RIGIDITY OF SCATTERING LENGTHS AND TRAVELLING TIMES FOR DISJOINT UNIONS OF STRICTLY CONVEX BODIES

LYLE NOAKES AND LUCHEZAR STOYANOV

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ABSTRACT. Obstacles K and L in $\mathbb{R}^d$ ($d \geq 2$) are considered that are finite disjoint unions of strictly convex domains with C^3 boundaries. We show that if K and L have (almost) the same scattering length spectrum, or (almost) the same travelling times, then $K = L$.

1. INTRODUCTION

Let K be a compact subset of $\mathbb{R}^d$ ($d \geq 2$) with C^3 boundary ∂K such that $\Omega_K = \overline{\mathbb{R}^d} \setminus K$ is connected. A *scattering ray* γ in Ω_K is an unbounded in both directions generalized geodesic (in the sense of Melrose and Sjöstrand [MS1], [MS2]). If K is a finite disjoint union of convex domains, then the scattering rays in Ω_K are simply billiard trajectories with finitely many common points with ∂K . This article concerns two types of problems related to recovering information about the obstacle K from certain measurements of scattering rays in the exterior of K . These problems have similarities with various problems on metric rigidity in Riemannian geometry – see [SU], [SUV] and the references there for more information.

1.1. The scattering length spectrum. The first type of problems deal with the so-called scattering length spectrum (SLS). Given a scattering ray γ in Ω_K , denote by T_γ the *sojourn time* of γ (cf. Sect. 2). If $\omega \in \mathbf{S}^{d-1}$ is the incoming direction of γ and $\theta \in \mathbf{S}^{d-1}$ its outgoing direction, γ will be called an (ω, θ) -ray. The *scattering length spectrum* of K is defined to be the family of sets of real numbers $SL_K = \{SL_K(\omega, \theta)\}_{(\omega, \theta)}$ where (ω, θ) runs over $\mathbf{S}^{d-1} \times \mathbf{S}^{d-1}$ and $SL_K(\omega, \theta)$ is the set of sojourn times T_γ of all (ω, θ) -rays γ in Ω_K . It is known (cf. [St3]) that for $d \geq 3$, d odd, and C^∞ boundary ∂K , we have $SL_K(\omega, \theta) = \text{sing supp } s_K(t, \theta, \omega)$ for almost all (ω, θ) . Here s_K is the *scattering kernel* related to the scattering operator for the wave equation in $\mathbb{R} \times \Omega_K$ with Dirichlet boundary condition on $\mathbb{R} \times \partial\Omega_K$ (cf. [LP1], [M]). Following [St4], we will say that two obstacles K and L have *almost the same SLS* if there exists a subset $\mathcal{R}$ of full Lebesgue measure in $\mathbf{S}^{d-1} \times \mathbf{S}^{d-1}$ such that $SL_K(\omega, \theta) = SL_L(\omega, \theta)$ for all $(\omega, \theta) \in \mathcal{R}$.

It is a natural and rather important problem in inverse scattering by obstacles to get information about the obstacle K from its SLS. It is known that various kinds of information about K can be recovered from its SLS, and for some classes of obstacles K is completely recoverable (see e.g. [Ma], [MaR], [LP2], [St2], [St4]),

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for example star-shaped obstacles are in this class. However, as an example of M. Livshits shows (cf. Ch. 5 in [M]; see also Figure 1), in general $\mathcal{SL}_K$ does not determine K uniquely.

FIGURE 1. Livshits' example (adapted from Ch. 5 of [M]): the internal upper part of the figure is half an ellipse with foci A and B . A ray entering the interior of the ellipse between the foci must exit between the foci after reflection. So, no scattering ray has a common point with the bold parts of the boundary.

1.2. Travelling times. The second type of problems deal with travelling times. Let $\mathcal{O}$ be a large ball in $\mathbb{R}^d$ containing K in its interior and set $S_0 := \partial\mathcal{O}$. For any pair of points $x, y \in S_0$ consider the scattering rays γ incoming through the point x and outgoing through the point y . Such rays will be called (x, y) -geodesics in Ω_K . Given such γ , let t_γ be the length of the part of γ from x to y . Let $\mathcal{T}_K(x, y)$ the set of travelling times t_γ of all (x, y) -geodesic in Ω_K . If K and L are two obstacles contained in the interior of S_0 , we will say that K and L have almost the same travelling times if $\mathcal{T}_K(x, y) = \mathcal{T}_L(x, y)$ for almost all $(x, y) \in S_0 \times S_0$ (with respect to the Lebesgue measure on $S_0 \times S_0$). Our second type of problems concern getting information about the obstacle K from its travelling times.

1.3. Unions of convex bodies. For either kind of problems, we consider obstacles K of the form

$$(1.1) \quad K = K_1 \cup K_2 \cup \dots \cup K_{k_0},$$

where K_i are strictly convex disjoint domains in $\mathbb{R}^d$ ($d \geq 2$) with C^3 smooth boundaries ∂K_i . In this case the so-called generalized Hamiltonian (or bicharacteristic) flow $\mathcal{F}_t^{(K)} : S^*(\Omega_K) \rightarrow S^*(\Omega_K)$ (see Sect. 2) coincides with the billiard flow, so it is easier to deal with.

A point $\sigma = (x, \omega) \in S^*(\Omega_K)$ is called *non-trapped* if both curves $\{\text{pr}_1(\mathcal{F}_t^{(K)}(\sigma)) : t \leq 0\}$ and $\{\text{pr}_1(\mathcal{F}_t^{(K)}(\sigma)) : t \geq 0\}$ in Ω_K are unbounded. Otherwise σ is called a *trapped point*. Here we use the notation $\text{pr}_1(y, \eta) = y$ and $\text{pr}_2(y, \eta) = \eta$. Denote by $\text{Trap}(\Omega_K)$ the set of all trapped points. It is well known that in general $\text{Trap}(\Omega_K)$ may have positive Lebesgue measure and a non-empty interior in $S^*(\Omega_K)$ (see e.g. Livshits' example). Set $\dot{T}^*(\Omega_K) = T^*(\Omega_K) \setminus \{0\}$.

Definition 1.1. Let K, L be two obstacles in $\mathbb{R}^d$. We will say that Ω_K and Ω_L have conjugate flows if there exists a homeomorphism

$$\Phi : \dot{T}^*(\Omega_K) \setminus \text{Trap}(\Omega_K) \rightarrow \dot{T}^*(\Omega_L) \setminus \text{Trap}(\Omega_L),$$

which has the following properties: (i) Φ defines a symplectic map on an open dense subset of $\dot{T}^*(\Omega_K) \setminus \text{Trap}(\Omega_K)$, (ii) $\Phi(S^*(\Omega_K) \setminus \text{Trap}(\Omega_K)) = S^*(\Omega_L) \setminus \text{Trap}(\Omega_L)$, (iii) $\mathcal{F}_t^{(L)} \circ \Phi = \Phi \circ \mathcal{F}_t^{(K)}$ for all $t \in \mathbb{R}$ and $\Phi = \text{id}$ on $\dot{T}^*(\mathbb{R}^d \setminus \mathcal{O}) \setminus \text{Trap}(\Omega_K) = \dot{T}^*(\mathbb{R}^d \setminus \mathcal{O}) \setminus \text{Trap}(\Omega_L)$.

It is known that for K, L in a very large (generic) class of obstacles in $\mathbb{R}^d$ ($d \geq 2$), if K and L have almost the same SLS or almost the same travelling times, then Ω_K and Ω_L have conjugate flows ([St4] and [NS]; see Sect. 2 below where these results are given in full details). In this paper we prove

Theorem 1.2. *Let $d \geq 2$ and let each of the obstacles K and L be a finite disjoint union of strictly convex domains in $\mathbb{R}^d$ with C^3 boundaries. If Ω_K and Ω_L have conjugate flows, then $K = L$.*

For the case where ∂K and ∂L are real analytic this was proved in [St2]. As an immediate consequence of Theorem 1.2 and results in [St4] and [NS], one gets the following.

