ASYMPTOTICS OF SHARP CONSTANTS OF MARKOV-BERNSTEIN INEQUALITIES IN INTEGRAL NORM WITH JACOBI WEIGHT

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ABSTRACT. The classical A. Markov inequality establishes a relation between the maximum modulus or the $L^\infty([-1, 1])$ norm of a polynomial Q_n and of its derivative: $\|Q'_n\| \leq M_n n^2 \|Q_n\|$, where the constant $M_n = 1$ is sharp. The limiting behavior of the sharp constants M_n for this inequality, considered in the space $L^2([-1, 1], w^{(\alpha, \beta)})$ with respect to the classical Jacobi weight $w^{(\alpha, \beta)}(x) := (1-x)^\alpha (x+1)^\beta$, is studied. We prove that, under the condition $|\alpha - \beta| < 4$, the limit is $\lim_{n \rightarrow \infty} M_n = 1/(2j_\nu)$ where j_ν is the smallest zero of the Bessel function $J_\nu(x)$ and $2\nu = \min(\alpha, \beta) - 1$.

1. INTRODUCTION

A quantity

$$(1.1) \quad M_n := \sup_{\deg Q_n \leq n} \frac{\|Q'_n\|_{\mathfrak{X}_1}}{\|Q_n\|_{\mathfrak{X}_2}}, \quad Q_n - \text{polynomial},$$

is called the sharp constant for the Markov-Bernstein inequality in functional spaces $\mathfrak{X}_1, \mathfrak{X}_2$ with norms $\|\cdot\|_{\mathfrak{X}_1}, \|\cdot\|_{\mathfrak{X}_2}$.

The remarkable classical inequality of A.A. Markov for $\mathfrak{X}_1 = \mathfrak{X}_2 = L^\infty([-1, 1])$

$$\|Q'_n\| \leq n^2 \|Q_n\|$$

is sharp [1]. We recall that the corresponding inequality for the trigonometric polynomials was first obtained by S.N. Bernshtein in [2]. His result was not sharp, and the sharp version is due to E. Landau (see [3]). For the weighted L^2 spaces $\mathfrak{X}_1 = \mathfrak{X}_2 := L^2([a, b], w)$, for some classical weights, the sharp constants (1.1) are known (see [4, pp. 570-571]):

1. $w = \exp(-x^2), x \in (-\infty, \infty), M_n = \sqrt{2n}$ (E. Schmidt, 1944),
2. $w = \exp(-x), x \in (0, \infty), M_n = \frac{1}{2 \sin(\frac{\pi}{4n+2})}$ (P. Turán, 1960).

However, for other classical weights, explicit expressions for the sharp constants are not known. In [8] results on the asymptotics $M_n \rightarrow ?$, $n \rightarrow \infty$ were discussed.

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In particular, for the Gegenbauer weight $w^{(\alpha)}(x) := (1-x^2)^\alpha$, $x \in [-1, 1]$, $\alpha > -1$, the following result was stated there

$$(1.2) \quad M_n = \frac{n^2}{2j_{\nu(\alpha)}} (1 + o(1)), \quad \nu(\alpha) := \frac{\alpha - 1}{2},$$

where j_ν is the smallest zero of the Bessel function $J_\nu(x)$ (we shall keep the notation $\nu(\alpha)$ and j_ν in what follows).

In the present paper we study the asymptotics of the sharp constant (1.1) for the classical Jacobi weight (the space $L^2([-1, 1], w^{(\alpha, \beta)})$ as defined in Section 2.1):

$$(1.3) \quad \mathfrak{X}_1 = \mathfrak{X}_2 := L^2([-1, 1], w^{(\alpha, \beta)}), \quad w^{(\alpha, \beta)}(x) := (1-x)^\alpha (x+1)^\beta, \quad \alpha, \beta > -1.$$

The main result of our paper is

Theorem 1.1. *Let the parameters of the Jacobi weight (1.3) satisfy the restriction*

$$(1.4) \quad w^{(\alpha, \beta)}(x) : \quad |\alpha - \beta| < 4.$$

Then, for the sharp constant (1.1) in the space (1.3), we have the asymptotics:

$$(1.5) \quad M_n = \frac{n^2}{2j_{\nu^*}} (1 + o(1)), \quad \nu^* = \min\{\nu(\alpha), \nu(\beta)\}.$$

We see that, for $\alpha = \beta$, asymptotics (1.5) match (1.2). When $\alpha \neq \beta$, then asymptotics (1.5) appear as a reasonable generalization of (1.2). The most surprising for us is the appearance of the restriction (1.4). At the moment we cannot prove or disapprove its necessity, however, we have to admit that this restriction is unavoidable in our proof strategy of Theorem 1.1.

The rest of the paper contains the proof of Theorem 1.1. Our approach consists of the following steps:

- 1) We start with an explicit representation of M_n as the eigenvalue of a linear operator in $\mathbb{R}^n$ defined by a five diagonal matrix.
- 2) Then we state a Finite Difference (FD) Boundary Value Problem (BVP) which is equivalent to the eigenvalue problem.
- 3) The next step is to determine a limiting (for FD problem) Differential Equation (DE) and its general solution.
- 4) Then we vanish the spectral parameter in the FD problem and find linearly independent Particular Solutions (PS) satisfying Boundary Conditions (BC) at the initial values of the discrete variable (the left end BC). For the small (with respect to n) indexes in FD and spectral parameter in the fixed range, the asymptotics of the solutions of FD does not depend on the spectral parameter. Therefore, the initial conditions can be rewritten as asymptotics conditions for the indices $1 \ll k \ll n$. Then this condition is exported to the boundary condition of DE.
- 5) Matching these FD problems, we get PS of the limiting DE.
- 6) Finally, taking these PS of DE as an approximation of the PS of FD, we satisfy the right end BC of FD BVP. It gives an approximation of the desired eigenvalue as in (1.2) or (1.5).

These steps are performed in Section 2. Some of these steps have already been studied before for various functional spaces in (1.1), see in [4, Chapter 6 Section 6.1.6], [9], [8], [10], [11].

However, to conclude a rigorous proof of Theorem 1.1, it remains to justify the final step, i.e. to prove that PS of DE which match the satisfying to the left end BC of FD problem, indeed are close to the PS of FD problem. In Section 3 we state and prove the corresponding result, see Theorem 3.1. This theorem establishes a new result on the local asymptotics of the powerlike growing solution of the high order recurrence relations. Previous results in this direction are in [12], [14], [13]. In all of the following we will use the factorial notation $(a)!$ for the value of $\Gamma(a+1)$.

