FUSION RULES OF VIRASORO VERTEX OPERATOR ALGEBRAS

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(Communicated by Kailash C. Misra)

ABSTRACT. In this paper we prove the fusion rules of Virasoro vertex operator algebras $L(c_{1,q}, 0)$, for $q \geq 1$. Roughly speaking, we consider $L(c_{1,q}, 0)$ as the limit of $L(c_{n,nq-1}, 0)$, for $n \rightarrow \infty$, and the fusion rules of $L(c_{1,q}, 0)$ follow as the limits of the fusion rules of $L(c_{n,nq-1}, 0)$.

1. INTRODUCTION

In classical representation theories (of compact groups or semi-simple Lie algebras), we can define the tensor product for two modules. The famous Clebsch-Gordan problem is: given irreducible representations U and V , describe the decomposition, with multiplicities, of the representation $U \otimes V$. For the representation theory of vertex operator algebras, the difference is that we cannot define the tensor product of two modules over a vertex operator algebra A . Nevertheless, we still have the analogue of the Clebsch-Gordan problem via the notion of intertwining operator. In particular, for three irreducible modules U , V and W over a vertex operator algebra A , we can define the fusion rule $\mathcal{N}_{U,V}^W$, the analogue of the Clebsch-Gordan coefficient. For the representation theory of vertex operator algebras, it is also very important to determine the fusion rules $\mathcal{N}_{U,V}^W$.

The Virasoro vertex operator algebras constitute one of the most important classes of vertex operator algebras. In [15] it was proved that the Virasoro vertex operator algebras $L(c_{p,q}, 0)$ are rational, where $(p, q) = 1$, $p, q > 1$ and $c_{p,q} = 13 - 6(\frac{p}{q} + \frac{q}{p})$. Furthermore, the fusion rules of $L(c_{p,q}, 0)$ were proved in [15] using the Frenkel-Zhu formula (cf. [7]). In the case of $L(c_{1,q}, 0)$, we cannot prove the fusion rules as in [15], for Frenkel-Zhu's formula cannot be applied to $L(c_{1,q}, 0)$ which is non-rational. The fusion rules of $L(c_{1,1}, 0)$ were first proved in [10] and further extended in [3].

In this paper we prove the fusion rules of $L(c_{1,q}, 0)$, for $q \geq 1$. Our method is totally different from those of [3, 10]. Roughly speaking, we consider $L(c_{1,q}, 0)$ as the limit of $L(c_{n,nq-1}, 0)$, for $n \rightarrow \infty$, and the fusion rules of $L(c_{1,q}, 0)$ follow as the limits of the fusion rules of $L(c_{n,nq-1}, 0)$. Formally, the main result about fusion rules of $L(c_{1,q}, 0)$ can be written as

$$L(c_{1,q}, h_{i_1, s_1}) \otimes L(c_{1,q}, h_{i_2, s_2}) = \bigoplus_{i \in A_{i_1, i_2} \quad s \in A_{s_1, s_2}} L(c_{1,q}, h_{i, s}),$$

Received by the editors February 24, 2014 and, in revised form, May 2, 2014 and May 17, 2014.
 2010 *Mathematics Subject Classification.* Primary 17B69.

Key words and phrases. Fusion rules, Virasoro vertex operator algebras, intertwining operator.
 This work was supported by the National Natural Science Foundation for young (no.11401098).

where $A_{m,n} = \{m+n-1, m+n-3, \dots, |m-n|+1\}$ for $m, n > 0$. Many special cases of this result have already been applied in several papers (cf. [1, 2, 13]).

This paper is structured as follows: In Section 2 we give some preliminaries about the representation theory of Virasoro vertex operator algebras. In Section 3, using the easy part of Frenkel-Zhu's formula, we get an upper bound for the fusion rules of $L(c_{1,q}, 0)$. In Section 4, we establish the fusion rules of $L(c_{1,q}, 0)$ by the limit method. In Section 5, we further extend the fusion rules of $L(c_{1,q}, 0)$ to include some other cases. Throughout this paper, we assume that the reader is familiar with the axiom theory of vertex operator algebras and modules. For more information, see [6, 11].