Corollary 1.3. *Assume that K and L are obstacles in $\mathbb{R}^d$ ($d \geq 2$) and each of them is a finite disjoint union of strictly convex domains with C^3 boundaries. If K and L have almost the same scattering length spectrum, or K and L have almost the same travelling times, then $K = L$.*

We remark that the above results are non-trivial. Indeed when K has a large number of connected components and they are densely packed (imagine the molecules of a gas in a container), then there are a great number of billiard trajectories with large numbers of reflections (and possibly with many tangencies) in the exterior of K . Moreover, in such cases many connected components of K can only be reached by billiard trajectories having many reflections. So, being able to completely recover the obstacle K by measuring sojourn times only, or travelling times only, is far from being a trivial matter. The assumption that ∂K and ∂L are C^3 smooth is required in order to be able to use some of the results in [St2] and [St4].

2. PRELIMINARIES

We refer the reader to [MS1], [MS2] (or Sect. 24.3 in [H]) for the definition of the generalized Hamiltonian (bicharacteristic) flow on a symplectic manifold with boundary. In the case of scattering by an obstacle K this is the *generalized geodesic flow* $\mathcal{F}_t^{(K)} : \dot{T}_b^*(\Omega_K) = T_b^*(\Omega_K) \setminus \{0\} \longrightarrow \dot{T}_b^*(\Omega_K)$ generated by the principal symbol of the wave operator in $\mathbb{R} \times \Omega_K$. Here $T_b^*(\Omega_K) = T^*(\Omega_K)/\sim$ is the quotient space with respect to the following equivalence relation on $T^*(\Omega_K)$: $(x, \xi) \sim (y, \eta)$ iff $x = y$ and either $\xi = \eta$ or $x = y \in \partial K$ and ξ and η are symmetric with respect to the tangent plane to ∂K at x . The image $S_b^*(\Omega_K)$ of the *unit cosphere bundle* $S^*(\Omega_K)$ under the natural projection is invariant with respect to $\mathcal{F}_t^{(K)}$. For simplicity of notation the subscript b will be suppressed, and it will be clear from the context exactly which second component we have in mind.

In general $\mathcal{F}_t^{(K)}$ is not a flow in the usual sense of dynamical systems, since there may exist different integral curves issued from the same point of the phase space (see [T]). Let $\mathcal{K}$ be the class of obstacles that have the following property: for each $(x, \xi) \in \dot{T}^*(\partial K)$ if the curvature of ∂K at x vanishes of infinite order in direction ξ , then all points (y, η) sufficiently close to (x, ξ) are *diffractive* (roughly speaking,

this means that ∂K is convex at y in the direction of η). It follows that for $K \in \mathcal{K}$ the flow $\mathcal{F}_t^{(K)}$ is well-defined and continuous ([MS2]).

We now describe the relevant results from [St4] and [NS] used in the proof of Corollary 1.3. Given $\xi \in \mathbf{S}^{d-1}$ denote by Z_ξ the hyperplane in $\mathbb{R}^d$ orthogonal to ξ and tangent to $\mathcal{O}$ such that $\mathcal{O}$ is contained in the open half-space R_ξ determined by Z_ξ and having ξ as an inner normal. For an (ω, θ) -ray γ in Ω , the *sojourn time* T_γ of γ is defined by $T_\gamma = T'_\gamma - 2a$, where T'_γ is the length of that part of γ which is contained in $R_\omega \cap R_{-\theta}$ and a is the radius of the ball $\mathcal{O}$. It is known (cf. [G]) that this definition does not depend on the choice of the ball $\mathcal{O}$. Following [PS2], given $\sigma = (x, \xi) \in S^*(\Omega_K)$ so that $\gamma_K(\sigma) = \{\text{pr}_1(\mathcal{F}_t^{(K)}(\sigma)) : t \in \mathbb{R}\}$ is a *simply reflecting ray*, i.e. it has no tangencies to ∂K , we will say that $\gamma_K(\sigma)$ is *non-degenerate* if for every $t \gg 0$ the map $\mathbb{R}^d \ni y \mapsto \text{pr}_2(\mathcal{F}_t^{(K)}(y, \xi)) \in \mathbf{S}^{d-1}$ is a submersion at $y = x$, i.e. its differential at $y = x$ has rank $d - 1$. Set $S_+^*(S_0) = \{(x, u) : x \in S_0, u \in \mathbf{S}^{d-1}, \langle \nu_x, u \rangle < 0\}$, where ν_x is the outward unit normal to S_0 at x . Denote by $\mathcal{K}_0$ the class of all obstacles $K \in \mathcal{K}$ contained in $\mathcal{O}$ satisfying the following non-degeneracy conditions: $\gamma_K(\sigma)$ is a non-degenerate simply reflecting ray for almost all $\sigma \in S_+^*(S_0) \setminus \text{Trap}(\Omega_K)$ such that $\gamma_K(\sigma) \cap \partial K \neq \emptyset$, and ∂K does not contain non-trivial open flat subsets (i.e. open subsets where the curvature is zero at every point). It can be derived without much difficulty from [PS1] (see Ch. 3 there) that $\mathcal{K}_0$ is of second Baire category in $\mathcal{K}$ with respect to the C^∞ Whitney topology in $\mathcal{K}$. That is, for every $K \in \mathcal{K}$, applying suitable arbitrarily small C^∞ deformations to ∂K , one gets obstacles from the class $\mathcal{K}_0$ and most deformations have this property.

Theorem 2.1 ([St4]). *If the obstacles $K, L \in \mathcal{K}_0$ have almost the same SLS, then Ω_K and Ω_L have conjugate flows. Conversely, if $K, L \in \mathcal{K}_0$ have conjugate flows, then K and L have the same SLS.*

There is a similar result for the travelling times spectrum. Consider the cross-sectional map $\mathcal{P}_K : S_+^*(S_0) \setminus \text{Trap}(\Omega_K) \rightarrow S^*(S_0)$ defined by the shift along the flow $\mathcal{F}_t^{(K)}$. Let γ be a (x_0, y_0) -geodesic in Ω_K for some $x_0, y_0 \in S_0$, which is a simply reflecting ray. Let $\omega_0 \in \mathbf{S}^{d-1}$ be the (incoming) direction of γ at x_0 . We will say that γ is *regular* if the differential of the map $\mathbf{S}^{d-1} \ni \omega \mapsto \mathcal{P}_K(x_0, \omega) \in S_0$ is a submersion at $\omega = \omega_0$, i.e. its differential at that point has rank $d - 1$. Denote by $\mathcal{L}_0$ the class of all obstacles $K \in \mathcal{K}$ such that ∂K does not contain non-trivial open flat subsets and $\gamma_K(x, u)$ is a regular simply reflecting ray for almost all $(x, u) \in S_+^*(S_0)$ such that $\gamma_K(x, u) \cap \partial K \neq \emptyset$. Using an argument from Ch. 3 in [PS1] one can show that $\mathcal{L}_0$ is of second Baire category in $\mathcal{K}$ with respect to the C^∞ Whitney topology in $\mathcal{K}$. That is, generic obstacles $K \in \mathcal{K}$ belong to the class $\mathcal{L}_0$.

Theorem 2.2 ([NS]). *If the obstacles $K, L \in \mathcal{L}_0$ have almost the same travelling times, then Ω_K and Ω_L have conjugate flows. Conversely, if $K, L \in \mathcal{L}_0$ have conjugate flows, then K and L have the same travelling times.*

Next, we describe four propositions from [St2] and [St4] that are needed in the proof of Theorem 1.2. In what follows we assume

Hypothesis SCC. K and L are finite disjoint unions of strictly convex domains in $\mathbb{R}^d$ ($d \geq 2$) with C^3 boundaries, and with conjugate generalized geodesic flows.

Given $\sigma \in S^*(\Omega_K)$ denote $\gamma_K^\pm(\sigma) := \{\text{pr}_1(\mathcal{F}_t^{(K)}(\sigma)) : t \geq 0\}$.