2. FINITE DIFFERENCE BVP FOR M_n AND ITS DIFFERENTIAL APPROXIMATION

2.1. A spectral representation for M_n in $\mathbb{R}^n$. We note from (1.1), that M_n is the norm of the operator differentiation in a finite dimensional space $\mathcal{P}_n$ of polynomials of degree at most equal to n . Let Q_n be an arbitrary polynomial of $\mathcal{P}_n$. We take the expansion of this polynomial Q_n and of its derivative Q'_n in the basis of monic Jacobi polynomials $P_k^{(\alpha, \beta)}$ ($P_k^{(\alpha, \beta)}(x) = x^k + \dots$). By using the $L^2([-1, 1], w^{(\alpha, \beta)})$ inner product

$$(2.1) \quad (g, f) := \int_{-1}^1 g(x) \overline{f(x)} w^{(\alpha, \beta)}(x) dx,$$

the square norm of $P_n^{(\alpha, \beta)}$ is

$$(2.2) \quad \|P_n^{(\alpha, \beta)}\|^2 = (P_n^{(\alpha, \beta)}, P_n^{(\alpha, \beta)}) = 2^{2n+\alpha+\beta-1} \frac{n!(n+\alpha)!(n+\beta)!(n+\alpha+\beta)!}{(2n+\alpha+\beta)!(2n+\alpha+\beta+1)!}.$$

Then, we have (in general $Q_n = \sum_{k=0}^n c_k P_k^{(\alpha, \beta)}$) but to solve (1.1), that it is sufficient to consider the case where $c_0 = 0$):

$$(2.3) \quad Q'_n(x) := \sum_{k=0}^{n-1} v_k P_k^{(\alpha, \beta)}, \quad Q_n(x) := \sum_{k=0}^{n-1} u_k P_{k+1}^{(\alpha, \beta)}.$$

Differentiating Q_n here, and using the property of Jacobi polynomials

$$\frac{d}{dx} P_k^{(\alpha, \beta)}(x) = k P_{k-1}^{(\alpha+1, \beta+1)}(x),$$

we arrive at

$$Q'_n = \sum_{k=0}^{n-1} u_k P'_{k+1}^{(\alpha, \beta)} = \sum_{k=0}^{n-1} (k+1) u_k P_k^{(\alpha+1, \beta+1)} = \sum_{k=0}^{n-1} v_k P_k^{(\alpha, \beta)}.$$

Then, applying the well-known α increasing (and β increasing) relation (see [5, Chapter 22]):

$$(2.4) \quad P_n^{(\alpha, \beta)} = P_n^{(\alpha+1, \beta)} - \frac{2n(n+\beta)P_{n-1}^{(\alpha+1, \beta)}}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)},$$

we obtain for the vectors from (2.3)

$$\vec{v} := (v_0, v_1, \dots, v_{n-1})^{Tr}, \quad \vec{u} := (u_0, u_1, \dots, u_{n-1})^{Tr},$$

the following relations

$$(2.5) \quad \mathbf{N}\vec{u} = \mathbf{C}_2 \mathbf{C}_1 \vec{v}, \quad \mathbf{N} := \text{diag}(1, 2, \dots, n),$$

and for the $n \times n$ matrices $\mathbf{C}_2, \mathbf{C}_1$ we have from (2.4)

$$\begin{aligned} \mathbf{C}_1 &:= \mathbf{I} - \text{diag} \left(\frac{2k(k+\beta)}{(2k+\alpha+\beta)(2k+\alpha+\beta+1)} \right)_{k=1}^n \mathbf{T} \\ \mathbf{C}_2 &:= \mathbf{I} + \text{diag} \left(\frac{2k(k+\alpha+1)}{(2k+\alpha+\beta+1)(2k+\alpha+\beta+2)} \right)_{k=1}^n \mathbf{T} \end{aligned}, \quad \mathbf{T} := \begin{pmatrix} 0 & 1 & & 0 \\ & 0 & 1 & \\ & & 0 & \ddots \\ 0 & & & \ddots & 1 \\ 0 & 0 & & & 0 \end{pmatrix}.$$

Now, we write norms for (2.3) by using the inner product (2.1):

$$\begin{aligned} \|Q'_n\|^2 &= \langle \vec{v}, \mathbf{D} \vec{v} \rangle, \quad \mathbf{D} := \text{diag} \left(\|P_k^{(\alpha, \beta)}\|^2 \right)_{k=0}^{n-1}, \\ \|Q_n\|^2 &= \langle \vec{u}, \mathbf{D}^+ \vec{u} \rangle, \quad \mathbf{D}^+ := \text{diag} \left(\|P_k^{(\alpha, \beta)}\|^2 \right)_{k=1}^n, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ is the usual vector inner product. Thus, for the sharp constant in (1.1) - (1.3) we have by using (2.5)

$$(2.6) \quad M_n^2 = \sup_{\vec{v}} \frac{\langle \vec{v}, \mathbf{D} \vec{v} \rangle}{\langle \mathbf{N}^{-1} \mathbf{C}_2 \mathbf{C}_1 \vec{v}, \mathbf{D}^+ \mathbf{N}^{-1} \mathbf{C}_2 \mathbf{C}_1 \vec{v} \rangle} = \sup_{\vec{v}} \frac{\langle \vec{v}, \mathbf{D} \vec{v} \rangle}{\langle \vec{v}, \mathbf{A} \vec{v} \rangle},$$

where we denote

$$(2.7) \quad \mathbf{A} := \mathbf{C}_1^{Tr} \mathbf{C}_2^{Tr} \mathbf{N}^{-1} \mathbf{D}^+ \mathbf{N}^{-1} \mathbf{C}_2 \mathbf{C}_1, \quad \mathbf{D} := \text{diag}(d_k)_{k=0}^{n-1}, \quad \mathbf{D}^+ := \text{diag}(d_k)_{k=1}^n.$$

For the purpose of (2.6) we can omit the factor $2^{\alpha+\beta-1}$ in (2.2), so we put

$$(2.8) \quad d_k := \frac{2^{2k} k! (k+\alpha)! (k+\beta)! (k+\alpha+\beta)!}{(2k+\alpha+\beta)! (2k+\alpha+\beta+1)!}.$$

Finally, from (2.6) we get by the arguments of a pencil of quadratic forms (see [6, Chapter 10.7]) the spectral radius representation for the exact constant:

$$(2.9) \quad M_n^2 = \lambda_{\min}^{-1}(\mathbf{A}, \mathbf{D}),$$

where $\lambda_{\min}(\mathbf{A}, \mathbf{D})$ is a root (with the minimal modulus) of the equation

$$(2.10) \quad \det(\mathbf{A} - \lambda \mathbf{D}) = 0,$$

and correspondingly the eigenvector $\vec{v}$

$$(2.11) \quad (\mathbf{A} - \lambda_{\min} \mathbf{D}) \vec{v} = 0$$

defines the extremal polynomial (2.3) in (1.1), (1.3).

2.2. Finite difference equation for the coordinates of $\vec{v}$. To simplify expressions (i.e. to cancel factorials) in what follows, we introduce a new variable for the coordinates of the vector $\vec{v}$ (see (2.3)):

$$(2.12) \quad v_k =: x_k \frac{(2k+\alpha+\beta+1)!}{2^k (k+\alpha)! (k+\beta)!}, \quad k = 0, \dots, n-1.$$

Next, taking the k -th coordinate of the equation (2.11)

$$(2.13) \quad [(\mathbf{A} - \lambda \mathbf{D}) \vec{v}]_k = 0, \quad k = 0, \dots, n-1,$$

we get a 5-term recurrence relation which connects the coordinates $\{x_s\}_{s=k-2}^{k+2}$, $k = 0, 1, 2, \dots, n-1$:

$$\begin{aligned} x_{k+2} \frac{(k+2)!}{(k-2)!} \frac{(2k+\alpha+\beta)!}{(2k+\alpha+\beta-3)!} \frac{(k+\alpha+\beta+1)!}{(k+\alpha+\beta-1)!} &= x_{k+1} \frac{(k+1)!}{(k-2)!} \frac{(2k+\alpha+\beta-1)!}{(2k+\alpha+\beta-3)!} \Xi_1 + \\ x_k \left(\Xi_2 - \lambda \frac{(k+1)!}{(k-2)!} \frac{(2k+\alpha+\beta+4)!}{(2k+\alpha+\beta-3)!} \frac{(k+\alpha+\beta)}{4} \right) &+ x_{k-2} \frac{(k+1)!}{(k-1)!} \frac{(2k+\alpha+\beta+4)!}{(2k+\alpha+\beta+1)!} \frac{(k+\alpha)!}{(k+\alpha-2)!} \frac{(k+\beta)!}{(k+\beta-2)!} \\ + x_{k-1} \frac{(2k+\alpha+\beta+4)!}{(2k+\alpha+\beta+2)!} (k^2-1)(k+\alpha)(k+\beta)(2k+\alpha+\beta-1)(\alpha+\beta-2)(\alpha-\beta), \end{aligned}$$

