FUSION RULES AMONG IRREDUCIBLE $V_{\sqrt{2}A_2}^\tau$ -MODULES OF TWISTED TYPE

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ABSTRACT. In this article, we first compute the quantum dimensions of irreducible $V_{\sqrt{2}A_2}^\tau$ -modules. These quantum dimensions give upper bounds on fusion rules among irreducible $V_{\sqrt{2}A_2}^\tau$ -modules. Together with the lower bounds obtained by Lam and the author, we determine explicitly fusion rules among all irreducible $V_{\sqrt{2}A_2}^\tau$ -modules of twisted type. This work completes the program for determining the fusion rules among irreducible $V_{\sqrt{2}A_2}^\tau$ -modules.

1. INTRODUCTION

Let $\sqrt{2}A_2$ be the $\sqrt{2}$ times the root lattice of type A_2 and τ a fixed point free isometry of order three of the lattice $\sqrt{2}A_2$. The orbifold VOA $V_{\sqrt{2}A_2}^\tau$ has been studied by many authors [KLY03, TY07, TY09] and is an important component for understanding the $\mathbb{Z}_3$ orbifold construction of the famous Moonshine VOA [DM94, KLY03, TY09]. In [TY07] the inequivalent irreducible modules for $V_{\sqrt{2}A_2}^\tau$ are classified using some W_3 -algebras, and fusion rules among some untwisted type modules are also obtained in [Tan05, TY09, TY13]. In [CL14], certain intertwining operators among irreducible modules of twisted type are constructed explicitly using a coset construction. Lower bounds on their fusion rules are also obtained.

In this paper, we use the notion of quantum dimension introduced by [DJX13] to explicitly determine the fusion rules among modules of twisted type. We first compute quantum dimensions of irreducible $V_{\sqrt{2}A_2}^\tau$ -modules of untwisted type using the fusion rules obtained in [TY13]. Knowing these quantum dimensions simplifies the calculation of S -matrix. Therefore we use S -matrix to find quantum dimension of irreducible $V_{\sqrt{2}A_2}^\tau$ -modules of twisted type. These quantum dimensions serve as upper bounds on fusion rules. By comparing with lower bounds obtained in [CL14] we can describe precisely fusion rules among all irreducible $V_{\sqrt{2}A_2}^\tau$ -modules of twisted type. This work completes our program for determining the fusion rules of $V_{\sqrt{2}A_2}^\tau$ -modules.

This article is organized as follows. In Section 2, we recall some basic facts about the orbifold VOA $V_{\sqrt{2}A_2}^\tau$ and quantum dimensions. In Section 3, we first determine quantum dimension of irreducible $V_{\sqrt{2}A_2}^\tau$ -modules. Finally, the fusion rules among irreducible modules of twisted type are computed.

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2. PRELIMINARIES

2.1. Orbifold VOA $V_{\sqrt{2}A_2}^\tau$. In this section we review some facts about the orbifold VOA $V_{\sqrt{2}A_2}^\tau$. For general background concerning lattice VOA, we refer to [FLM88, LL03].

Let α_1, α_2 be the simple roots of type A_2 and set $\alpha_0 = -(\alpha_1 + \alpha_2)$. Then $\langle \alpha_i, \alpha_i \rangle = 2$ and $\langle \alpha_i, \alpha_j \rangle = -1$ if $i \neq j$. Set $\beta_i = \sqrt{2}\alpha_i$ and let $L = \mathbb{Z}\beta_1 + \mathbb{Z}\beta_2$ be the lattice spanned by β_1 and β_2 . We know L is isometric to $\sqrt{2}A_2$.

We adopt the same notation as in [KLY03, DLT⁺04] and denote the cosets of L in the dual lattice $L^\perp = \{\alpha \in \mathbb{Q} \otimes_{\mathbb{Z}} L \mid \langle \alpha, L \rangle \subset \mathbb{Z}\}$, as follows:

$$L^0 = L, \quad L^1 = \frac{-\beta_1 + \beta_2}{3} + L, \quad L^2 = \frac{\beta_1 - \beta_2}{3} + L,$$

$$L_0 = L, \quad L_a = \frac{\beta_2}{2} + L, \quad L_b = \frac{\beta_0}{2} + L, \quad L_c = \frac{\beta_1}{2} + L,$$

and

$$L^{(i,j)} = L_i + L^j,$$

for $i = 0, a, b, c$ and $j = 0, 1, 2$, where $\{0, a, b, c\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Then, $L^{(i,j)}, i \in \{0, a, b, c\}, j \in \{0, 1, 2\}$, are all the cosets of L in $L^\perp$ and $L^\perp/L \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$. It is shown in [Don93] that there are exactly 12 isomorphism classes of irreducible V_L -modules, which are represented by $V_{L^{(i,j)}}$, $i = 0, a, b, c$ and $j = 0, 1, 2$.