Proposition 2.3 ([St2]). (a) *There exists a countable family $\{M_i\} = \{M_i^{(K)}\}$ of codimension 1 submanifolds of $S_+^*(S_0) \setminus \text{Trap}(\Omega_K)$ such that every $\sigma \in S_+^*(S_0) \setminus (\text{Trap}(\Omega_K) \cup_i M_i)$ generates a simply reflecting ray in Ω_K . Moreover the family $\{M_i\}$ is locally finite, that is any compact subset of $S_+^*(S_0) \setminus \text{Trap}(\Omega_K)$ has common points with only finitely many of the submanifolds M_i .*

(b) *There exists a countable locally finite family $\{R_i\}$ of codimension 2 smooth submanifolds of $S_+^*(S_0)$ such that for any $\sigma \in S_+^*(S_0) \setminus (\cup_i R_i)$ the trajectory $\gamma_K(\sigma)$ has at most one tangency to ∂K .*

(c) *There exists a countable family $\{Q_i\}$ of codimension 2 smooth submanifolds of $S_{\partial K}^*(\Omega_K)$ such that for any $\sigma \in S_+^*(S_0) \setminus (\cup_i Q_i)$ the trajectory $\gamma_K(\sigma)$ has at most one tangency to ∂K .*

It follows from the conjugacy of flows and Proposition 4.3 in [St4] that the submanifolds M_i are the same for K and L , i.e. $M_i^{(K)} = M_i^{(L)}$ for all i .

Remark. Different submanifolds M_i and M_j may have common points (these generate rays with more than one tangency to ∂K) and in general are not transversal to each other. However, as we see from part (b), if $M_i \neq M_j$ and $\sigma \in M_i \cap M_j$, then locally near σ , $M_i \neq M_j$, i.e. there exist points in $M_i \setminus M_j$ arbitrarily close to σ .

For the present case we also have some information about the size of the set $\text{Trap}(\Omega_K)$.

Proposition 2.4 ([St2]). *Let $d \geq 2$ and let K have the form (1.1). Then $S_+^*(S_0) \setminus \text{Trap}(\Omega_K)$ is an arc connected open dense subset of $S_+^*(S_0)$.*

Proof. This is essentially proved in [St2] assuming $d \geq 3$, however a small modification of the argument works for $d = 2$ as well. We sketch it here for completeness. So, assume that $d \geq 2$. Clearly $S_+^*(S_0) \setminus \text{Trap}(\Omega_K)$ is open in $S_+^*(S_0)$.

Consider an arbitrary $\sigma'_0 = (x'_0, \xi'_0) \in S_+^*(S_0) \setminus \text{Trap}(\Omega_K)$. We will find a continuous curve $\sigma'(t)$ in $S^*(\mathbb{R}^d \setminus \mathcal{O}) \setminus \text{Trap}(\Omega_K)$ such that $\sigma'(0) = \sigma'_0$ and $\sigma'(1)$ generates a free trajectory in $S^*(\mathbb{R}^n \setminus \mathcal{O})$, i.e. a trajectory without any reflections at ∂K . We will assume that the trajectory $\gamma_K^+(\sigma_0)$ has a common point with K ; otherwise there is nothing to prove.

There exists a strictly convex smooth hypersurface X in $\mathbb{R}^n \setminus \mathcal{O}$ with a continuous unit normal field ν_X such that for some $x_0 \in X$ and $t > 0$, we have $x'_0 = x_0 + t\nu_X(x_0)$ and $\nu_X(x_0) = \xi'_0$, and there exists $x_1 \in X$ such that the ray $\{x_1 + t\nu_X(x_1) : t > 0\}$ has no common points with K – e.g. take the boundary sphere X of a very large ball in $\mathbb{R}^d \setminus \mathcal{O}$ with exterior unit normal field ν_X . Clearly it is enough to construct a C^1 curve $\sigma(t)$ in $S^*(X) \setminus \text{Trap}(\Omega_K)$ such that $\sigma(0) = (x_0, \nu_X(x_0))$ and $\sigma(1) = (x_1, \nu_X(x_1))$.

Set $\tilde{X} = \{(x, \nu_X(x)) : x \in X\}$. Considering trajectories $\gamma_K^+(\sigma)$ with $\sigma \in \tilde{X}$ and using the strict convexity of X will allow us to use the strong hyperbolicity properties of the billiard flow in Ω_K ([Si1], [Si2]; see also [St1]).

Take a very large open ball U_1 that contains the orthogonal projection of the convex hull $\hat{K}$ of K onto X and the point x_1 as well. Since $\text{Trap}(\Omega_K) \cap S^*(U_1)$ is compact and $\sigma_0 = (x_0, \nu_X(x_0)) \notin \text{Trap}(\Omega_K)$, there exists an open connected neighbourhood V_0 of σ_0 in $S^*(U_1)$ with $V_0 \cap \text{Trap}(\Omega_K) = \emptyset$.

It follows from Proposition 2.3(b) that there exists a countable family $\{Q'_i\}$ of smooth codimension 2 submanifolds of $S^*(U_1)$ such that for any $\sigma \in S^*(U_1) \setminus (\bigcup_i Q'_i)$, the trajectory $\gamma_K(\sigma)$ has at most one tangency to ∂K . The submanifolds Q'_i are obtained from the submanifolds Q_i in Proposition 2.3(b) by translation along the second (vector) component, so, locally they are invariant under the flow $\mathcal{F}_t^{(K)}$.

Using Thom's Transversality Theorem (cf. e.g. [Hi]), and applying an arbitrarily small in the C^3 Whitney topology deformation to X , as in the proof of Proposition 5.1 in [St2], without loss of generality, we may assume that $\tilde{X}$ is transversal to each of the submanifolds Q'_i . When $d = 2$, this simply means that $\tilde{X} \cap Q'_i$ is a discrete subset of $\tilde{X}$. When $d \geq 3$, $\tilde{X} \cap Q'_i$ is a submanifold of $\tilde{X}$ with $\dim(\tilde{X} \cap Q'_i) = d - 3$. By the Sum Theorem for the topological dimension $\dim$ (cf. e.g. Theorem 15 in [F]), for $X' = \tilde{X} \cap (\bigcup_i Q_i)$ we get $\dim(X') \leq d - 3$.

Next, denote by X_0 the set of those $\sigma \in \tilde{X} \cap \text{Trap}(\Omega_K)$ such that the trajectory $\gamma_K^+(\sigma)$ has no tangencies to ∂K . Given integers p, q such that $q \in \{1, \dots, k_0\}$ and $p \geq 1$, denote by $X(p, q)$ the set of those $\sigma \in \tilde{X} \cap \text{Trap}(\Omega_K)$ such that $\gamma_K^+(\sigma)$ has exactly one tangency to ∂K which is its p th reflection point and it belongs to ∂K_q . As in the proof of Proposition 5.1 in [St2] (see also the proof of Lemma 3.1 below where we repeat this argument), it follows that each of the subspaces X_0 and $X(p, q)$ of $\tilde{X}$ has topological dimension 0. Assuming $d \geq 3$, the Sum Theorem for topological dimension (cf. Theorem 15 in [F]) shows that

$$\dim(\tilde{X} \cap \text{Trap}(\Omega_K)) \leq \dim(X' \cup X_0 \cup \bigcup_{p,q} X(p, q)) \leq d - 3.$$

In the case $d = 2$ we simply have $\dim(\tilde{X} \cap \text{Trap}(\Omega_K)) = 0$. In both cases a theorem by Mazurkiewicz (see e.g. Theorem 25 in [F]) implies that $\tilde{X} \setminus \text{Trap}(\Omega_K)$ is arc connected. Thus, there exists a C^1 curve $\sigma(t)$ in $\tilde{X} \setminus \text{Trap}(\Omega_K)$ (and so in $S^*(X) \setminus \text{Trap}(\Omega_K)$) such that $\sigma(0) = (x_0, \nu_X(x_0))$ and $\sigma(1) = (x_1, \nu_X(x_1))$. This proves that $S_+^*(S_0) \setminus \text{Trap}(\Omega_K)$ is arc connected.