2. VERTEX OPERATOR ALGEBRAS AND MODULES ASSOCIATED TO VIRASORO ALGEBRA

In this section, we give a short review of vertex operator algebras and modules associated to Virasoro algebra; details can be found in [4, 5, 8, 9, 11]. First, recall that Virasoro algebra is the Lie algebra Vir with basis

$$\{L_n \mid n \in \mathbb{Z}\} \cup \{C\},$$

satisfying

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3-m}{12}\delta_{m+n,0}C$$

and

$$[Vir, C] = \{0\}.$$

Define the following subalgebras of Vir :

$$Vir^{\pm} = \bigoplus_{\pm n > 0} \mathbb{C}L_n; \quad Vir^0 = L_0 \oplus C;$$

$$Vir^{\geq 0} = Vir^+ \oplus Vir^0; \quad Vir^{\geq -1} = Vir^{\geq 0} \oplus \mathbb{C}L_{-1}.$$

Let c and h be two complex numbers and let $\mathbb{C}v_{c,h}$ be the one dimensional $Vir^{\geq 0}$ -module with C and L_0 acting respectively as the scalars c and h , and with Vir^+ acting trivially. Set

$$M(c, h) = U(Vir) \otimes_{U(Vir^{\geq 0})} \mathbb{C}v_{c,h}$$

and call it the Verma module with central charge c and highest weight h . For any $\mathbb{I} = (1^{r_1} 2^{r_2} \dots n^{r_n}) \in \mathcal{P}_n$, set

$$e_{\mathbb{I}} = L_{-n}^{r_n} \dots L_{-2}^{r_2} L_{-1}^{r_1} \in U(Vir^-)_{-n}.$$

Then, $\{e_{\mathbb{I}} \cdot v_{c,h} \mid \mathbb{I} \in \mathcal{P}_n\}$ forms a basis of the weight subspace $M(c, h)_{h+n}$. Let $M'(c, h)$ be the largest proper submodule of $M(c, h)$. Then

$$L(c, h) = M(c, h)/M'(c, h)$$

is an irreducible Vir -module.

We recall the following proposition.

Proposition 2.1 (cf. [14]). *Set $n = \alpha\beta$, $c = c(t) = 13-6t-6t^{-1}$ and $h = h_{\alpha,\beta}(t) = \frac{1}{4}(\alpha^2-1)t - \frac{1}{2}(\alpha\beta-1) + \frac{1}{4}(\beta^2-1)t^{-1}$ for $\alpha, \beta \in \mathbb{Z}_{>0}$. Then there exists*

$$S_n = \sum_{\mathbb{I} \in \mathcal{P}_n} f_{\mathbb{I}}(c, h) e_{\mathbb{I}} \in U(Vir^-)_{-n}$$

such that $S_n v_{c,h} \in M(c, h)_{h+n}$ is a singular vector, where $f_{\mathbb{I}}(x, y) \in \mathbb{C}[x, y]$, and $f_{\mathbb{I}_0}(x, y) = 1$ for $\mathbb{I}_0 = (1^n)$.

Let p, q, r and s be positive integers, satisfying $(p, q) = 1$, $r < p$ and $s < q$. Let

$$c = c_{p,q} = 13 - 6\left(\frac{q}{p} + \frac{p}{q}\right)$$

and

$$h = h_{p,q;r,s} = \frac{(sp - rq)^2 - (p - q)^2}{4pq}.$$

Using Proposition 2.1, it can be shown that $M(c, h)$ has two singular vectors $u_{p,q}^{r,s}$ and $v_{p,q}^{r,s}$, of weights $h + rs$ and $h + (p - r)(q - s)$ respectively. Moreover, the maximal submodule of $M(c, h)$ is generated by $u_{p,q}^{r,s}$ and $v_{p,q}^{r,s}$.

Similarly, when $p = 1$ and

$$h = h_{i,s} = \frac{(iq - s)^2 - (q - 1)^2}{4q}$$

for some $i > 0$, $0 < s \leq q$, $M(c_{1,q}, h)$ has a singular vector of weight $h + is$ which generates the maximal proper submodule of $M(c_{1,q}, h)$. Moreover, $M(c_{1,q}, h) = L(c_{1,q}, h)$ when $h \neq h_{i,s} = \frac{(iq - s)^2 - (q - 1)^2}{4q}$ for any $i > 0$, $0 < s \leq q$.

