sylow q -subgroup Q for some prime q . Also, $q \neq p$ since $\mathbf{O}_p(H) = 1$, and thus Q is a full Sylow q -subgroup of G . If $Q \triangleleft G$, then since $|G : K|$ is a power of p , we see that $Q \subseteq K$. Then Q is the unique Sylow q -subgroup of K , and it follows that $\theta(Q) = Q$. Since Q is nontrivial, this is a contradiction, and we conclude that Q is not normal in G .

Now $H \subseteq \mathbf{N}_G(Q) < G$, and we deduce that $H = \mathbf{N}_G(Q)$. Since $|\theta(Q)| = |Q|$, we see that $\theta(Q)$ is a Sylow q -subgroup of G , and thus $\theta(Q) = Q^g$ for some element $g \in G$. Then $Q^g = \theta(Q) \triangleleft \theta(H) = K$, so $K \subseteq \mathbf{N}_G(Q^g) = \mathbf{N}_G(Q)^g = H^g$. But $|K| = |H| = |H^g|$, and we deduce that $K = H^g$, and thus K is maximal, as required. $\square$

In order to prove Theorem B, we need to quote two known results. The first of these is Theorem 4, below, due to H. Lausch [2]. To state it, we say that a nilpotent subgroup $U \subseteq G$ is **maximally nilpotent** in G if U is not contained in any properly larger nilpotent subgroup of G .

Theorem 4 (Lausch). *Let U and V be maximally nilpotent subgroups of a finite group G , and suppose that $\mathbf{C}_G(U \cap V) \subseteq U \cap V$. Then U and V are conjugate in G .*

Recall that a **nilpotent injector** of a solvable group G can be defined to be a maximally nilpotent subgroup of G containing the Fitting subgroup $\mathbf{F}(G)$. Clearly, G has nilpotent injectors, and in fact, G has a unique conjugacy class of such subgroups. (This well-known result is the special case of the following corollary obtained by taking $H = G$.)

Corollary 5. *Let $H \subseteq G$, where G is solvable, and suppose that $|G : H|$ and $|\mathbf{F}(G)|$ are coprime. Let U and V be nilpotent injectors of H and G , respectively. Then U and V are conjugate in G .*

Proof. Let $F = \mathbf{F}(G)$, and write π to denote the set of prime divisors of $|F|$. Since $F \subseteq V$ and V is nilpotent, the Hall π' -subgroup Q of V satisfies $Q \subseteq \mathbf{C}_G(F) \subseteq F$, and thus $Q = 1$, and we see that V is a π -group. Now $|G : H|$ is a π' -number, so H contains a Hall π -subgroup of G , and thus H contains some conjugate W of V in G . It suffices to show that U and W are conjugate.

Since V is maximally nilpotent in G and contains F , we see that W is also maximally nilpotent in G , and it too contains F . Then $F \subseteq H$, and hence $F \subseteq \mathbf{F}(H) \subseteq U$. Now U and W are maximally nilpotent in H and $F \subseteq U \cap W$. We have

$$\mathbf{C}_H(U \cap W) \subseteq \mathbf{C}_G(F) \subseteq F \subseteq U \cap W,$$

and thus U and W are conjugate in H by Theorem 4. $\square$

We also need the following version of the Glauberman ZJ-Theorem [1]. To state it, we recall that for a p -group U , there are several different definitions in the literature of a characteristic subgroup denoted $\mathbf{J}(U)$, and unfortunately, these definitions often yield different subgroups. For our purposes, we define $\mathbf{J}(U)$ to be the subgroup generated by all abelian subgroups of U having largest possible order, and we do not require that U be a p -group.

Theorem 6 (Glauberman). *Let $P \in \text{Syl}_p(G)$, where $p > 2$, and assume that a Sylow 2-subgroup of G is abelian. Suppose that $\mathbf{O}_p(G) \supseteq \mathbf{C}_G(\mathbf{O}_p(G))$. Then $\mathbf{Z}(\mathbf{J}(P))$ is characteristic in G .*

Our application of Theorem 6 will be to solvable groups, so by the Hall-Higman lemma, the condition that $\mathbf{O}_p(G) \supseteq \mathbf{C}_G(\mathbf{O}_p(G))$ will automatically be satisfied if $\mathbf{O}_{p'}(G) = 1$.

Theorem 7. *Let I be a nilpotent injector of a solvable group G , and suppose that a Sylow 2-subgroup of G is abelian. Then $\mathbf{Z}(\mathbf{J}(I))$ is characteristic in G .*

Proof. For each prime p , we show that $\mathbf{Z}(\mathbf{J}(P)) \triangleleft G$, where P is the Sylow p -subgroup of I . To see this, decompose $\mathbf{F}(G) = A \times B$, where A is a p -group and B is a p' -group. Then $A \subseteq \mathbf{F}(G) \subseteq I$, and so $A \subseteq P$. Also, since I is nilpotent and P and B are subgroups with coprime orders, we have $P \subseteq \mathbf{C}_G(B)$.

If $p = 2$, then P is abelian, and thus P centralizes A as well as B , and we have $P \subseteq \mathbf{C}_G(\mathbf{F}(G)) \subseteq \mathbf{F}(G)$. Then $P = A$, so $\mathbf{Z}(\mathbf{J}(P)) = A \triangleleft G$, as wanted. We can thus assume that $p \neq 2$.