It remains to show that $S_+^*(S_0) \setminus \text{Trap}(\Omega_K)$ is dense in $S_+^*(S_0)$. Consider an arbitrary $\sigma'_0 = (x'_0, \xi'_0) \in S_+^*(S_0)$ and construct X and $\tilde{X}$ as above. Then the above argument shows that $\dim(\tilde{X} \cap \text{Trap}(\Omega_K)) \leq d - 3$ when $d \geq 3$, and $\dim(\tilde{X} \cap \text{Trap}(\Omega_K)) = 0$ when $d = 2$. In both cases there exist points $\sigma \in \tilde{X} \setminus \text{Trap}(\Omega_K)$ arbitrarily close to σ'_0 . This implies that $S_+^*(S_0) \setminus \text{Trap}(\Omega_K)$ is dense in $S_+^*(S_0)$. $\square$

Since $S_+^*(S_0)$ is a manifold and $\text{Trap}(\Omega_K) \cap S_+^*(S_0)$ is compact, it follows from Proposition 2.4 that any two points in $S_+^*(S_0) \setminus \text{Trap}(\Omega_K)$ can be connected by a C^1 curve lying entirely in $S_+^*(S_0) \setminus \text{Trap}(\Omega_K)$. Denote by S_K the set of the points $\sigma \in S_+^*(S_0) \setminus \text{Trap}(\Omega_K)$ such that $\gamma_K(\sigma)$ is a simply reflecting ray. It follows from [MS2] (cf. also Sect. 24.3 in [H]) and Proposition 2.4 in [St2]) that S_K is open and dense and has full Lebesgue measure in $S_+^*(S_0) \setminus \text{Trap}(\Omega_K)$. Proposition 4.3 in [St4] shows that if K, L have conjugate flows, then $S_K = S_L$. Moreover, using Proposition 6.3 in [St4] and the above Proposition 2.4 we get the following.

Proposition 2.5. *Let K, L satisfy Hypothesis SCC. Then*

$$(2.1) \quad \#(\gamma_K(\sigma) \cap \partial K) = \#(\gamma_L(\sigma) \cap \partial L)$$

for all $\sigma \in S_K = S_L$.

Definition 2.6. A C^1 path $\sigma(s)$, $0 \leq s \leq a$ (for some $a > 0$), in $S_+^*(S_0) \setminus \text{Trap}(\Omega_K)$ will be called *K-admissible* if it has the following properties:

- (i) $\sigma(0)$ generates a *free ray* in Ω_K , i.e. a ray without any common points with ∂K ;
- (ii) if $\sigma(s) \in M_i$ for some i and $s \in [0, a]$, then σ is transversal to M_i at $\sigma(s)$ and $\sigma(s) \notin M_j$ for any submanifold $M_j \neq M_i$.

Thus, if a curve $\sigma(s)$, $0 \leq s \leq a$, in $S_+^*(S_0) \setminus \text{Trap}(\Omega_K)$ is admissible, then the first point $\sigma(0)$ of this curve generates a free ray $\gamma_K^+(\sigma(0))$ in Ω_K , i.e. a straight-line ray without common points with K . When s moves gradually over the interval $[0, a]$ the ray $\gamma_K^+(\sigma(s))$ may start having common points with ∂K and the number of common points may increase substantially¹ by the time s becomes equal to a . Moreover, it follows from (ii) that if for some $s \in (0, a]$ the scattering ray $\gamma_K^+(\sigma(s))$ has a tangent point to ∂K , then it has only one tangent point to ∂K and the tangency is of first order only. It is also clear from the above Definition that if the curve $\sigma(s)$ ($0 \leq s \leq a$) is admissible and $\rho(s)$ ($0 \leq s \leq a$) is uniformly close to $\sigma(s)$ (i.e. $\rho(s)$ and $\sigma(s)$ and their derivatives are ϵ -close for all s for some sufficiently small $\epsilon > 0$), then $\rho(s)$ is also admissible.

Notice that under Hypothesis SCC, we have $S_+^*(S_0) \setminus \text{Trap}(\Omega_K) = S_+^*(S_0) \setminus \text{Trap}(\Omega_L)$ and a path is *K-admissible* iff it is *L-admissible*. So, in what follows we will just call such paths *admissible*.

Proposition 2.7. *For any $\rho \in S_+^*(S_0) \setminus \text{Trap}(\Omega_K)$ there exists an admissible path $\sigma(s)$, $0 \leq s \leq a$, with $\sigma(a) = \rho$.*

Proof. This follows from Proposition 6.3 in [St4] (or rather its proof) and Propositions 2.3 and 2.4 above. We sketch the proof here for completeness.

Fix an arbitrary $\rho_0 \in S_+^*(S_0) \setminus \text{Trap}(\Omega_K)$ such that $\gamma_K^+(\rho_0)$ is a free ray in Ω_K , i.e. it has no common points with K .

Let $\rho \in S_+^*(S_0) \setminus \text{Trap}(\Omega_K)$. Since $S_+^*(S_0) \setminus \text{Trap}(\Omega_K)$ is open in $S_+^*(S_0)$ and arc connected by Proposition 2.4, there exists a smooth curve $\sigma(s)$, $0 \leq s \leq 1$, in $S_+^*(S_0) \setminus \text{Trap}(\Omega_K)$ such that $\sigma(0) = \rho_0$ and $\sigma(1) = \rho$. Now the compactness of $\text{Trap}(\Omega_K) \cap S_+^*(S_0)$ implies the existence of a compact neighbourhood W_0 of $\gamma = \{\sigma(s) : s \in [0, 1]\}$ in $S_+^*(S_0)$ such that $W_0 \cap \text{Trap}(\Omega_K) = \emptyset$. Slightly perturbing the curve γ in the interior of W_0 , we may assume that γ is transversal to any of the submanifolds M_i and R_i from Proposition 2.3 (cf. [Hi]). Since the codimension of R_i in $S_+^*(S_0)$ is 2, this implies $\gamma \cap R_i = \emptyset$ for each i . Moreover, $W_0 \cap (\bigcup_i R_i)$ is compact, since $\{R_i\}$ is locally finite. Then shrinking W_0 we may assume that $W_0 \cap R_i = \emptyset$ for all i . Now the fact that the family $\{M_i\}$ is locally finite in $S_+^*(S_0) \setminus \text{Trap}(\Omega_K)$ implies that $\gamma \cap M_i \neq \emptyset$ only for finitely many i . So, there exist only finitely many $s_1, \dots, s_k \in [0, 1]$ such that $\gamma_K^+(\sigma(s_j))$ contains a tangent point to ∂K , and moreover (since $\sigma(s_j) \notin R_i$ for any i), $\sigma(s_j)$ belongs to only one member of the family $\{M_i\}$ for all $j = 1, \dots, k$. $\square$

3. PROOF OF THEOREM 1.2

Let K and L be as in Theorem 1.2. We will show that they coincide. A point $y \in \partial K$ will be called *regular* if $\partial K = \partial L$ in an open neighbourhood of y in ∂K .

¹Although in general this number is not a monotone function of s , so it may change in many different fashions.

Otherwise y will be called irregular. A point $y \in \partial K$ will be called *accessible* if there exists $\rho \in S_+^*(S_0) \setminus \text{Trap}(\Omega_K)$ such that the trajectory $\gamma_K^+(\rho)$ contains the point y . Denote by $\mathcal{A}_K$ the set of all accessible points $y \in \partial K$.

Lemma 3.1. *Let $d \geq 2$ and let K have the form (1.1). Then $\mathcal{A}_K$ is dense in ∂K .*

Proof. We will use a slight modification of an argument contained in the proof of Proposition 5.1 in [St2] (see also Proposition 2.4 above) which assumes $d \geq 3$. We will sketch this proof here dealing with the case $d = 2$ as well.

First, it follows from Proposition 2.3(c) that there exists a countable locally finite family $\{Q_i\}$ of codimension 2 smooth submanifolds of $S_{\partial K}^*(\Omega_K)$ such that for any $\sigma \in S_{\partial K}^*(\Omega_K) \setminus (\bigcup_i Q_i)$ the trajectory $\gamma_K(\sigma)$ has at most one tangency to ∂K . Notice that when $d = 2$, we have $\dim(S_{\partial K}^*(\Omega_K)) = 2$, and then each Q_i is a finite subset of $S_{\partial K}^*(\Omega_K)$, so $\bigcup_i Q_i$ is a discrete subset of $S_{\partial K}^*(\Omega_K)$.