Consider the isometry $\tau : L \rightarrow L$ defined by

$$\beta_1 \mapsto \beta_2 \mapsto \beta_0 \mapsto \beta_1.$$

Then τ is fixed point free of order three and can be lifted naturally to an automorphism of V_L by mapping

$$a^1(-n_1) \cdot a^k(-n_k)e^b \mapsto (\tau a^1)(-n_1) \cdot (\tau a^k)(-n_k)e^{\tau b}.$$

By abuse of notation, we also use τ to denote the lift.

In [KLY03], it was shown that there are exactly three isomorphism classes of irreducible τ -twisted V_L -modules and three isomorphism classes of τ^2 -twisted V_L -modules. They are represented by $V_L^{T,j}(\tau)$ or $V_L^{T,j}(\tau^2)$ for $j = 0, 1, 2$.

The irreducible V_L^τ -modules are also classified by Tanabe and Yamada [TY07]. For any τ -invariant space U , we denote $U[\varepsilon] = \{u \in U \mid \tau u = \xi^\varepsilon u\}$, $\varepsilon = 0, 1, 2$, where $\xi = \exp(2\pi\sqrt{-1}/3)$. The following result was proved by Tanabe and Yamada.

Proposition 2.1 ([TY07]). *The VOA V_L^τ is a simple, rational, C_2 -cofinite, and of CFT type. There are exactly 30 inequivalent irreducible V_L^τ -modules. They are represented as follows.*

- (1) $V_{L^{(0,j)}}[\varepsilon]$, $j, \varepsilon = 0, 1, 2$.
- (2) $V_{L^{(c,j)}}[\varepsilon]$, $j = 0, 1, 2$.
- (3) $V_L^{T,j}(\tau^i)[\varepsilon]$, $i = 1, 2$ and $j, \varepsilon = 0, 1, 2$.

In [TY13] Tanabe and Yamada also compute some fusion rules, which we will recall in Section 3, Proposition 3.4. These fusion rules gives us a short cut to compute the quantum dimension of V_L^τ -modules of untwisted type.

2.2. Quantum dimension. In this section we recall the notions of trace functions and quantum dimensions.

Let V be a VOA of central charge c and $M = \bigoplus_{n \in \mathbb{Z}_+} M_{\lambda+n}$, where λ is the conformal weight of M . For any homogeneous element $v \in V$, a *trace function* associated to v is defined as

$$Z_M(v, z) = \mathrm{tr}_M o(v) q^{L(0)-c/24} = q^{\lambda-c/24} \sum_{n \in \mathbb{Z}_+} \mathrm{tr}_{M_{\lambda+n}} o(v) q^n,$$

where $o(v) = v(\mathrm{wt} v - 1)$ is the degree zero operator of v . Here and below $z = x + \sqrt{-1}y$ is in the complex upper half-plane $\mathbb{H}$ and $q = e^{2\pi\sqrt{-1}z}$. When $v = \mathbb{1}$ is the vacuum vector, we have

$$Z_M(\mathbb{1}, z) = q^{\lambda-c/24} \sum_{n \in \mathbb{Z}_+} \dim M_{\lambda+n} q^n,$$

which is usually called the *normalized character* and denoted by $Z_M(z)$.

The trace function $Z_M(v, z)$ was studied extensively by [Zhu96, DLM00]. In particular, we have the following transformation property.

Theorem 2.2 ([Zhu96, DLM00]). *Let V be a simple, rational, C_2 -cofinite VOA and $\{V \cong M^0, \dots, M^d\}$ be the set of all inequivalent irreducible V -modules. Then*

$$(2-1) \quad Z_{M^i}\left(\frac{-1}{z}\right) = \sum_{j=0}^d S_{i,j} Z_{M^j}(z),$$

for some $S_{i,j} \in \mathbb{C}$. The matrix $S = (S_{i,j})$ is called *S-matrix*.