where

$$\begin{aligned} \Xi_1 &= (k+\alpha+\beta)(2k+\alpha+\beta+3)(\alpha+\beta-2)(\alpha-\beta), \quad \Xi_2 = \frac{k^4}{2} + k^3(1+\alpha+\beta) \\ &+ k^2 \frac{2\alpha+2\beta+2\alpha^2+3\alpha\beta+2\beta^2+1}{2} + k \frac{(1+\alpha+\beta)(\alpha^2+\alpha\beta+\beta^2)}{2} + O_{\alpha, \beta}(1). \end{aligned}$$

This finite difference equation can be considered as a spectral equation for the problem (2.11). We obtain a nontrivial solution of (2.11) if $x_{-1} = x_{-2} = 0$ and $x_n = x_{n+1} = 0$. These boundary conditions will be widely used in the paper. The 5-term recurrence equation can be rewritten in a matrix form for the bundle $\vec{X}_k$:

$$(2.14) \quad \vec{X}_{k+2} = \mathbf{M}_2(k, \lambda) \vec{X}_k, \quad \vec{X}_k := (x_{k-2}, x_{k-1}, x_k, x_{k+1})^{Tr}, \quad k \in \mathbb{Z}_+.$$

The matrix $\mathbf{M}_2$ can be divided in two terms (one is linearly dependent on λ , the other one is independent on λ): $\mathbf{M}_2(k, \lambda) = \lambda \mathbf{M}_2^{(1)} + \mathbf{M}_2^{(0)}$.

The leading coefficients of the expansion of the matrices $\mathbf{M}_2^{(1)}$ and $\mathbf{M}_2^{(0)}$ are

$$\mathbf{M}_2^{(1)} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & (\tilde{b} - 8k - 2a - 3) & 0 \\ 0 & 0 & 4(2-a)(\alpha - \beta) & (\tilde{b} - 16k - 6a - 15) \end{pmatrix} + O\left(\frac{1}{k}\right),$$

where $a := \alpha + \beta$, $\tilde{b} := -4k^2 - 4ak - 2a^2$, and

$$\mathbf{M}_2^{(0)} := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 - \frac{2}{k} & 0 & 2 + \frac{2}{k} & 0 \\ 0 & -1 - \frac{2}{k} & 0 & 2 + \frac{2}{k} \end{pmatrix} + O\left(\frac{1}{k^2}\right).$$

The relations (2.14) can be rewritten as a finite difference equation involving the vectors $\vec{X}_k$:

$$\frac{\vec{X}_{k+2} - \vec{X}_k}{2/n} = \frac{n}{2} [\mathbf{M}_2 - \mathbf{I}] \vec{X}_k, \quad k = 0, 2, \dots$$

In order to work with a better-structured matrix, we pass from $\vec{X}_k$ to $\vec{Y}_k$:

$$(2.15) \quad \vec{Y}_k := \mathbf{U}_k \vec{X}_k, \quad \mathbf{U}_k := \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -k & -k & k & k \\ -k & k & k & -k \end{pmatrix} \quad k > 1,$$

$$\mathbf{U}_0 := \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 \end{pmatrix}$$

satisfying

$$(2.16) \quad \vec{Y}_{k+2} = \widehat{\mathbf{M}}_2^{(\alpha, \beta)}(k, \lambda) \vec{Y}_k, \quad \widehat{\mathbf{M}}_2^{(\alpha, \beta)} := \mathbf{U}_{k+2} \mathbf{M}_2 \mathbf{U}_k^{-1}.$$

Then, we arrive at a finite-difference system, $k = 2, 4, \dots$,

$$(2.17) \quad \frac{\vec{Y}_{k+2} - \vec{Y}_k}{2/n} = \frac{n}{k} \mathbf{M}_3(k, \lambda) \vec{Y}_k, \quad \vec{Y}_2 = \vec{Y}_0 + 2\mathbf{M}_3(0, \lambda) \vec{Y}_0, \quad \vec{Y}_0 = (0, 0, C_1, C_2)^{Tr}$$

where $\mathbf{M}_3(0, \lambda) = (1/2)[U_2 \mathbf{M}_2(0, \lambda) U_0^{-1} - \mathbf{I}]$ and the matrix $\mathbf{M}_3(k, \lambda) = \frac{k}{2} [\mathbf{U}_{k+2} \mathbf{M}_2 \mathbf{U}_k^{-1} - \mathbf{I}] = \lambda \mathbf{M}_3^{(1)}(k) + \mathbf{M}_3^{(0)}(k)$ ($k > 0$) has expansions $\mathbf{M}_3^{(0)} :=$

$$\begin{pmatrix} 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 \\ 2\alpha(\alpha-2) - \frac{\alpha \diamond(\alpha, \beta)}{k} & \frac{2\beta(\beta-2)}{k} & 2 + \frac{\square(\alpha, \beta)}{2k} & \frac{\triangle(\alpha, \beta)}{2k} \\ \frac{2\alpha(\alpha-2)}{k} & 2\beta(\beta-2) - \frac{\beta \diamond(\beta, \alpha)}{k} & \frac{\triangle(\beta, \alpha)}{2k} & 2 + \frac{\square(\beta, \alpha)}{2k} \end{pmatrix} + O\left(\frac{1}{k^2}\right),$$

and

$$\mathbf{M}_3^{(1)} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2k^4 - (2a+10)k^3 & 2k^3 & -2k^3 & 2k^2 \\ 2k^3 & -2k^4 - (2a+10)k^3 & 2k^2 & -2k^3 \end{pmatrix} + O(k).$$

Here we denoted

$$\diamond(\alpha, \beta) = 4 - 6\alpha + 2\alpha^2 + 2\alpha\beta - \beta, \quad \square(\alpha, \beta) = 4\alpha^2 - 9\alpha + 2\alpha\beta - \beta + 4,$$

$$\triangle(\alpha, \beta) = \alpha^2 - \beta^2 - 2\alpha + 2\beta + 1, \quad \text{and} \quad a := \alpha + \beta.$$

2.3. General solution of the limiting system of ODEs. Now, we take a formal limit (under an appropriate scaling) of the Finite Difference (FD) problem (2.17) to arrive at a limiting system of ordinary differential equations (ODEs). Indeed, if we denote $\vec{y} = (y_1, y_2, y_3, y_4)^{Tr}$:

$$(2.18) \quad \vec{y}(t, l) := \lim_{\substack{n \rightarrow \infty \\ \frac{k}{n} \rightarrow t}} \vec{Y}_k(\lambda) \Big|_{\lambda=l/n^4},$$

(we shall investigate the existence of this limit later), then we arrive from (2.17) to the system of ODEs:

$$(2.19) \quad \frac{d}{dt} \vec{y}(t, l) = \frac{1}{t} \widetilde{\mathbf{M}}_3(t, l) \vec{y}(t, l), \quad \widetilde{\mathbf{M}}_3(t, l) = \lim_{\frac{k}{n} \rightarrow t} \mathbf{M}_3(k, \frac{l}{n^4}),$$

where

$$\widetilde{\mathbf{M}}_3(t, l) = \begin{pmatrix} 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 \\ -2lt^4 + 2\alpha(\alpha-2) & 0 & 2 & 0 \\ 0 & -2lt^4 + 2\beta(\beta-2) & 0 & 2 \end{pmatrix}.$$