Consider an arbitrary $x_0 \in \partial K$ and let $\delta > 0$. We will show that there exist points $x \in \mathcal{A}_K$ which are δ -close to x_0 . Take an arbitrary $u_0 \in \mathbf{S}^{d-1}$ so that $\sigma_0 = (x_0, u_0) \in S_{\partial K}^*(\Omega_K)$. Fix a small $\epsilon > 0$ and set $X = \{x_0 + \epsilon u : \|u - u_0\| < \epsilon\}$ and $\tilde{X} = \{(x_0 + \epsilon u, u) : \|u - u_0\| < \epsilon\}$. As in the proof of Proposition 2.4, using Thom's Transversality Theorem (cf. e.g. [Hi]), and applying an arbitrarily small in the C^3 Whitney topology deformation to X , we get a C^3 convex surface Y in Ω_K which is C^3 -close to X and so that $\tilde{Y}$ is transversal to each of the submanifolds Q'_i . We take Y so close to X that for every $(y, v) \in \tilde{Y}$ for the point $x \in \partial K$ with $x + tv = y$ for some $t > 0$ close to ϵ we have $\|x_0 - x\| < \delta$. As in the proof of Proposition 2.4, when $d = 2$, $\tilde{Y} \cap Q'_i$ is a discrete subset of $\tilde{Y}$, and when $d \geq 3$, for $Y' = \tilde{Y} \cap (\bigcup_i Q_i)$ we have $\dim(Y') \leq d - 3$.

Let Y_0 the set of those $\sigma \in \tilde{Y} \cap \text{Trap}(\Omega_K)$ such that the trajectory $\gamma_K^+(\sigma)$ has no tangencies to ∂K , and let $\tilde{F} = \prod_{r=0}^{\infty} F$, where $F = \{1, 2, \dots, k_0\}$. Fix an arbitrary $\theta \in (0, 1)$ and consider the metric d on $\tilde{F}$ defined by $d((x_i), (y_i)) = 0$ if $x_i = y_i$ for all i , $d((x_i), (y_i)) = 1$ if $x_0 \neq y_0$, and $d((x_i), (y_i)) = \theta^N$ if $x_i = y_i$ for all $i = 0, 1, \dots, N - 1$ and $N \geq 1$ is maximal with this property. Then $\tilde{F}$ is a compact totally disconnected metric space. Consider the map $f : Y_0 \rightarrow \tilde{F}$, defined by $f(\sigma) = (i_0, i_1, \dots)$, where the j th reflection point of $\gamma_K^+(\sigma)$ belongs to ∂K_{i_j} for all $j = 0, 1, \dots$. Clearly, the map f is continuous and, using the strict convexity of Y , it follows from [St1] that $f^{-1} : f(Y_0) \rightarrow Y_0$ is also continuous (it is in fact Lipschitz with respect to an appropriate metric on $\tilde{F}$). Thus, Y_0 is homeomorphic to a closed subspace of $\tilde{F}$ and therefore Y_0 is a compact totally disconnected subset of $\tilde{Y}$.

Given integers p, q such that $q \in \{1, \dots, k_0\}$ and $p \geq 1$, denote by $Y(p, q)$ the set of those $\sigma \in \tilde{Y} \cap \text{Trap}(\Omega_K)$ such that $\gamma_K^+(\sigma)$ has exactly one tangency to ∂K which is its p th reflection point and it belongs to ∂K_q . Given p, q , define $f : Y(p, q) \rightarrow \tilde{F}$ as above and notice that in the present case we have $i_q = p$ for any $\sigma \in Y(p, q)$, where $f(\sigma) = (i_j)$. This and the definition of $Y(p, q)$ imply that f is continuous on $Y(p, q)$ and using [St1] again, we get that $Y(p, q)$ is homeomorphic to a closed subspace of $\tilde{F}$ and so $Y(p, q)$ is a compact totally disconnected subset of $S_{\partial K}^*(\Omega_K)$.

Thus, each of the sets Y' , Y_0 and $Y(p, q)$ is a subset of first Baire category in $\tilde{Y}$. It now follows from Baire's category theorem that $Y' \cup Y_0 \cup \bigcup_{p,q} Y(p, q)$ is nowhere dense in $\tilde{Y}$. Thus, $\mathcal{A}_K \cap \tilde{Y} \supseteq \tilde{Y} \setminus (Y' \cup Y_0 \cup \bigcup_{p,q} Y(p, q))$ is dense in $\tilde{Y}$.

Thus, there exists $(y, v) \in \widetilde{Y}$ which generates a non-trapped trajectory. By the choice of Y , if $x \in \partial K$ is so that $x + tv = y$ for some $t > 0$ close to ϵ , then we have $\|x_0 - x\| < \delta$. Clearly, $x \in \mathcal{A}_K$, so x_0 is δ -close to a point in $\mathcal{A}_K$. This proves the assertion. $\square$

Definition 3.2. For any integer $n \geq 1$ let Z_n be the set of those irregular points $x \in \partial K$ for which there exists an admissible path $\sigma(s)$, $0 \leq s \leq a$, in $S_+^*(S_0) \setminus \text{Trap}(\Omega_K)$ such that $\sigma(0)$ generates a free ray in $\mathbb{R}^d$, x belongs to the billiard trajectory $\gamma_K^+(\sigma(a))$, and for any $s \in [0, a]$ the trajectory $\gamma_K^+(\sigma(s))$ has at most n irregular common points with ∂K .

Notice that in the above definition, the billiard trajectory $\gamma_K^+(\sigma(s))$ may have more than n common points with ∂K – at most n of them will be irregular and all the others must be regular. We will prove by induction on n that $Z_n = \emptyset$ for all $n \geq 1$.

First, we will show that $Z_1 = \emptyset$. Let $x \in Z_1$. Then there exists a C^1 path $\sigma(s)$, $0 \leq s \leq a$, in $S_+^*(S_0)$ as in the definition of Z_1 . In particular, x lies on $\gamma_K^+(\sigma(a))$ and for each $s \in [0, a]$ the trajectory $\gamma_K^+(\sigma(s))$ has at most 1 irregular point. As in the argument below dealing with the inductive step, we may assume that $a > 0$ is the smallest number for which $\gamma_K^+(\sigma(a))$ has an irregular point, i.e. for all $s \in [0, a)$ the trajectory $\gamma_K^+(\sigma(s))$ contains no irregular points² at all. Set $\rho = \sigma(a)$ and $\gamma = \gamma_K^+(\sigma(a))$.

Before continuing let us remark that $\gamma = \gamma_K^+(\rho)$ and $\gamma' = \gamma_L^+(\rho)$ must have the same number of common points with ∂K and ∂L , respectively. Indeed, if γ has a tangent point to ∂K (then $\rho \in M_i$ for some, unique, i), then γ has only one tangent point to ∂K and γ' also has a unique tangent point to ∂L . Let k be the number of proper reflection points of γ at ∂K . Then we can find $\rho' \in S_+^*(S_0) \setminus \text{Trap}(\Omega_K)$ arbitrarily close to ρ so that $\gamma_K^+(\rho')$ has exactly k common points with ∂K , all of them being proper reflection points. By Proposition 2.5, $\gamma_L^+(\rho')$ also has exactly k common points with ∂L , all of them being proper reflection points. Thus, $\gamma_L^+(\rho')$ also has at most k proper reflection points, so

$$\#(\text{proper reflection points of } \gamma_K^+(\rho)) \geq \#(\text{proper reflection points of } \gamma_L^+(\rho)).$$

By symmetry, it follows that $\gamma_K^+(\rho)$ and $\gamma_L^+(\rho)$ have the same number of proper reflection points, and therefore the same number of common points with ∂K and ∂L , respectively.