It is proved in [Hua08] that if V is of CFT-type and V is self-dual (i.e. V is isomorphic to V' its contragredient dual), then the matrix S is symmetric.

The notion of quantum dimension is introduced by Dong et al. [DJX13].

Definition 2.3. Suppose $Z_V(z)$ and $Z_M(z)$ exist. The *quantum dimension* of M over V is defined as

$$(2-2) \quad \mathrm{qdim}_V M = \lim_{y \rightarrow 0^+} \frac{Z_M(\sqrt{-1}y)}{Z_V(\sqrt{-1}y)},$$

where y is a real and positive.

Fundamental properties of quantum dimension are also proved in their paper.

Proposition 2.4 ([DJX13, Sec. 4]). *Let V be a simple, rational, C_2 -cofinite VOA of CFT-type and $V \cong V'$. Let $\{V \cong M^0, \dots, M^d\}$ be the set of all inequivalent irreducible V -modules. Let W^1, W^2 be V -modules. Then*

(i) $0 < \mathrm{qdim}_V M^i < \infty$ for any $0 \leq i \leq d$. In fact, it relates to S -matrix via

$$(2-3) \quad \mathrm{qdim}_V M^i = \frac{S_{i,0}}{S_{0,0}}.$$

(ii) qdim_V is multiplicative, that is, $\mathrm{qdim}_V(W^1 \times W^2) = \mathrm{qdim}_V W^1 \cdot \mathrm{qdim}_V W^2$, where $W^1 \times W^2$ denotes the fusion product.

(iii) A V -module W^1 is a simple current if and only if $\mathrm{qdim}_V W^1 = 1$.

Remark 2.5. Recall that for a simple V -module M is a *simple current* module if and only if for every simple V -module W , $M \times W$ exists and is also a simple V -module.

3. MAIN RESULTS

In this section, we will determine the fusion rules among irreducible V_L^τ -modules. First, we prove that for $L = \sqrt{2}A_2$ the lattice VOA V_L is self-dual.

Lemma 3.1. *The lattice VOA V_L is self-dual.*

Proof. It is clear that $L(1)V_1 = 0$. By [Li94, Cor. 3.2], V_L has a symmetric contragradient form and hence $V_L \cong V_{L'}$. $\square$

Quantum dimension of modules of untwisted type. The following lemma is an easy application of Dong et al.'s result.

Lemma 3.2. *For $\varepsilon = 0, 1, 2$, we know*

$$(3-1) \quad \text{qdim}_{V_L[0]} V_{L(0,0)}[\varepsilon] = 1.$$

Proof. It is proved in [DJX13, Thm. 6.3] that $\text{qdim}_{V_L[0]} V_{L(0,0)}[\varepsilon] = \chi(1)$, where χ is the linear character of the group $G := \langle \tau \rangle$. Since G is a cyclic group of order 3 and the quantum dimension is a positive real number, we have $\chi(1) = 1$ and our lemma follows. $\square$

Remark 3.3. Miyamoto shows in [Miy13, Prop. 2] that $V_{L(0,0)}[\varepsilon]$ is a simple current $V_{L(0,0)}[0]$ -module. Together with Proposition 2.4(iii), this lemma also follows.

From now on, we use $\text{qdim } W = \text{qdim}_{V_L^\tau} W$ to denote the quantum dimension of V_L^τ -modules W . Recall Tanabe and Yamada's work on fusion rules.

Proposition 3.4 ([TY13, Prop.4.5]). *Let $\varepsilon, \varepsilon_1, \varepsilon_2, j, j_1, j_2, k \in \mathbb{Z}_3$ and $i = 1, 2$. Then*

$$(3-2) \quad V_{L(0,j_1)}[\varepsilon_1] \times V_{L(0,j_2)}[\varepsilon_2] = V_{L(0,j_1+j_2)}[\varepsilon_1 + \varepsilon_2],$$

$$(3-3) \quad V_{L(0,j_1)}[\varepsilon] \times V_{L(c,j_2)} = V_{L(c,j_1+j_2)},$$

$$(3-4) \quad V_{L(c,j_1)} \times V_{L(c,j_2)} = \sum_{\rho=0}^2 V_{L(0,j_1+j_2)}[\rho] + 2V_{L(c,j_1+j_2)},$$

$$(3-5) \quad V_{L(0,j)}[\varepsilon_1] \times V_L^{T,k}(\tau^i)[\varepsilon_2] = V_L^{T,k-ij}(\tau^i)[i\varepsilon_1 + \varepsilon_2],$$

$$(3-6) \quad V_{L(c,j)} \times V_L^{T,k}(\tau^i)[\varepsilon] = \sum_{\rho=0}^2 V_L^{T,k-ij}(\tau^i)[\rho].$$