Due to the special structure of the main term of asymptotics of the matrix M_3 in (2.17), this system is split in two second order independent scalar ODEs:

$$\begin{cases} t \frac{d}{dt} y_j(t) = \frac{z_j(t)}{2}, \\ t \frac{d}{dt} z_j(t) = 2[(b_j(b_j - 2) - lt^4)y_j(t) + z_j(t)], \end{cases} \quad j = 1, 2,$$

and

$$(2.20) \quad \frac{d^2}{dt^2} y_j(t, l) = \frac{1}{t} \frac{d}{dt} y_j(t, l) - \left(t^2 l - \frac{b_j(b_j - 2)}{t^2} \right) y_j, \quad j = 1, 2.$$

Here and in what follows, we use the notation:

$$(2.21) \quad b_j := \begin{cases} \alpha, & j = 1 \\ \beta, & j = 2 \end{cases}.$$

The ODE (2.20) is a modified Bessel equation (so-called Bowman form, see [7, Section 104]); its general solution is

$$(2.22) \quad y_j(t, l) = \tilde{C}_{1,j} t J_{\nu(b_j)} \left(\sqrt{l} \frac{t^2}{2} \right) + \tilde{C}_{2,j} t Y_{\nu(b_j)} \left(\sqrt{l} \frac{t^2}{2} \right), \quad j = 1, 2$$

(we use the notation $\nu(\cdot)$ defined in (1.2)). Thus, the general solution of (2.19) is

$$\vec{y}(t, l) = (y_1(t, l), y_2(t, l), 2ty_1'(t, l), 2ty_2'(t, l))^T.$$

2.4. Approximate general solution of the FD problem and further plan.

Thus, in the regime

$$(2.23) \quad \begin{cases} n \rightarrow \infty \\ \frac{k}{n} \rightarrow t \in K \Subset (0, 1] \end{cases},$$

the general solutions (2.22) could be a good approximation for general solutions $C_1 \vec{Y}_k^{(1)} + C_2 \vec{Y}_k^{(2)}$ of (2.15) - (2.14), for $k \in \mathbb{Z}$:

$$(2.24) \quad \vec{Y}_k^{(1)}(\lambda) \approx \begin{pmatrix} y_1\left(\frac{k}{n}, \lambda n^4\right) \\ 0 \\ \frac{2k}{n} y_1'\left(\frac{k}{n}, \lambda n^4\right) \\ 0 \end{pmatrix}, \quad \vec{Y}_k^{(2)}(\lambda) \approx \begin{pmatrix} 0 \\ y_2\left(\frac{k}{n}, \lambda n^4\right) \\ 0 \\ \frac{2k}{n} y_2'\left(\frac{k}{n}, \lambda n^4\right) \end{pmatrix},$$

here $(\cdot)'$ denotes the derivative with respect to the first variable.

Now, we recall (see Introduction) the further steps we need to proceed in order to obtain in the regime (2.23) asymptotics of the exact constant (2.9) and the vector $\vec{v}$ - (2.11) which defines the extremal polynomial Q'_n . To choose, from the general (approximate) solution (2.24) for $k \in \mathbb{Z}$, a solution which corresponds to the boundary conditions (BC)

$$(2.25) \quad v_{-2} = v_{-1} = 0 \quad \text{and} \quad v_n = v_{n+1} = 0,$$

we proceed as in [14]. First we find a set of two particular solutions of (2.14) $x_k(\lambda)$ for $\lambda = 0$ which correspond to the boundary conditions (2.25) at the left end:

$$(2.26) \quad x_{-2}^{(j)}, x_{-1}^{(j)} = 0, \quad j = 1, 2.$$

The second step is to choose constants C_1, C_2 for (2.24) such that asymptotics of

$y_j(t, l)$, $j = 1, 2$, when $t \rightarrow 0$, would match with asymptotics of $\vec{Y}_k^{(j)}(0)$ $j = 1, 2$ when $k \rightarrow \infty$, which correspond to the particular solutions $x_k^{(j)}(0)$. It defines the unknown constants for (2.24), (2.22). The last step is to obtain a linear combination of two approximate discrete solutions $\tilde{v}_k^{(j)}(\lambda)$, $j = 1, 2$ (we get them from (2.24)), satisfying the left end BC in (2.25), such that this combination satisfies the right end BC in (2.25). It is possible to do this if

$$(2.27) \quad \det \begin{vmatrix} \tilde{v}_n^{(1)}(\lambda) & \tilde{v}_{n+1}^{(1)}(\lambda) \\ \tilde{v}_n^{(2)}(\lambda) & \tilde{v}_{n+1}^{(2)}(\lambda) \end{vmatrix} = 0.$$

This equation (in λ) has the same meaning as (2.10) and its solutions with approximate $\tilde{v}_n^{(j)}(\lambda)$ gives an approximation of the eigenvalues $\lambda(\mathbf{A}, \mathbf{D})$.

2.5. Two particular solutions of the FD problem for $\lambda = 0$. Here we find solutions of the recurrence equation associated with (2.13) for $\lambda = 0$, satisfying the BC (2.25) at the left end

$$(2.28) \quad v_{-2} = v_{-1} = 0.$$

We are looking for the solutions of recurrences

$$(2.29) \quad [\mathbf{C}_1^{Tr} \mathbf{C}_2^{Tr} \mathbf{N}^{-1} \mathbf{D}^+ \mathbf{N}^{-1} \mathbf{C}_2 \mathbf{C}_1 \vec{v}]_k = 0, \quad k = 0, 1, \dots, n-3.$$

First we consider the equation

$$(2.30) \quad [\mathbf{C}_2 \vec{h}]_k = 0, \quad \text{with } (\mathbf{C}_2 \vec{h})_{k-1} = h_{k-1} + \frac{2k(k+\alpha+1)h_k}{(2k+\alpha+\beta+1)(2k+\alpha+\beta+2)}.$$

The structure of this equation is such that $h_{-1} = 0$ and h_{-2} can take any values, and we can put it to be zero. It is easy to check that the homogeneous equation (2.30) has a solution

$$(2.31) \quad h_{-2} = h_{-1} = 0, \quad h_k = \frac{(2k+\alpha+\beta+2)!}{(-2)^k k! (k+\alpha+1)!}, \quad k = 0, 1, \dots, n-1.$$

Next, we consider the equation:

$$(2.32) \quad [\mathbf{C}_1 \vec{h}]_k = 0, \quad \text{with } (\mathbf{C}_1 \vec{h})_{k-1} = h_{k-1} + \frac{-2k(k+\beta)}{(2k+\alpha+\beta)(2k+\alpha+\beta+1)} h_k.$$

Again we can check that this homogeneous equation has a solution

$$h_{-2} = h_{-1} = 0, \quad h_k = \frac{(2k+\alpha+\beta+1)!}{2^k k! (k+\beta)!}, \quad k = 0, 1, \dots, n-1,$$

which can be taken as the first particular solution of (2.29), (2.28):

$$(2.33) \quad v_k^{(1)}(0) = \frac{(2k+\alpha+\beta+1)!}{2^k k! (k+\beta)!}, \quad k = 0, 1, \dots, n-1.$$