Let $x_1, \dots, x_m$ be the common points of $\gamma_K^+(\rho)$ and ∂K and let $x_i = x$ for some i . Then for each $j \neq i$ there exists an open subset U_j of ∂K such that $U_j = U_j \cap \partial L$. Let $\rho = (x_0, u_0)$ and let $\omega = (x_{m+1}, u_{m+1}) \in S^*(S_0)$ be the last common point of $\gamma_K^+(\rho)$ with $S^*(S_0)$ before it goes to ∞ . Let $\mathcal{F}_{t_j}^{(K)}(\rho) = (x_j, u_j)$, $1 \leq j \leq m+1$. It then follows from the above that $\mathcal{F}_t^{(K)}(\rho) = \mathcal{F}_t^{(L)}(\rho)$ for $0 \leq t < t_i$ and also, $\mathcal{F}_\tau^{(K)}(\omega) = \mathcal{F}_\tau^{(L)}(\omega)$ for all $-(t_{m+1} - t_i) < \tau \leq 0$. So, the trajectories $\gamma_K^+(\rho)$ and $\gamma_L^+(\rho)$ both pass through x_{i-1} with the same (reflected) direction u_{i-1} and through x_{i+1} with the same (reflected) direction u_{i+1} . Thus, $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_m$ are common points of $\gamma_L^+(\rho)$ and ∂L . As observed above, $\gamma_L^+(\rho)$ must have exactly m

²We should probably remark that γ may have more than one common point with ∂K , and for some $s < a$, $\gamma_K^+(\sigma(s))$ may have common points with ∂K , however all of them will be regular points.

common points with ∂L , so it has a common point y_i with ∂L ‘between’ x_{i-1} and x_{i+1} .

Next, we consider two cases.

Case 1. x_i is a proper reflection point of γ at ∂K . It then follows immediately from the above that x_i lies on γ' and γ' has a proper reflection point at x_i , so in particular $x_i \in \partial L$. Moreover for any $y \in \partial K$ sufficiently close to x_i there exists $\rho' \in S_+^*(S_0) \setminus \text{Trap}(\Omega_K)$ close to ρ so that $\gamma_K^+(\rho')$ has a proper reflection point at y . Then repeating the above argument and using again $U_j = U_j \cap \partial L$ for $j \neq i$, shows that $y \in \partial L$. Thus, $\partial K = \partial L$ in an open neighbourhood of $x = x_i$ in ∂K , which is a contradiction with the assumption that x is an irregular point.

Case 2. x_i is a tangent point of γ to ∂K . Then γ (and so γ') has $m - 1$ proper reflection points and y_i is a point on the segment $[x_{i-1}, x_{i+1}]$. Assume for a moment that $y_i \neq x_i$. Clearly we can choose $x'_i \in \partial K$ arbitrarily close to x_i and $u'_i \in \mathbf{S}^{d-1}$ close to u_i so that u'_i is tangent to ∂K at x'_i and the straight line determined by x'_i and u'_i intersects ∂L transversally near y_i . Let $\rho' \in S_+^*(S_0) \setminus \text{Trap}(\Omega_K)$ be the point close to ρ which determines a trajectory $\gamma_K^+(\rho')$ passing through x'_i in direction u'_i , i.e. tangent to ∂K at x'_i . Then $\gamma_K^+(\rho')$ has $m - 1$ proper reflection points and one tangent point, while $\gamma_L^+(\rho')$ has m proper reflection points and no tangent points at all. This is impossible, so we must have $y_i = x_i$. Moreover, using a similar argument one shows that every $x' \in \partial K$ sufficiently close to x_i belongs to ∂L , as well. So, x_i is a regular point, contradicting our assumption.

This proves that $Z_1 = \emptyset$. Next, suppose that for some $n > 1$ we have $Z_1 = \dots = Z_{n-1} = \emptyset$ and that $Z_n \neq \emptyset$. Let $x \in Z_n$. Then there exists an admissible C^1 path $\sigma(s)$, $0 \leq s \leq a$, in $S_+^*(S_0)$ as in the definition of Z_n . In particular, x lies on $\gamma_K^+(\sigma(a))$ and for each $s \in [0, a]$ the trajectory $\gamma_K(\sigma(s))$ has at most n irregular points. Set

$$A = \{s \in (0, a] : \gamma_K^+(\sigma(s)) \text{ has } n \text{ irregular points}\}.$$

Clearly $A \neq \emptyset$, since by assumption $x \in Z_n$ and $Z_j = \emptyset$ for $j < n$. Set $b = \inf A$. Then there exists a decreasing sequence $\{s_m\} \subset (0, a]$ with $s_m \searrow b$ and such that for each $m \geq 1$ the trajectory $\gamma_K^+(\sigma(s_m))$ contains n distinct irregular (consecutive) points $x_1^{(m)}, \dots, x_n^{(m)}$. Since K has the form (1.1), these points belong to distinct connected components of K . Choosing an appropriate subsequence of $\{s_m\}$, we may assume that $x_i^{(m)}$ belongs to the same connected component K_{j_i} for all $m \geq 1$ and there exists $x_i = \lim_{m \rightarrow \infty} x_i^{(m)}$ for all $i = 1, \dots, n$. Then $x_1, \dots, x_n$ are distinct common points of $\gamma_K^+(\sigma(b))$ with ∂K . Moreover, if some x_i is a regular point, then $\partial K = \partial L$ in an open neighbourhood of x_i in ∂K , so $x_i^{(m)}$ would be regular for large m , a contradiction. Thus, all $x_1, \dots, x_n$ are irregular points.

The above reasoning shows that we may assume $b = a$, $\gamma_K^+(\sigma(a))$ contains exactly n irregular points, and for any $s \in [0, a)$ the trajectory $\gamma_K^+(\sigma(s))$ has $< n$ irregular points. Now the inductive assumption implies that for any $s \in [0, a)$ the trajectory $\gamma_K^+(\sigma(s))$ contains no irregular points at all. Since $Z_j = \emptyset$ for $j < n$, every irregular point of $\gamma_K^+(\sigma(a))$ must belong to Z_n .

Set $\gamma = \gamma_K^+(\sigma(a))$ for brevity. Let $x_1, \dots, x_n$ be the consecutive irregular points of γ (which may have some other common points with ∂K) and for $s < a$ close to a , let $x_1(s), \dots, x_n(s)$ be the consecutive common points of $\gamma_K^+(\sigma(s))$ with ∂K such that $x_i(s)$ lies on the connected component K_{j_i} of K containing x_i ($i = 1, \dots, n$).

Then for every i , $\partial K = \partial L$ in an open neighbourhood of $x_i(s)$ in ∂K for $s < a$ close to a , so there exists an open subset U_i of ∂K with $x_i \in \overline{U_i}$ and $\partial K \cap U_i = \partial L \cap U_i$. Next, we consider two cases.

Case 1. γ contains no tangent points to ∂K .

Choose a small $\delta > 0$ (how small will be determined later). We will replace the path $\sigma(s)$ by another one $\tilde{\sigma}(s)$, $0 \leq s \leq a$, such that $\tilde{\sigma}(s) = \sigma(s)$ for $s \in [0, a - 2\delta]$. Let $\mathcal{F}_{t_1}^{(K)}(x_1, u_1) = (x_2, u_2)$ for some $t_1 > 0$, where $u_1 \in \mathbf{S}^{d-1}$ is the (reflected) direction of γ at x_1 and $u_2 \in \mathbf{S}^{d-1}$ is the reflected direction of γ at x_2 . Take a small $\epsilon > 0$ and set

$$X := \{x_1 + \epsilon u : u \in \mathbf{S}^{d-1}, \|u - u_1\| < \delta\}$$

and

$$\tilde{X} := \{(x_1 + \epsilon u, u) : u \in \mathbf{S}^{d-1}, \|u - u_1\| < \delta\}.$$

Let $u_1(s)$ be the (reflected) direction of the trajectory $\sigma(s)$ at $x_1(s)$. Take $t' < t_1$ close to t_1 and set $\tilde{Y} = \mathcal{F}_{t'}^{(K)}(\tilde{X})$, $Y = \text{pr}_1(\tilde{Y})$. Then

$$\tilde{Y} = \{(y, \nu_Y(y)) : y \in Y\},$$

where $\nu_Y(y)$ is the *unit normal* to Y at y in the direction of the flow $\mathcal{F}_t^{(K)}$, and $(y_1, v_1) = \mathcal{F}_{t'}^{(K)}(x_1, u_1) \in \tilde{Y}$, so $y_1 \in Y$.