Now consider $\mathbb{C}$ as a $Vir^{\geq -1}$ -module with C acting as the scalar c , and with $Vir^+ \oplus L_0 \oplus L_{-1}$ acting trivially. Set

$$V_c = U(Vir) \otimes_{U(Vir^{\geq -1})} \mathbb{C}.$$

It is well known that V_c has a canonical structure of vertex operator algebra of central charge c and with $\omega = L(-2)\mathbf{1}$ as conformal vector. In this way, $M(c, h)$ and $L(c, h)$ are modules for the vertex operator algebra V_c . Furthermore, $L(c, 0)$, as a quotient of V_c , is a simple vertex operator algebra.

Note that $L(c, 0) = V_c$ when $c \neq c_{p,q}$, where $p, q > 1$ and $(p, q) = 1$. If $c = c_{p,q}$ for some p, q as above, then $L(c, 0) \neq V_c$, and $L(c, h)$ is an $L(c, 0)$ -module if and only if $h = h_{p,q;r,s} = \frac{(sp - rq)^2 - (p - q)^2}{4pq}$ for some positive integer r, s satisfying $r < p$ and $s < q$ (cf. [15]). By the representation theory of Virasoro vertex operator algebras (cf. [11, 15]), we have

Theorem 2.2. *$L(c_{1,q}, h)$ is an irreducible module for the vertex operator algebra $L(c_{1,q}, 0)$, and $L(c_{1,q}, h)$ for $h \in \mathbb{C}$ exhaust all the irreducible $L(c_{1,q}, 0)$ -modules. For the vertex operator algebra $L(c_{p,q}, 0)$ ($p, q > 1$ and $(p, q) = 1$), $L(c_{p,q}, h)$ for $h = h_{p,q;r,s} = \frac{(sp - rq)^2 - (p - q)^2}{4pq}$ $0 < r < p$ and $0 < s < q$ exhaust all the irreducible $L(c_{p,q}, 0)$ -modules.*

Now we introduce the notions of intertwining operator and fusion rule for a triple of modules of vertex operator algebra (cf. [6]).

Definition 2.3. Let W_1, W_2 and W_3 be three modules over a vertex operator algebra V . A linear map $W_1 \otimes W_2 \rightarrow W_3\{x\}$, or equivalently,

$$\begin{aligned} W_1 &\rightarrow Hom(W_2, W_3)\{x\} \\ w &\mapsto \mathcal{Y}(w, x) = \sum_{n \in \mathbb{Q}} w_n x^{-n-1}, \end{aligned}$$

where $w_n \in Hom(W_2, W_3)$, is called an intertwining operator of type $\left(\begin{smallmatrix} W_3 \\ W_1 \ W_2 \end{smallmatrix}\right)$ if it satisfies:

- (The truncation property) For any $w_1 \in W_1$, $w_2 \in W_2$, $w_1 \cdot_n w_2 = 0$ for n sufficiently large;

2 (The L_{-1} -derivative formula) For any $w \in W_1$,

$$\mathcal{Y}(L_{-1}w, x) = \frac{d}{dx}\mathcal{Y}(w, x);$$

3 (The Jacobi identity) For any $v \in V$ and $w_1 \in W_1$,

$$\begin{aligned} & x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right)Y(v, x_1)\mathcal{Y}(w_1, x_2) \\ & - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right)\mathcal{Y}(w_1, x_2)Y(v, x_1) \\ & = x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right)\mathcal{Y}(Y(v, x_0)w_1, x_2). \end{aligned}$$

Set $I\left(\begin{smallmatrix} W_3 \\ W_1 \ W_2 \end{smallmatrix}\right)$ to be the vector space of all intertwining operators of type $\left(\begin{smallmatrix} W_3 \\ W_1 \ W_2 \end{smallmatrix}\right)$. Its dimension $\mathcal{N}_{W_1, W_2}^{W_3}$ is called the fusion rule of type $\left(\begin{smallmatrix} W_3 \\ W_1 \ W_2 \end{smallmatrix}\right)$.

The main result of this paper is the following:

Theorem 2.4. *Let $i_n > 0$, $0 < s_n \leq q$ ($n = 1, 2, 3$), and $A_{m,n} = \{m + n - 1, m + n - 3, \dots, |m - n| + 1\}$. Then*

$$\mathcal{N}_{L(c_{1,q}, h_{i_1, s_1}), L(c_{1,q}, h_{i_2, s_2})}^{L(c_{1,q}, h_{i_3, s_3})} \leq 1,$$

and equality holds if and only if $h_{i_3, s_3} = h_{i, s}$ for some $i \in A_{i_1, i_2}$ and $s \in A_{s_1, s_2}$.