We find the second particular solution of (2.29), (2.28) as a solution of the nonhomogeneous equation (2.32) with right hand side h_{k-1} from (2.31)

$$v_{k-1}^{(2)}(0) - \frac{2k(k+\beta)}{(2k+\alpha+\beta)(2k+\alpha+\beta+1)} v_k^{(2)}(0) = \frac{-2k(2k+\alpha+\beta)!}{(-2)^k k! (k+\alpha)!}.$$

It is easy to check that this nonhomogeneous equation has a solution

$$(2.34) \quad v_k^{(2)}(0) = \frac{(2k+\alpha+\beta+1)!}{(-2)^k k! (k+\alpha)!}, \quad k = 0, 1, \dots, n-1.$$

Note that for $v^{(1)}, v^{(2)}$ one has

$$\mathbf{C}_1^{Tr} \mathbf{C}_2^{Tr} \mathbf{N}^{-1} \mathbf{D}^+ \mathbf{N}^{-1} \mathbf{C}_2 \mathbf{C}_1 \vec{v} = \text{const}_1 \vec{e}_{n-1} + \text{const}_2 \vec{e}_{n-2}.$$

where $\vec{e}_{n-2}$ and $\vec{e}_{n-1}$ are the following vectors of $\mathbb{R}^n$: $\vec{e}_{n-2} = (0, \dots, 0, 1, 0)^{Tr}$, $\vec{e}_{n-1} = (0, \dots, 0, 0, 1)^{Tr}$.

2.6. Matching of the particular solutions of FD and ODEs problems. We obtain the particular solutions $\vec{X}_k^{(1)}(0), \vec{X}_k^{(2)}(0)$ of (2.14) from (2.33), (2.34), (2.12):

$$(2.35) \quad x_k^{(1)}(0) := \frac{(k+\alpha)!}{k!}, \quad x_k^{(2)}(0) := \frac{(k+\beta)!}{(-1)^k k!}.$$

Using (2.15) we have

$$(2.36) \quad \vec{Y}_k^{(j)}(0) = \begin{pmatrix} x_{k-2}^{(j)} + x_{k-1}^{(j)} \\ x_{k-2}^{(j)} - x_{k-1}^{(j)} \\ k \left(-(x_{k-2}^{(j)} + x_{k-1}^{(j)}) + (x_k^{(j)} + x_{k+1}^{(j)}) \right) \\ k \left(-(x_{k-2}^{(j)} - x_{k-1}^{(j)}) + (x_k^{(j)} - x_{k+1}^{(j)}) \right) \end{pmatrix}, \quad j = 1, 2.$$

Substituting the expansions of (2.35) in (2.36), we get for $\vec{Y}_k^{(1)}(0)$ and $\vec{Y}_k^{(2)}(0)$

$$k^\alpha \left[\begin{pmatrix} 2 + \frac{(\alpha-2)\alpha}{k} \\ -\frac{\alpha}{k} \\ 4\alpha + O\left(\frac{1}{k}\right) \\ O\left(\frac{1}{k}\right) \end{pmatrix} + O\left(\frac{1}{k^2}\right) \right], \quad k^\beta \left[\begin{pmatrix} -\frac{\beta}{k} \\ 2 + \frac{(\beta-2)\beta}{k} \\ O\left(\frac{1}{k}\right) \\ 4\beta + O\left(\frac{1}{k}\right) \end{pmatrix} + O\left(\frac{1}{k^2}\right) \right],$$

correspondingly. We conclude for $k \rightarrow \infty$, $j = 1, 2$ (recalling the notation (2.21)):

$$(2.37) \quad \vec{Y}_k^{(j)}(0) = k^{b_j} \left(\vec{C}_0^{(j)} + O\left(\frac{1}{k}\right) \right), \quad \vec{C}_0^{(1)} := \begin{pmatrix} 2 \\ 0 \\ 4\alpha \\ 0 \end{pmatrix}, \quad \vec{C}_0^{(2)} := \begin{pmatrix} 0 \\ 2 \\ 0 \\ 4\beta \end{pmatrix}.$$

Now, we state a “Matching Condition” for the choice of the particular solutions of the differential problem (2.19)-(2.22) when $t \rightarrow 0$:

$$(2.38) \quad \vec{y}^{(j)}(t, l) = t^{b_j} \left(\vec{C}_0^{(j)} + o(1) \right).$$

If this condition is satisfied, then, for $\lambda = l n^{-4}$, we expect that in the regime (2.23)

$$(2.39) \quad \vec{Y}_k^{(j)}(l n^{-4}) = n^{b_j} \vec{y}^{(j)}\left(\frac{k}{n}, l\right) + o(k^{b_j}).$$

This assertion will be proved later (see Theorem 3.1).

Using the well-known power series expansion of the Bessel functions for $x \rightarrow 0$ (see [7]),

$$J_\nu(x) = \hat{c}_\nu x^\nu (1 + \tilde{c}_\nu x^2 + O(x^4)), \quad \hat{c}_\nu := \frac{1}{2^\nu \nu!},$$

and matching the conditions (2.38), (2.37), we obtain expressions for the constants $\tilde{C}_{k,j}$ in the presentation (2.22) of the general solution of (2.19):

$$\tilde{C}_{2,j} = 0, \quad \tilde{C}_{1,j} = 2^{\nu_j+1} \left(\hat{c}_{\nu_j} l^{\nu_j/2} \right)^{-1}, \quad \nu_j = \nu(b_j), \quad j = 1, 2.$$

Thus, the particular solutions of (2.19), satisfying the condition (2.38), are

$$(2.40) \quad \vec{y}^{(1)}(t, l) = \begin{pmatrix} y_1(t, l) \\ 0 \\ 2t \frac{d}{dt} y_1(t, l) \\ 0 \end{pmatrix}, \quad \vec{y}^{(2)}(t, l) = \begin{pmatrix} 0 \\ y_2(t, l) \\ 0 \\ 2t \frac{d}{dt} y_2(t, l) \end{pmatrix},$$

where

$$(2.41) \quad y_j(t, l) = \frac{2^{b_j} \nu_j!}{l^{\nu_j/2}} t J_{\nu_j} \left(\frac{\sqrt{l} t^2}{2} \right), \quad j = 1, 2.$$

2.7. Matching of the right end BC for $\{v_k^{(j)}\}_{k=0}^{n-1}$. Now, substituting (2.40)-(2.41) into (2.39), (2.15), and (2.12), we arrive at the two particular approximate sequences $\{v_k^{(s)}(\lambda)\}_{k \in \mathbb{Z}_+}$, satisfying the left end BC in (2.25):

$$v_n^{(j)}(\lambda) \Big|_{\lambda = \frac{l}{n^4}} = (-1)^{(j-1)n} \frac{(2n + \alpha + \beta)! \nu_j! n^{b_j}}{2^{n-2\nu_j} (n + \alpha)! (n + \beta)! l^{\nu_j/2}} \left(J_{\nu_j} \left(\frac{\sqrt{l}}{2} \right) + o(1) \right).$$

Thus, the right end BC (see (2.27)) when $n \rightarrow \infty$ and $\lambda = \frac{l}{n^4}$, is equivalent to

$$C(l, n) \left(\det \begin{vmatrix} J_{\nu(\alpha)} \left(\frac{\sqrt{l}}{2} \right) & J_{\nu(\alpha)} \left(\frac{\sqrt{l}}{2} \right) \\ -J_{\nu(\beta)} \left(\frac{\sqrt{l}}{2} \right) & J_{\nu(\beta)} \left(\frac{\sqrt{l}}{2} \right) \end{vmatrix} + o(1) \right) = 0, \quad C(l, n) \neq 0, \quad l > 0.$$