Basic arithmetic argument on these fusion rules gives us quantum dimension of modules of untwisted type.

Theorem 3.5. *We have*

- (i) $\text{qdim } V_{L(0,j)}[\varepsilon] = 1$ for $j, \varepsilon = 0, 1, 2$.
- (ii) $\text{qdim } V_{L(c,j)} = 3$ for $j = 0, 1, 2$.
- (iii) $\text{qdim } V_L^{T,0}(\tau)[0] = \text{qdim } V_L^{T,k}(\tau^i)[\varepsilon]$ for $i = 1, 2$ and $k, \varepsilon = 0, 1, 2$.

Proof. (i) From (3-2) we have

$$\text{qdim } V_{L(0,j)}[\varepsilon_1] \cdot \text{qdim } V_{L(0,0)}[\varepsilon_2] = \text{qdim } V_{L(0,j)}[\varepsilon_1 + \varepsilon_2].$$

Using (3-1) and picking suitable $\varepsilon_1, \varepsilon_2 \in \mathbb{Z}_3$ give $\text{qdim } V_{L(0,j)}[0] = \text{qdim } V_{L(0,j)}[1] = \text{qdim } V_{L(0,j)}[2]$. Using (3-2) again we have

$$\text{qdim } V_{L(0,1)}[0] \cdot \text{qdim } V_{L(0,1)}[0] = \text{qdim } V_{L(0,2)}[0],$$

and

$$\text{qdim } V_{L(0,2)}[0] \cdot \text{qdim } V_{L(0,2)}[0] = \text{qdim } V_{L(0,1)}[0].$$

This gives

$$(\text{qdim } V_{L(0,1)}[0])^4 = \text{qdim } V_{L(0,1)}(0).$$

Since $\text{qdim } V_{L(0,1)}[0] > 0$ by Proposition 2.4, we have

$$(3-7) \quad 1 = \text{qdim } V_{L(0,1)}[0] = \text{qdim } V_{L(0,2)}[0] = \text{qdim } V_{L(0,j)}[\varepsilon],$$

for any $j, \varepsilon \in \mathbb{Z}_3$.

(ii) Using (3-3) and (3-7) we know $\text{qdim } V_{L(c,1)} = \text{qdim } V_{L(c,2)} = \text{qdim } V_{L(c,0)}$. Using (3-4) we have

$$[\text{qdim } V_{L(c,1)}]^2 = 3 + 2 \text{qdim } V_{L(c,1)}.$$

Thus

$$\text{qdim } V_{L(c,1)} = \text{qdim } V_{L(c,2)} = \text{qdim } V_{L(c,0)} = 3.$$

(iii) Similarly, using (3-5) and (3-7) we have

$$\text{qdim } V_L^{T,0}(\tau^i)[0] = \text{qdim } V_L^{T,k}(\tau^i)[\varepsilon],$$

for $k, \varepsilon \in \mathbb{Z}_3$ and $i = 1, 2$. In fact, by the definition of twisted modules, $V_L^{T,k}(\tau)[\varepsilon]$ and $V_L^{T,k}(\tau^2)[\varepsilon]$ have the same trace function and hence the same quantum dimension. Thus we have

$$\text{qdim}_{V_L[0]} V_L^{T,0}(\tau)[0] = \text{qdim } V_L^{T,k}(\tau^i)[\varepsilon],$$

for $i = 1, 2$ and $k, \varepsilon = 0, 1, 2$. □

Quantum dimensions of module of twisted type. We now use S -matrix (2-3) to obtain quantum dimensions of irreducible V_L^τ -module of twisted type. For convenience, we number the irreducible V_L^τ -modules as follows.