Write $C = \mathbf{C}_G(B)$, and recall that $P \subseteq C$. We argue that, in fact, P is a full Sylow p -subgroup of C . To see this, let $P \subseteq S \in \text{Syl}_p(C)$, and note that

since $S \subseteq C$, the p -group S centralizes the nilpotent p' -group B . Then SB is nilpotent, and so SB is contained in some maximally nilpotent subgroup U of G . Now $A \subseteq P \subseteq S$, so $\mathbf{F}(G) = AB \subseteq SB \subseteq U$, and thus U is a nilpotent injector of G . Then U is conjugate to I , so S is a p -subgroup of a conjugate of I . Since P is a Sylow p -subgroup of I , it follows that $|S| \leq |P|$, and hence $S = P$ and $P \in \text{Syl}_p(C)$, as claimed.

Let $M = \mathbf{O}_{p'}(C)$, so M centralizes B since $C = \mathbf{C}_G(B)$. Also, A is a normal p -subgroup of C , so M centralizes A , and thus $M \subseteq \mathbf{C}_G(\mathbf{F}(G)) \subseteq \mathbf{F}(G)$. It follows that $M \subseteq B$, and hence $M \subseteq \mathbf{Z}(C)$.

Write $Z = \mathbf{Z}(\mathbf{J}(P))$, and recall that our goal is to show that $Z \triangleleft G$. The image of Z under the natural isomorphism from P to PM/M is ZM/M , and it follows that $ZM/M = \mathbf{Z}(\mathbf{J}(PM/M))$. Since PM/M is a Sylow p -subgroup of C/M and $\mathbf{O}_{p'}(C/M)$ is trivial, we can apply Theorem 6 to deduce that ZM/M is characteristic in C/M . Also, $C \triangleleft G$, so $C/M \triangleleft G/M$, and thus $ZM/M \triangleleft G/M$, and we have $ZM \triangleleft G$. But Z is a normal Sylow p -subgroup of ZM since M is central in C , and thus $Z \triangleleft G$, as wanted.

Now I is nilpotent, so $\mathbf{J}(I)$ is the product of the subgroups $\mathbf{J}(P)$, as P runs over the Sylow subgroups of I . Then $\mathbf{Z}(\mathbf{J}(I))$ is the product of the subgroups $\mathbf{Z}(\mathbf{J}(P))$, and since each of these factors is normal in G , we see that $\mathbf{Z}(\mathbf{J}(I)) \triangleleft G$. Since the nilpotent injectors of G are conjugate, it follows that the subgroup $\mathbf{Z}(\mathbf{J}(I))$ is independent of the choice of the particular nilpotent injector I , and hence $\mathbf{Z}(\mathbf{J}(I))$ is characteristic in G . $\square$

We can now prove Theorem B.

Proof of Theorem B. We are given a maximal subgroup $H \subseteq G$, where G is solvable, and we assume that a Sylow 2-subgroup of H is abelian. Suppose that $K \subseteq G$ and $\theta : H \rightarrow K$ is an isomorphism. We proceed by induction on $|G|$ to show that K is maximal in G .

Using exactly the same reasoning as in the proof of Theorem A, we can assume that there exists no nontrivial subgroup $N \triangleleft G$ such that $N \subseteq H$ and $\theta(N) = N$. Also, as in the proof of Theorem A, we can assume that $\mathbf{O}_p(G) = 1$, where p is the unique prime divisor of $|G : H| = |G : K|$. Then $\mathbf{F}(G)$ is a p' -group, and it follows by Corollary 5 that the nilpotent injectors of H and of K are nilpotent injectors of G .

Now let U be a nilpotent injector of H , and write $V = \theta(U)$ so that V is a nilpotent injector of K . Then U and V are nilpotent injectors of G , and hence they are conjugate in G , and we can write $V = U^g$ for some element $g \in G$. We now have

$$\theta(\mathbf{Z}(\mathbf{J}(U))) = \mathbf{Z}(\mathbf{J}(\theta(U))) = \mathbf{Z}(\mathbf{J}(V)) = \mathbf{Z}(\mathbf{J}(U^g)) = \mathbf{Z}(\mathbf{J}(U))^g,$$

so if $\mathbf{Z}(\mathbf{J}(U)) \triangleleft G$, then $\theta(\mathbf{Z}(\mathbf{J}(U))) = \mathbf{Z}(\mathbf{J}(U))$, and this is a contradiction since $\mathbf{Z}(\mathbf{J}(U))$ is nontrivial. By Theorem 7, however, we have $\mathbf{Z}(\mathbf{J}(U)) \triangleleft H$, and thus $H = \mathbf{N}_G(\mathbf{Z}(\mathbf{J}(U)))$ by the maximality of H . Then

$$H^g = \mathbf{N}_G(\mathbf{Z}(\mathbf{J}(U))^g) = \mathbf{N}_G(\mathbf{Z}(\mathbf{J}(V))) \supseteq K,$$

where the final containment follows by Theorem 7. Since $|H^g| = |H| = |K|$, we have $H^g = K$, and thus K is maximal in G . $\square$

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