From here we conclude that the roots of the equation

$$(2.42) \quad J_{\nu(\alpha)} \left(\frac{\sqrt{l}}{2} \right) J_{\nu(\beta)} \left(\frac{\sqrt{l}}{2} \right) + o(1) = 0$$

give approximate values of $\lambda = \frac{l}{n^4}$, for which the BC (2.25) are fulfilled. Indeed, if $\alpha < \beta$ and l_0 is the smallest zero of the Bessel function $J_{\nu(\alpha)} \left(\frac{\sqrt{l}}{2} \right)$, then the determinant in (2.27) changes the sign from $l_0 - \epsilon$ to $l_0 + \epsilon$. This gives a correct interval for the smallest λ for which (2.27) vanishes. In the case $\alpha = \beta$ the equation (2.42) takes the form $J_{\nu(\alpha)}^2 \left(\frac{\sqrt{l}}{2} \right) + o(1) = 0$. If l_0 is the smallest zero of the Bessel function $J_{\nu(\alpha)} \left(\frac{\sqrt{l}}{2} \right)$, then by Rouché's theorem the determinant (2.27) has two zeros in the complex plane in the neighborhood of l_0 . By symmetry of matrices of generalized eigenvalue problem (2.10) these zeros have to be real. This gives a correct interval for the smallest λ for which (2.27) vanishes in the case $\alpha = \beta$. Therefore the minimal root j_ν of the Bessel function J_ν gives the main term of asymptotics of the exact Markov-Bernstein constant

$$(2.43) \quad M_n = \frac{n^2}{2j_{\nu^*}} (1 + o(1)).$$

3. MATCHING AND CONVERGENCE OF FD AND DE PROBLEMS

3.1. Statements of the results. Our derivation of the asymptotics (2.43) contained one assumption which requires a special rigorous treatment. It is the convergence in the regime (2.23) of the discrete solution to the continuous solution, see (2.39). Here we state a theorem which establishes (2.39) under a restriction on (α, β) .

Theorem 3.1. Let $\{\vec{Y}_k^{(j)}(\lambda)\}_{k=0}^\infty$, $j = 1, 2$, be a set of the particular solutions of the FD problem, i.e. the recurrence (2.16) with the matrix $\widehat{\mathbf{M}}_2^{(\alpha, \beta)}(k, \lambda)$ such that $\{\vec{Y}_k^{(j)}(\lambda)\}_{k=0}^\infty$, $j = 1, 2$, are given by (2.36), (2.35), $\vec{Y}_0^{(j)}(\lambda) = \vec{Y}_0^{(j)}(0)$.

Let the parameters (α, β) in $\widehat{\mathbf{M}}_2^{(\alpha, \beta)}$ satisfy the condition:

$$(3.1) \quad |\alpha - \beta| < 4, \quad \alpha, \beta > -1.$$

Then, for $\lambda = \frac{l}{n^x}$, $\frac{k}{n} \rightarrow t$ and $n \rightarrow \infty$, uniformly for $l \in \tilde{K} \Subset \mathbb{C}$, $t \in K \Subset (0, 1]$,

$$(3.2) \quad \vec{Y}_k^{(j)}(\lambda) = n^{b_j} \left(\vec{y}^{(j)}(t, l) + o(t^{b_j}) \right), \quad b_j = \begin{cases} \alpha, & j = 1 \\ \beta, & j = 2 \end{cases},$$

holds true. Here $\vec{y}^{(j)}$, $j = 1, 2$, are the particular solutions (2.41), (2.40) of the DE problem (2.19) satisfying the matching condition (2.38).

Taking into account the procedure of derivation of (2.43) we obtain the validity of Theorem 1.1 as a corollary of Theorem 3.1.

3.2. Proof of Theorem 3.1. In our proof we use an approach proposed in [14]. The comparison of $\vec{Y}_k^{(j)}$ and $n^{b_j} \vec{y}^{(j)}$, in the regime (2.23), will be performed by means of the following relations. In what follows we denote by $|v|$ a norm of the vector v in the vector space $\mathbb{R}^n$ and by $\|S\|$ the associated matrix norm. For two recurrent sequences $\{\vec{v}_k\}$, $\{\vec{w}_k\}$, defined by

$$\vec{v}_{k+2} = S_k^{(1)} \vec{v}_k, \quad \vec{w}_{k+2} = S_k^{(2)} \vec{w}_k, \quad \vec{v}_k, \vec{w}_k \in \mathbb{R}^N, \quad k \in \mathbb{N} * 2,$$

we have for $k > k_0$

$$(3.3) \quad \left[|\vec{w}_k - \vec{v}_k| \leq \left[|\vec{w}_{k_0} - \vec{v}_{k_0}| + \sum_{m=k_0, 2}^{k-2} \left| (S_m^{(2)} - S_m^{(1)}) \vec{v}_m \right| E(k_0, m, S^{(2)}) \right] E^{-1}(k_0, k-2, S^{(2)}), \right.$$

where the summation is performed with Step 2, and

$$E(k_0, m, S) := \exp \left(\sum_{i=k_0, 2}^m (1 - \|S_i\|) \right).$$

The estimate (3.3) easily follows by induction from

$$|\vec{w}_{n+2} - \vec{v}_{n+2}| = \left| S_n^{(2)} \vec{w}_n - S_n^{(1)} \vec{v}_n \right| \leq \|S_n^{(2)}\| |\vec{w}_n - \vec{v}_n| + \left| (S_n^{(2)} - S_n^{(1)}) \vec{v}_n \right|$$

and from the inequality $xe^{1-x} \leq 1$ applied to $\|S_n^{(2)}\|$.

1) We are going to use the estimate (3.3) to compare the solution $\vec{Y}_k^{(j)}(\lambda)$ of the finite difference problem (2.17) with the solution $\vec{y}^{(j)}(t, l)$ (see (2.40)-(2.41)) of the differential problem (2.19). To do this (having in mind (3.3)) we define a difference operator which connects values of $\vec{y}^{(j)}(t)$ taken on the discrete grid $t := \frac{k}{n}$, $k \in \mathbb{N}_+$:

$$(3.4) \quad \vec{y}^{(j)} \left(\frac{k+2}{n}, l \right) = \widetilde{\mathbf{M}}_2(k, l, n) \vec{y}^{(j)} \left(\frac{k}{n}, l \right), \quad \widetilde{\mathbf{M}}_2 =: \left(\mathbf{I} + \frac{2}{k} \widetilde{\mathbf{M}}_3 \right).$$

We apply the following lemma from [14] (see Lemma 3.2).

Lemma 3.2. *Let $M(t)$ be a matrix-valued function solving the Cauchy problem with smooth matrix-valued coefficient F :*

$$\frac{d}{dt}M(t) = F(t)M(t), \quad M(t_1) = \mathbf{I}.$$

Then the following estimate holds for $t_2 > t_1$, $\int_{t_1}^{t_2} \|F(t)\| dt < 1$:

$$\|M(t_2) - \mathbf{I} - (t_2 - t_1)F(t_1)\| < 2(t_2 - t_1)^2 \max_{[t_1; t_2]} \|F'(t) + F^2(t)\|.$$

Applying this lemma to $M = \widetilde{\widetilde{\mathbf{M}}}_2$, $t_1 = \frac{k}{n}$, $t_2 = \frac{k+2}{n}$ we get an estimate which we shall use in our analysis later.

$$(3.5) \quad \left\| \widetilde{\widetilde{\mathbf{M}}}_3(k, l, n) - \widetilde{\mathbf{M}}_3\left(\frac{k}{n}, l\right) \right\| = O\left(\frac{1}{k}\right),$$

here $\widetilde{\widetilde{\mathbf{M}}}_3$ is defined in (2.19) and $k \rightarrow \infty$, $\frac{k}{n} \in K_1 \Subset \mathbb{R}_+$, $l \in K_2 \Subset \mathbb{C}$.