$$(3-8) \quad \begin{aligned} M^{3i+j} &:= V_{L(0,i)}[j]; & M^{9+j} &:= V_{L(c,j)}; \\ M^{12+3i+j} &:= V_L^{T,i}(\tau)[j]; & M^{21+3i+j} &:= V_L^{T,i}(\tau^2)[j], \end{aligned}$$

for $i, j = 0, 1, 2$. The matrix S involves 30 variables and is usually difficult to solve. However, Proposition 3.5 tells us that $S_{i,0} = S_{0,0}$ for $0 \leq i \leq 8$, $S_{i,0} = 3S_{0,0}$ for $9 \leq i \leq 11$. Moreover, we can define $d = \frac{S_{i,0}}{S_{0,0}}$ for $12 \leq i \leq 29$. Then $d = \text{qdim } V_L^{T,j}(\tau^i)[\varepsilon]$ for $i = 1, 2$ and $j, \varepsilon \in \mathbb{Z}_3$. Therefore, we only need to deal with two variables. It makes the calculation not so hopeless. Recall, S is symmetric and hence by (2-1) we have

$$\begin{aligned} Z_{M^0}\left(\frac{-1}{z}\right) &= \sum_{i=0}^{29} S_{0,i} Z_{M^i}(z) \\ &= S_{0,0} \sum_{i=0}^8 Z_{M^i}(z) + 3S_{0,0} \sum_{i=9}^{11} Z_{M^i}(z) + dS_{0,0} \sum_{i=12}^{29} Z_{M^i}(z). \end{aligned}$$

By the definition of trace functions and the construction of the modules M^i , we know

$$\begin{aligned} \sum_{i=0}^8 Z_{M^i}(z) &= \sum_{j=0}^2 Z_{V_L^{(0,j)}}(z), & \sum_{i=9}^{11} Z_{M^i}(z) &= \sum_{j=0}^2 Z_{V_L^{(c,j)}}(z), \\ \sum_{i=12}^{20} Z_{M^i}(z) &= \sum_{j=0}^2 Z_{V_L^{T,j}(\tau)}(z), & \sum_{i=21}^{29} Z_{M^i}(z) &= \sum_{j=0}^2 Z_{V_L^{T,j}(\tau^2)}(z). \end{aligned}$$

Therefore,

$$\begin{aligned} (3-9) \quad Z_{M^0}\left(\frac{-1}{z}\right) &= S_{0,0} \sum_{j=0}^2 Z_{V_L^{(0,j)}}(z) + 3S_{0,0} \sum_{j=0}^2 Z_{V_L^{(c,j)}}(z) \\ &\quad + dS_{0,0} \left(\sum_{j=0}^2 Z_{V_L^{T,j}(\tau)}(z) + \sum_{j=0}^2 Z_{V_L^{T,j}(\tau^2)}(z) \right). \end{aligned}$$

The characters appear on the right hand side of the above equation are well-known. For an even integral lattice E of rank d and a coset $\lambda + E \in E^\perp/E$, the normalized character of the VOA module $V_{\lambda+E}$ is $Z_{\lambda+E}(z) = \frac{q^{-d/24} \theta_{\lambda+E}(z)}{\eta(z)^d}$, where

$$\begin{aligned} \theta_{\lambda+E}(z) &= \sum_{n \geq 0} \#\{\alpha \in \lambda + E \mid \frac{1}{2}\langle \alpha, \alpha \rangle = n\} q^n, \\ \eta(z) &= q^{1/24} \prod_{i=1}^{\infty} (1 - q^i). \end{aligned}$$

A direct computation gives the following lemma.

Lemma 3.6. *We have*

$$\begin{aligned} Z_{V_L^{(0,0)}}(z) &= q^{-1/12} + 2q^{7/12} + 4q^{19/12} + 3q^{23/12} + \dots, \\ Z_{V_L^{(0,1)}}(z) &= Z_{V_L^{(0,2)}}(z) = 2q^{7/12} + q^{11/12} + 4q^{19/12} + 4q^{23/12} + \dots, \\ Z_{V_L^{(c,0)}}(z) &= 2q^{5/12} + 6q^{17/12} + 14q^{29/12} + 34q^{41/12} + \dots, \\ Z_{V_L^{(c,1)}}(z) &= Z_{V_L^{(c,2)}}(z) = q^{1/12} + 4q^{13/12} + 11q^{25/12} + 26q^{37/12} + \dots. \end{aligned}$$

Next, we compute the characters of irreducible twisted modules of V_L .