Set

$$\hat{Y} = \{(y, v) : y \in Y, v \in \mathbf{S}^{d-1}, \|v - \nu_Y(y)\| < \epsilon\}.$$

There exists an open neighbourhood V of (x_1, u_1) in $S_{\partial K}^*(\Omega_K)$ such that the shift $\Phi : V \rightarrow \hat{Y}$ along the flow $\mathcal{F}_t^{(K)}$ is well-defined and smooth. Assuming δ is sufficiently small, $(y_1(s), v_1(s)) = \Phi(x_1(s), u_1(s))$ is well-defined for $s \in [a - 2\delta, a]$. Then $y_1(s)$, $s \in [a - 2\delta, a]$, is a C^1 curve on Y and the ray issued from $y_1(s)$ in direction $v_1(s)$ hits ∂K_{j_2} at $x_2(s)$. Moreover, $v_1(a) = \nu_Y(y_1)$.

Assuming $\delta > 0$ is sufficiently small, for all $s, s' \in [a - 2\delta, a]$, there exists a unique vector $v_1(s, s') \in \mathbf{S}^{d-1}$ such that

$$y_1(s) + t(s, s')v_1(s, s') = x_2(s')$$

for some $t(s, s')$ close to $t_1 - t'$. Moreover, $v_1(s, s')$ and $t(s, s')$ are smoothly (C^1) depending on s and s' . Clearly, $v_1(s, s) = v_1(s)$.

Since Y is a strictly convex surface with a unit normal field $\nu_Y(y)$, it is clear that for any $s \in (0, a]$ sufficiently close to a there exists $y \in Y$ close to y_1 such that $y + t\nu_Y(y) = x_2(s)$ for some t close to $t_1 - t'$. Fix a small $\delta > 0$, set $s_0 = a - \delta/2$ and let $\tilde{y} \in Y$ be so that

$$(3.1) \quad \tilde{y} + t\nu_Y(\tilde{y}) = x_2(s_0) \in U_2$$

for some $t > 0$ close to $t_1 - t'$. Take a C^1 curve $\tilde{y}_1(s)$, $s \in [a - 2\delta, a]$, such that $\tilde{y}_1(s) = y_1(s)$ for $s \in [a - 2\delta, a - 3\delta/2]$ and $\tilde{y}_1(a) = \tilde{y}$.

Next, define $\hat{s}(s)$, $s \in [0, a]$, by $\hat{s}(s) = s$ for $0 \leq s \leq a - \delta$ and

$$\hat{s}(s) = \frac{s}{2} + s_0 - \frac{a}{2} = s_0 - \frac{a-s}{2} \in [a - \delta, s_0], \quad s \in [a - \delta, a].$$

Then $\hat{s}$ is continuous however not differentiable at $s = a - \delta$. Take a C^1 function $\tilde{s}(s)$, $s \in [0, a]$, which coincides with $\hat{s}$ on $[0, a - \delta - \delta_1]$ and on $[a - \delta + \delta_1, a]$ for

some $\delta_1 < \delta/2$ so that the range of $\tilde{s}$ is the same as that of $\hat{s}$, i.e. it coincides with the interval $[0, s_0]$. Now for any $s \in [a - 2\delta, a]$ take $\tilde{v}_1(s)$ so that

$$\tilde{y}_1(s) + \tilde{t}(s)\tilde{v}_1(s) = x_2(\tilde{s}(s)), \quad s \in [a - 2\delta, a],$$

for some $\tilde{t}(s)$ close to $t_1 - t'$. For $s \in [a - 2\delta, a - \delta - \delta_1]$ we have $\tilde{y}_1(s) = y_1(s)$ and $\tilde{s}(s) = s$, which imply $\tilde{v}_1(s) = v_1(s)$. When $s = a$ we have $\tilde{y}_1(a) = \tilde{y}$ and $\tilde{s}(a) = \hat{s}(a) = s_0$, so by (3.1) we must have $\tilde{v}_1(a) = \nu_Y(\tilde{y})$, i.e. $(\tilde{y}_1(a), \tilde{v}_1(a)) \in \tilde{Y}$, and therefore

$$(3.2) \quad \text{pr}_1(\Phi^{-1}(\tilde{y}_1(a), \tilde{v}_1(a))) = \text{pr}_1(\mathcal{F}_{-t'}^{(K)}(\tilde{y}_1(a), \tilde{v}_1(a))) = x_1.$$

Set $(\tilde{x}_1(s), \tilde{u}_1(s)) = \Phi^{-1}(\tilde{y}_1(s), \tilde{v}_1(s))$, $s \in [a - 2\delta, a]$. For $s \in [a - 2\delta, a - \delta - \delta_1]$, we have $\tilde{s}(s) = \hat{s}(s) = s$ and therefore $\tilde{y}_1(s) = y_1(s)$ and $\tilde{v}_1(s) = v_1(s)$, which gives $(\tilde{x}_1(s), \tilde{u}_1(s)) = (x_1(s), u_1(s))$.

Now define the path $\tilde{\sigma}(s)$, $0 \leq s \leq a$, on $S_+^*(S_0)$ by $\tilde{\sigma}(s) = \sigma(s)$ for $s \in [0, a - 2\delta]$, and for $s \in [a - 2\delta, a]$ let $\tilde{\sigma}(s)$ be the unique point such that $\mathcal{F}_{\tilde{t}_0(s)}^{(K)}(\tilde{\sigma}(s)) = (\tilde{x}_1(s), \tilde{u}_1(s))$ for some $\tilde{t}_0(s)$ close to $t_0(s)$, where $\mathcal{F}_{t_0(s)}^{(K)}(\sigma(s)) = (x_1(s), u_1(s))$. It follows from (3.2) that $\tilde{\sigma}(a)$ contains the point x_1 . Moreover the construction of $\tilde{\sigma}$ shows that it is a C^1 path. Assuming δ is sufficiently small, $\tilde{\sigma}(s)$ has no tangent points to ∂K for all $s \in [a - 2\delta, a]$, so $\tilde{\sigma}(s)$ is an admissible path. For $s \in [0, a - 2\delta]$, $\gamma_K(\tilde{\sigma}(s)) = \gamma_K(\sigma(s))$ contains no irregular points. For $s \in [a - 2\delta, a]$, $\gamma_K(\tilde{\sigma}(s))$ has at most n irregular points, close to $x_1, x_2, \dots, x_n$ (if any). If for some $s \in [a - 2\delta, a]$, $\gamma_K(\tilde{\sigma}(s))$ has exactly n irregular points, then the second of these must be $x_2(\tilde{s}(s))$. However, $\tilde{s}(s) < a$ for all such s , so $x_2(\tilde{s}(s)) \in U_2$, and therefore that cannot be the case. Thus, for all $s \in [a - 2\delta, a]$, $\gamma_K(\tilde{\sigma}(s))$ has not more than $n - 1$ irregular points, and therefore x_1 (and every other irregular point on $\gamma_K(\tilde{\sigma}(a))$) belongs to Z_{n-1} . This is a contradiction with the inductive assumption that $Z_j = \emptyset$ for all $j = 1, \dots, n - 1$.

Case 2. γ contains a tangent point y_0 to ∂K (this may be one of the irregular points x_i).

Let $y_0 \in \partial K_p$ for some p . According to the definition of an admissible path, γ has only one tangent point to ∂K , so all other common points are proper reflection points. Since $n > 1$, at least one of the irregular points on γ is a proper reflection point, and at least one of them is different from y_0 . We will assume $x_1 \neq y_0$; the general case is very similar. As in Case 1, we will replace the path $\sigma(s)$ by another one $\tilde{\sigma}(s)$, $0 \leq s \leq a$, such that $\tilde{\sigma}(s) = \sigma(s)$ for $s \in [0, a - 2\delta]$ for some small $\delta > 0$.