2) Now, we can rewrite (3.3) for our purpose

$$(3.6) \quad \begin{aligned} & \left| \vec{Y}_k^{(j)}(\lambda) - n^b \vec{y}^{(j)}\left(\frac{k}{n}, \lambda n^4\right) \right| \lesssim \left\{ \left| \vec{Y}_{k_0}^{(j)}(\lambda) - n^b \vec{y}^{(j)}\left(\frac{k_0}{n}, \lambda n^4\right) \right| \right. \\ & * \sum_{m=k_0, 2}^{k-2} \left| \left(\widehat{\mathbf{M}}_2^{(\alpha, \beta)}(m, \lambda) - \widetilde{\widetilde{\mathbf{M}}}_2(m, \lambda n^4) \right) n^b \vec{y}^{(j)}\left(\frac{k}{n}, \lambda n^4\right) \right| \\ & \left. * E\left(k_0, m, \widehat{\mathbf{M}}_2^{(\alpha, \beta)}\right) \right\} E^{-1}\left(k_0, k-2, \widehat{\mathbf{M}}_2^{(\alpha, \beta)}\right), \end{aligned}$$

where $\widehat{\mathbf{M}}_2, \widetilde{\widetilde{\mathbf{M}}}_2$ are defined in (2.16), (3.4), correspondingly.

3) To proceed with (3.6) we start with an estimation of the initial deviation. We put $k_0 := n^{1-\delta}$, where $\delta > 0$ will be fixed later. We shall compare the initial data in (3.6), estimating their deviation from the corresponding particular solution $Y_{k_0}^{(j)}(0)$, which we know explicitly (2.36), (2.35).

$$(3.7) \quad \begin{aligned} & \left| \vec{Y}_{k_0}^{(j)}(\lambda) - n^b \vec{y}^{(j)}\left(\frac{k_0}{n}, \lambda n^4\right) \right| < \\ & \left| \vec{Y}_{k_0}^{(j)}(\lambda) - \vec{Y}_{k_0}^{(j)}(0) \right| + \left| \vec{Y}_{k_0}^{(j)}(0) - n^b \vec{y}^{(j)}\left(\frac{k_0}{n}, \lambda n^4\right) \right|, \quad k_0 = n^{1-\delta}. \end{aligned}$$

To estimate the second term in the right hand side of (3.7) we use the matching condition (2.38)

$$(3.8) \quad \begin{aligned} & \left| \vec{Y}_{k_0}^{(j)}(0) - n^{b_j} \vec{y}^{(j)}\left(\frac{k_0}{n}, l\right) \right| \\ & = \left| \left(\vec{C}_0^{(j)} k_0^{b_j} + O\left(\frac{1}{k_0}\right) \right) - \left(\vec{C}_0^{(j)} k_0^{b_j} + O\left(\frac{k_0^4}{n^4}\right) \right) \right| \\ & = O\left(k_0^{b-1}\right) + O\left(k_0^b \left(\frac{k_0}{n}\right)^4\right). \end{aligned}$$

To estimate the first term in the right hand side of (3.7) we again use (3.3)

$$(3.9) \quad \left| \vec{Y}_{k_0}^{(j)}(\lambda) - \vec{Y}_{k_0}^{(j)}(0) \right| \leq \left\{ \left| \vec{Y}_2^{(j)}(\lambda) - \vec{Y}_2^{(j)}(0) \right| \right. \\ \left. * \sum_{m=2,2}^{k_0-2} \left| \left(\widehat{\mathbf{M}}_2^{(\alpha, \beta)}(m, \lambda) - \widehat{\mathbf{M}}_2^{(\alpha, \beta)}(m, 0) \right) \vec{Y}_m^{(j)}(0) \right| \right. \\ \left. * E \left(k_0, m, \widehat{\mathbf{M}}_2^{(\alpha, \beta)}(m, \lambda) \right) \right\} E^{-1} \left(k_0, m, \widehat{\mathbf{M}}_2^{(\alpha, \beta)}(m, \lambda) \right).$$

For further estimations we shall use $\| \cdot \|_\delta$ -norm introduced in [14]. This norm is related to a basis in which the operator $\mathbf{M}_3^{(0)}(\infty)$ defined in (2.16), (2.17) has a matrix (with eigenvalues on the diagonal) which is arbitrarily close to a diagonal matrix

$$\lim_{m \rightarrow \infty} \widehat{\mathbf{M}}_2^{(\alpha, \beta)}(m, 0) =: \widehat{\mathbf{M}}_2^{(\alpha, \beta)}(\infty, 0) = \left(\mathbf{I} + \frac{1}{k} \mathbf{M}_3^{(0)}(\infty) \right).$$

In such a basis the operator $\widehat{\mathbf{M}}_2^{(\alpha, \beta)}(\infty, 0)$ will have a norm close to $(1 + \frac{1}{k} B)$, where

$$(3.10) \quad B := \max\{\alpha, 2 - \alpha, \beta, 2 - \beta\}$$

is the maximal real part of eigenvalues of the matrix

$$\mathbf{M}_3^{(0)}(\infty) := \widetilde{\mathbf{M}}_3(0, 0) = \begin{pmatrix} 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 \\ 2\alpha(\alpha - 2) & 0 & 2 & 0 \\ 0 & 2\beta(\beta - 2) & 0 & 2 \end{pmatrix}.$$

That is, for each $\delta > 0$ there exist $k_\delta \in \mathbb{N}$ and a basis such that, with respect to the Euclidean norm $\| \cdot \|_\delta$ associated with this basis (see [14] for the details about the δ -norm), the corresponding operator norm $\| \cdot \|_\delta$ has the estimate

$$(3.11) \quad \forall k > k_\delta, \quad \left\| \widehat{\mathbf{M}}_2^{(\alpha, \beta)}(k, 0) \right\| = \left\| \mathbf{I} + \frac{1}{k} \mathbf{M}_3^{(0)}(k) \right\|_\delta < 1 + \frac{B + \delta}{k}.$$

Then, we estimate the terms from the right hand side of (3.9). We have (see (2.17))

$$\left| \left(\widehat{\mathbf{M}}_2^{(\alpha, \beta)}(m, \lambda) - \widehat{\mathbf{M}}_2^{(\alpha, \beta)}(m, 0) \right) \vec{Y}_m^{(j)}(0) \right|_\delta \\ = \frac{1}{m} \left| (\mathbf{M}_3(m, \lambda) - \mathbf{M}_3(m, 0)) \vec{Y}_m^{(j)}(0) \right|_\delta = \frac{1}{m} \left| \mathbf{M}_3^{(1)} \lambda \vec{Y}_m^{(j)}(0) \right| \leq O \left(|\lambda| \frac{m^4}{m} m^{b_j} \right).$$