Lemma 3.7. *We have*

$$Z_{V_L^{T,j}(\tau^i)}(z) = q^{1/36} + q^{13/36} + 2q^{25/36} + 2q^{37/36} + \dots,$$

for $i = 1, 2$ and $j = 0, 1, 2$.

Proof. Recall the construction of twisted V_L -modules:

$$V_L^{T,j}(\tau^i) \cong S[\tau^i] \otimes T_{\chi_j},$$

where as a vector space $S[\tau^i]$ is isomorphic to a polynomial algebra of variables $h_1(1/3 + n)$ and $h_2(2/3 + n)$ for $n \in \mathbb{Z}_{<0}$. Its weight is given by $\text{wt } 1 = 1/9$ and $\text{wt } h_k(k/3 + n) = -(k/3 + n)$. The module T_{χ_j} is one-dimensional of weight 0.

Thus,

$$\begin{aligned} Z_{V_L^{T,j}(\tau^i)}(z) &= q^{-1/12+1/9} \frac{\prod_{n=1}^{\infty} (1-q^n)}{\prod_{n=1}^{\infty} (1-q^{n/3})} \\ &= q^{1/36} + q^{13/36} + 2q^{25/36} + 2q^{37/36} + \dots, \end{aligned}$$

for $i = 1, 2$ and $j = 0, 1, 2$. □

To compute $Z_{M^0}(\frac{-1}{z})$, we need to a better understanding about the orbifold VOA $M^0 = V_L^\tau$. It is shown in [KLY03] that V_L admits a decomposition

$$V_L \cong M_k^0 \otimes M_t^0 \oplus W_k^0 \otimes W_t^0,$$

where as Virasoro VOA modules

$$\begin{aligned} M_t^0 &\cong L(\frac{4}{5}, 0) \oplus L(\frac{4}{5}, 3), & M_k^0 &\cong L(\frac{1}{2}, 0) \otimes L(\frac{7}{10}, 0) \oplus L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{7}{10}, \frac{3}{2}), \\ W_t^0 &\cong L(\frac{4}{5}, \frac{2}{5}) \oplus L(\frac{4}{5}, \frac{7}{5}), & W_k^0 &\cong L(\frac{1}{2}, 0) \otimes L(\frac{7}{10}, \frac{3}{5}) \oplus L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{7}{10}, \frac{1}{10}). \end{aligned}$$

According, V_L^τ admits a decomposition

$$V_L^\tau \cong M_k^0[0] \otimes M_t^0 \oplus W_k^0[0] \otimes W_t^0,$$

and hence

$$(3-10) \quad Z_{M^0}(z) = Z_{M_k^0[0]}(z)Z_{M_t^0}(z) + Z_{W_k^0[0]}(z)Z_{W_t^0}(z).$$

The characters of the VOA modules above are computed in [DLT⁺04].

Proposition 3.8 ([DLT⁺04, Thm. 6.2]). *We have the following characters.*

$$\begin{aligned} Z_{M_t^0}(z) &= \Xi_{1,1}^{(3)} + \Xi_{4,1}^{(3)}, \\ Z_{W_t^0}(z) &= \Xi_{2,1}^{(3)} + \Xi_{3,1}^{(3)}, \\ Z_{M_k^0[0]}(z) &= \frac{1}{3} \left(\Xi_{1,1}^{(1)} \Xi_{1,1}^{(2)} + \Xi_{2,1}^{(1)} \Xi_{3,1}^{(2)} + 2 \frac{\eta(z)}{\eta(3z)} \Xi_{3,3}^{(3)} \right), \\ Z_{W_k^0[0]}(z) &= \frac{1}{3} \left(\Xi_{1,1}^{(1)} \Xi_{3,2}^{(2)} + \Xi_{2,1}^{(1)} \Xi_{3,3}^{(2)} - 2 \frac{\eta(z)}{\eta(3z)} \Xi_{4,3}^{(3)} \right), \end{aligned}$$

where $\eta(z) = q^{1/24} \prod_{i=1}^{\infty} (1-q^i)$ is the Dedekind η -function, and