Let $\mathcal{F}_\tau^{(K)}(x_1, u_1) = (y_0, v_0)$ for some $\tau \in \mathbb{R}$ (which may be positive or negative), where $u_1 \in \mathbf{S}^{d-1}$ is the (reflected) direction of γ at x_1 and v_0 is the direction of γ at y_0 . Take a small $\epsilon > 0$ and let

$$X := \{x_1 + \epsilon u : u \in \mathbf{S}^{d-1}, \|u - u_1\| < \delta\}$$

and

$$\tilde{X} := \{(x_1 + \epsilon u, u) : u \in \mathbf{S}^{d-1}, \|u - u_1\| < \delta\}.$$

Assuming $\delta > 0$ is small enough, for $s \in [a - 2\delta, a]$ the trajectory $\gamma_K(\sigma(s))$ has a proper reflection point $x_1(s) \in \partial K_{j_1}$. Let $u_1(s)$ be the (reflected) direction of the trajectory $\sigma(s)$ at $x_1(s)$.

Clearly in the present case we have $\sigma(a) \in M_i$ for some i . By the definition of an admissible path, $\sigma(a) \notin M_j$ for any $j \neq i$ and moreover σ is transversal

to M_i at $\sigma(a)$. Let $\sigma(s) = (x_0(s), u_0(s)) \in S_+^*(S_0)$ and let $\mathcal{F}_{t_0(s)}^{(K)}(x_0(s), u_0(s)) = (x_1(s), u_1(s))$. The shift $\Phi : S_+^*(S_0) \rightarrow S_{\partial K}^*(\Omega_K)$ along the flow $\mathcal{F}_t^{(K)}$ is well-defined on an open neighbourhood V_0 of $(x_0, u_0) = (x_0(a), u_0(a))$ and defines a diffeomorphism $\Phi : V_0 \rightarrow V = \Phi(V_0)$ for some small open neighbourhood V of (x_1, u_1) in $S_{\partial K}^*(\Omega_K)$. Since M_i is a submanifold of $S_+^*(S_0)$ of codimension one, we can take V_0 so that $V_0 \setminus M_i$ has two (open) connected components (separated by M_i) – each of them diffeomorphic to an open half-ball. We take V_0 so small that $V_0 \cap M_j = \emptyset$ for all $j \neq i$.

Setting $M'_i = \Phi(M_i \cap V_0)$, we get a similar picture in V , namely $V \setminus M'_i$ has two (open) connected components (separated by M'_i) – each of them diffeomorphic to an open half-ball. Assuming δ is sufficiently small, we have $\sigma(s) \in V_0$ for all $s \in [a - 2\delta, a]$ and σ is transversal to M_i at $\sigma(a) = (x_0, u_0)$. Thus, the curve $(x_1(s), u_1(s))$, $s \in [a - 2\delta, a]$, in V is transversal to M'_i at (x_1, u_1) , so it must be contained in one of the connected components of $V \setminus M'_i$. Denote by V_+ the connected component of $V \setminus M'_i$ that contains $(x_1(s), u_1(s))$ for $s \in [a - 2\delta, a]$.

Finally, to get this picture near the tangent point y_0 , take $\tau' \in (\tau - \delta, \tau - \delta/2)$ and set

$$F = \{(x_1, u) : u \in \mathbf{S}^{d-1}, \|u - u_1\| < \delta\} \subset V, \quad \tilde{Y} = \mathcal{F}_{\tau'}^{(K)}(F),$$

and $Y = \text{pr}_1(\tilde{Y})$. Then $\tilde{Y} = \{(y, \nu_Y(y)) : y \in Y\}$, where $\nu_Y(y)$ is the *unit normal* to Y at y in the direction of the flow $\mathcal{F}_t^{(K)}$, and $(y_1, v_1) = \mathcal{F}_{\tau'}^{(K)}(x_1, u_1) \in \tilde{Y}$, so $y_1 \in Y$. Let $\Psi : F \rightarrow \tilde{Y}$ be the shift along the flow $\mathcal{F}_t^{(K)}$. Assuming that δ is sufficiently small, this defines a diffeomorphism $\Psi : F \rightarrow G$ between F and an open subset G of $\tilde{Y}$. Set $M''_i = \Psi(F \cap M'_i)$. Then M''_i consists of those $(y, \nu_Y(y)) \in G$ that generate trajectories tangent to ∂K (and this can only happen in the vicinity of y_0 on ∂K_p). It is clear (by a direct observation using the convexity of K_p) that $G \setminus M''_i$ has two connected components. Let G_+ be the one with $\Psi^{-1}(G_+) \subset V_+$. Thus, there exists $v'_1 \in \mathbf{S}^{d-1}$, $\|v'_1 - v_1\| < \delta$ such that $(y_1, v'_1) \in G_+$. Applying Ψ^{-1} , this gives $u'_1 \in \mathbf{S}^{d-1}$ with $\|u'_1 - u_1\| < \delta$ such that $(x_1, u'_1) \in V_+$.

Since V_+ is connected (in fact diffeomorphic to an open half-ball), there exists a C^1 curve $(\tilde{x}_1(s), \tilde{u}_1(s))$, $s \in [a - 2\delta, a]$, in V_+ such that $(\tilde{x}_1(s), \tilde{u}_1(s)) = (x_1(s), u_1(s))$ for $s \in [a - 2\delta, a - \delta]$ and $(\tilde{x}_1(a), \tilde{u}_1(a)) = (x_1, u'_1)$. Define the path $\tilde{\sigma}(s)$ by $\tilde{\sigma}(s) = \sigma(s)$ for $s \in [0, a - 2\delta]$ and $\tilde{\sigma}(s) = \Phi^{-1}(\tilde{x}_1(s), \tilde{u}_1(s))$ for $s \in [a - 2\delta, a]$. It is clear from the construction that $\tilde{\sigma}$ is a C^1 path in $S_+^*(S_0)$. From the properties of σ , we have that for $s \in [0, a - 2\delta]$ the trajectory $\gamma_K(\tilde{\sigma}(s))$ does not contain any irregular points. The choice V_0 and that of the curve $(\tilde{x}_i(s), \tilde{u}_1(s))$, $s \in [a - 2\delta, a]$, show that $\gamma_K(\tilde{\sigma}(s))$ has no tangencies to ∂K for any $s \in [a - 2\delta, a]$. On the other hand, $\gamma_K(\tilde{\sigma}(a))$ contains the irregular point x_1 (and therefore must have n irregular points, since $Z_j = \emptyset$ for $j < n$). Now repeating the argument from Case 1 we get a contradiction. Thus, we must have $Z_n = \emptyset$.

This completes the induction process and proves that $Z_n = \emptyset$ for all $n \geq 1$.

We will now prove that $\partial K \subseteq \partial L$. By Lemma 3.1, $\mathcal{A}_K$ is dense in ∂K , so it is enough to show that $\mathcal{A}_K \subseteq \partial L$. Given $x \in \mathcal{A}_K$, by the definition of $\mathcal{A}_K$ (see the beginning of Sect. 3), x is accessible, i.e. there exists $\rho \in S_+^*(S_0) \setminus \text{Trap}(\Omega_K)$ such that $\gamma_K^+(\rho)$ has a reflection point at x . Then by Propositions 2.4 and 2.7, there exists an admissible path $\sigma(s)$, $0 \leq s \leq a$, for some $a > 0$, with $\rho' = \sigma(a)$ arbitrarily close to ρ . Then $\gamma_K^+(\sigma(\rho'))$ has a reflection point x' at ∂K near x . Since there are

no irregular points on $\sigma(a)$, it follows that $\partial K = \partial L$ on an open neighbourhood of x' . Thus, x is arbitrarily close to ∂L , so we must have $x \in \partial L$. This proves that $\partial K \subseteq \partial L$.

By symmetry, $\partial L \subseteq \partial K$, so we have $\partial K = \partial L$. $\square$

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UNIVERSITY OF WESTERN AUSTRALIA, CRAWLEY, WESTERN AUSTRALIA 6009, AUSTRALIA
E-mail address: `lyle.noakes@uwa.edu.au`

UNIVERSITY OF WESTERN AUSTRALIA, CRAWLEY, WESTERN AUSTRALIA 6009, AUSTRALIA
E-mail address: `luchezar.stoyanov@uwa.edu.au`