Analogously

$$\| \mathbf{M}_3(m, \lambda) - \mathbf{M}_3(m, 0) \|_\delta = |\lambda| \| \mathbf{M}_3^{(1)} \| = O(\lambda m^4),$$

and

$$\left\| \mathbf{I} + \frac{1}{m} \mathbf{M}_3(m, \lambda) \right\|_\delta \leq 1 + \frac{B + \delta}{m} + \frac{1}{m} \| \mathbf{M}_3(m, \lambda) - \mathbf{M}_3(m, 0) \|_\delta \\ = 1 + \frac{B + \delta + O(|\lambda| m^4)}{m}.$$

Now, we proceed with (3.9). For $k = k_0 = n^{1-\delta}$ we have

$$\begin{aligned} \left| \vec{Y}_k^{(j)}(\lambda) - \vec{Y}_k^{(j)}(0) \right| &\leq O(1) \exp \left\{ \sum_{\tilde{k}=2,2}^{k-2} \frac{B+\delta+O(|l|n^{-4})\tilde{k}^4}{\tilde{k}} \right\} \\ &\quad * \sum_{\tilde{k}=2,2}^{k-2} \frac{O(|l|n^{-4})\tilde{k}^{4+b_j}}{\tilde{k}} \exp \left\{ - \sum_{\tilde{k}=2,2}^{\tilde{k}-2} \frac{B+\delta+O(|l|n^{-4})\tilde{k}^4}{\tilde{k}} \right\} \\ &= \frac{|l|}{n^4} O \left(k^{B+\delta} \exp \left[O \left(|l| \left(\frac{k}{n} \right)^4 \right) \right] \right) \sum_{\tilde{k}=2,2}^{k-2} \tilde{k}^{4+b_j-B-\delta-1} \exp \left[O \left(|l| \left(\frac{\tilde{k}}{n} \right)^4 \right) \right]. \end{aligned}$$

At this point we assume that

$$4 + b_j > B + \delta.$$

This assumption (see (3.10)) implies the restriction (3.1) in our theorems. Thus, we obtained

$$\left| \vec{Y}_{k_0}^{(j)}(\lambda) - \vec{Y}_{k_0}^{(j)}(0) \right| \leq O \left(\left(\frac{k_0}{n} \right)^4 k_0^{b_j} \right),$$

and finally, for the initial deviation (3.7) in (3.6) we get from here and (3.8)

$$(3.12) \quad \left| \vec{Y}_{k_0}^{(j)}(\lambda) - n^{b_j} \vec{y} \left(\frac{k_0}{n}, \lambda n^4 \right) \right| \leq O \left(k_0^{b_j-1} \right) + O \left(\left(\frac{k_0}{n} \right)^4 k_0^{b_j} \right).$$

4) Now, we come back to (3.6). Using (3.5) and the triangle inequality one has

$$\begin{aligned} k \left\| \widetilde{\mathbf{M}}_2(k, \lambda n^4) - \widehat{\mathbf{M}}_2^{(\alpha, \beta)}(k, \lambda) \right\|_\delta &= \left\| \widetilde{\mathbf{M}}_3(k, l, n) - \mathbf{M}_3 \left(k, \frac{l}{n^4} \right) \right\|_\delta \\ &\leq O \left(\frac{1}{k} \right) + \frac{l}{n^4} (k^4 + \text{const } k^3 + \dots) - \frac{l}{n^4} k^4 = O \left(\frac{1}{k} \right) + O \left(\frac{l}{n^4} k^3 \right) \\ &= o \left(\left(\frac{k}{n} \right)^4 k^{-\varepsilon_1} \right) + o(k^{-\varepsilon}), \quad \forall \varepsilon, \varepsilon_1 \in (0, 1). \end{aligned}$$

Substituting this estimate and (3.12), (3.11) in (3.6), we proceed:

$$\begin{aligned} \left| n^{b_j} \vec{y}^{(j)} \left(\frac{k}{n}, \lambda n^4 \right) - \vec{Y}_k^{(j)}(\lambda) \right| &\leq \left\{ O \left(k_0^{b_j-1} \right) + O \left(k_0^{b_j} \left(\frac{k_0}{n} \right)^4 \right) \right. \\ &+ \sum_{m=k_0,2}^{k-2} \frac{1}{m} \left[o(m^{-\varepsilon_1} \left(\frac{m}{n} \right)^4) + o(m^{-\varepsilon}) \right] n^{b_j} O \left(\frac{m}{n} \right)^{b_j} \exp \left(- \sum_{s=k_0,2}^m \frac{B+\delta+O(|l|n^{-4}s^4)}{s} \right) \Bigg\} \\ &\quad * \exp \left(\sum_{s=k_0,2}^m \frac{B+\delta+O(|l|n^{-4}s^4)}{s} \right). \end{aligned}$$

Assuming again from (3.1)

$$4 + b_j > B + \delta + \varepsilon_1 ,$$

we continue:

$$\begin{aligned} & \left| n^{b_j} \bar{y}^{(j)}\left(\frac{k}{n}, \lambda n^4\right) - \bar{Y}_k^{(j)}(\lambda) \right| \leq \left\{ o\left(k_0^{b_j - \tilde{\varepsilon}}\right) + o\left(k_0^{b_j} \left(\frac{k_0}{n}\right)^{B + \delta - b_j}\right) \right. \\ & + \sum_{m=k_0, 2}^k \frac{1}{m} \left[o(m^{b_j - \varepsilon_1} \left(\frac{m}{n}\right)^4) + o(m^{b_j - \varepsilon}) \right] * \left(\frac{k_0}{m}\right)^{B + \delta} \exp\left(O\left[\left(\frac{m}{n}\right)^4 - \left(\frac{k_0}{n}\right)^4\right]\right) \Big\} \\ & * \left(\frac{k}{k_0}\right)^{B + \delta} \exp\left(O\left[\left(\frac{k}{n}\right)^4 - \left(\frac{k_0}{n}\right)^4\right]\right) \leq \left[o\left(k_0^{b_j - \tilde{\varepsilon}}\right) \left(\frac{k}{k_0}\right)^{B + \delta} + o(n^{b_j}) \right. \\ & + \sum_{m=k_0, 2}^{k-2} \frac{1}{m} \left[o\left(m^{b_j - \varepsilon_1} \left(\frac{m}{n}\right)^4\right) + o(m^{b_j - \varepsilon}) \right] \left(\frac{k}{m}\right)^{B + \delta} \exp(O(1)) \Big] \exp(O(1)) \\ & \leq o(n^b) + o\left(k_0^{b_j - \tilde{\varepsilon}} \left(\frac{k}{k_0}\right)^{B + \delta}\right) + o(k^{b_j - \varepsilon_1}) + o\left(k_0^{b_j - \tilde{\varepsilon}} \left(\frac{k}{k_0}\right)^{B + \delta}\right) . \end{aligned}$$

Taking into account that

$$k_0 = n^{1 - \delta}, \quad \frac{k}{n} = t \in K \Subset \mathbb{R}_+,$$

we see that the right hand side of this estimation is equal to

$$o\left(n^{\max(b_j, b_j - \tilde{\varepsilon} + (B - b_j + \tilde{\varepsilon} + \delta)\delta, b_j - \varepsilon_1, b_j - \varepsilon + (B - b_j + \varepsilon + \delta)\delta)}\right) .$$

Since $\tilde{\varepsilon} < \varepsilon$, we choose δ :

$$\delta(B - b + \tilde{\varepsilon} + \delta) < \tilde{\varepsilon} ,$$

which yields

$$\left| n^{b_j} \bar{y}^{(j)}\left(\frac{k}{n}, l\right) - \bar{Y}_k^{(j)}\left(\frac{l}{n^4}\right) \right| = o(n^{b_j}) .$$

Theorem is proved.

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