$$\begin{aligned} \Xi_{r,s}^{(m)}(z) &= q^{-c_m/24} \frac{\sum_{k \in \mathbb{Z}} (q^{b(k)} - q^{a(k)})}{\prod_{i=1}^{\infty} (1-q^i)}, \\ c_m &= 1 - \frac{6}{(m+2)(m+3)}, \\ a(k) &= \frac{(2(m+2)(m+3)k + (m+3)r + (m+2)s)^2 - 1}{4(m+2)(m+3)}, \\ b(k) &= \frac{(2(m+2)(m+3)k + (m+3)r - (m+2)s)^2 - 1}{4(m+2)(m+3)}. \end{aligned}$$

The calculation of $Z_{M^0}(\frac{-1}{z})$ relies on the following transformation properties:

$$(3-11) \quad \begin{aligned} \eta\left(\frac{-1}{z}\right) &= (-\sqrt{-1}z)^{1/2} \eta(z), \\ \Xi_{r,s}^{(m)}\left(\frac{-1}{z}\right) &= \sqrt{\frac{8}{(m+2)(m+3)}} \\ &\quad \cdot \sum_{1 \leq j \leq i \leq m+1} (-1)^{(r+s)(i+j)} \sin \frac{\pi r i}{m+2} \sin \frac{\pi s j}{m+3} \Xi_{i,j}^{(m)}(z). \end{aligned}$$

Note that we have

$$\frac{\eta(\frac{-1}{z})}{\eta(\frac{-3}{z})} = \sqrt{3} \frac{\eta(z)}{\eta(z/3)}.$$

Thus by computer calculation we obtain the following proposition.

Proposition 3.9. *Under the transformation $z \mapsto \frac{-1}{z}$, the character $Z_{M^0}(z)$ becomes*

$$(3-12) \quad Z_{M^0}\left(\frac{-1}{z}\right) = \frac{1}{6\sqrt{3}}q^{-1/12} + \frac{2}{\sqrt{3}}q^{1/36} + \frac{1}{\sqrt{3}}q^{1/12} + \frac{2}{\sqrt{3}}q^{13/36} + \dots.$$

We can now determine the quantum dimensions of modules of twisted type.

Theorem 3.10. *We have $\text{qdim } V_L^{T,j}(\tau^i)[\varepsilon] = 2$ for $i = 1, 2$ and $j, \varepsilon \in \mathbb{Z}_3$.*

Proof. Using Lemma 3.6 and 3.7, equation (3-9) becomes

$$Z_{M^0}\left(\frac{-1}{z}\right) = S_{0,0}q^{-1/12} + dS_{0,0}6q^{1/36} + 6S_{0,0}q^{1/12} + dS_{0,0}6q^{13/36} + \dots.$$

Comparing with (3-12), we have $S_{0,0} = \frac{1}{6\sqrt{3}}$ and $d = \text{qdim } V_L^{T,j}(\tau^i)[\varepsilon] = 2$ for $i = 1, 2$ and $j, \varepsilon \in \mathbb{Z}_3$. $\square$

Fusion rules. Knowing quantum dimension of these irreducible V_L^T -modules, we can now prove the main theorem of this article.

Theorem 3.11. *We have the following fusion rules among irreducible V_L^T -modules of twisted type.*

$$\begin{aligned} (i) \quad & V_L^{T,i}(\tau^l)[\varepsilon] \times V_L^{T,j}(\tau^l)[\varepsilon'] = V_L^{T, -(i+j)}(\tau^{2l})[-(\varepsilon + \varepsilon')] \\ & + V_L^{T, -(i+j)}(\tau^{2l})[2 - (\varepsilon + \varepsilon')]; \\ (ii) \quad & V_L^{T,i}(\tau)[\varepsilon] \times V_L^{T,j}(\tau^2)[\varepsilon'] = V_{L(0, i+2j)}[\varepsilon + 2\varepsilon'] + V_{L(c, i+2j)}, \end{aligned}$$

where $l \in \{1, 2\}$, $i, j, \varepsilon, \varepsilon' \in \{0, 1, 2\}$.

Proof. It is proved in [CL14, Prop. 7.10] that we have the $\geq$ inequalities in above equations. Applying quantum dimensions to all above terms, we get the desired equalities. $\square$

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