3. FRENKEL-ZHU'S FORMULA

Recall that to a vertex operator algebra V , we can associate Zhu's algebra $A(V)$, and for each V -module M , the space of lowest weight vectors $M(0)$ has a natural structure of $A(V)$ -module. More generally, for each V -module M , define $O(M) \subset M$ to be the linear span of elements of type

$$Res_z Y(a, z) \frac{(1+z)^{\deg a}}{z^2} m$$

where $a \in V$ and $m \in M$, and let $A(M)$ be the quotient space $M/O(M)$. Then $A(M)$ has a natural structure of an $A(V)$ -bimodule. We recall the following useful result (cf. [7]).

Proposition 3.1. *For each submodule M_1 of M , $A(M_1)$ is a submodule of the $A(V)$ -bimodule $A(M)$, and the quotient $A(M)/A(M_1)$ is isomorphic to the bimodule $A(M/M_1)$.*

For Virasoro vertex operator algebras and Verma modules, we have the following results (cf. [7, 8]).

Proposition 3.2. *Let $\mathcal{L}$ be the subalgebra of Vir^- spanned by*

$$L_{-n-2} + 2L_{-n-1} + L_{-n},$$

for $n \geq 1$. Then $O(V_c) = \mathcal{L}.V_c$ and $A(V_c) \cong H_0(\mathcal{L}, V_c)$. In the case of $M(c, h)$ (resp. the irreducible quotient $L(c, h)$), we also have

$$O(M(c, h)) = \mathcal{L}.M(c, h)$$

$$(resp. O(L(c, h)) = \mathcal{L}.L(c, h)),$$

and

$$A(M(c, h)) \cong H_0(\mathcal{L}, M(c, h))$$

$$(resp. A(L(c, h)) \cong H_0(\mathcal{L}, L(c, h))).$$

Proposition 3.3. *We have an isomorphism of associative algebra:*

$$A(V_c) \cong \mathbb{C}[x]; \quad [\omega]^n \mapsto x^n, \quad n \in \mathbb{Z}_{\geq 0}.$$

For Verma module $M(c, h)$, the $A(V_c)$ -bimodule $A(M(c, h))$ is isomorphic to $\mathbb{C}[x, y]$, where the highest weight vector $v_{c,h}$ represents $\mathbf{1} \in \mathbb{C}[x, y]$, and the left and the right actions of $A(V_c)$ are given by

$$x \cdot f(x, y) = xf(x, y),$$

$$f(x, y) \cdot x = yf(x, y),$$

for any $f(x, y) \in \mathbb{C}[x, y]$.

Proposition 3.4. *The left and right actions of $A(V_c)$ on $A(M(c, h))$ are given by*

$$[\omega][v] = [(L_{-2} + 2L_{-1} + L_0)v],$$

$$[v][\omega] = [(L_{-2} + L_{-1})v],$$

for any $v \in M(c, h)$, where $\omega = L_{-2}\mathbf{1}$.

From now on, $W_i = \bigoplus_{n \in \mathbb{N}} W_i(n)$ ($i = 1, 2, 3$) will always be irreducible V -modules, where $W_i(n)$ is the L_0 -eigenspace of W_i with eigenvalue $n + h_i$.

Proposition 3.5 (cf. [7]). *Let $\mathcal{Y}(\cdot, x)$ be an intertwining operator of type $\begin{pmatrix} W_3 \\ W_1 W_2 \end{pmatrix}$. Then $\mathcal{Y}(\cdot, x)$ has the following form:*

$$\mathcal{Y}(w, x) = \sum_{n \in \mathbb{Z}} w(n) x^{-n-1} x^{-h_1-h_2+h_3},$$

such that for any $w \in W_1(k)$,

$$w(n)W_2(m) \subset W_3(m+k-n-1).$$

We also need the symmetry property of fusion rules, i.e., $\mathcal{N}_{W_1, W_2}^{W_3} = \mathcal{N}_{W_2, W_1}^{W_3}$ (cf. [6]).

Let $\mathcal{Y}(\cdot, x)$ be an intertwining operator of type $\begin{pmatrix} W_3 \\ W_1 W_2 \end{pmatrix}$. By Proposition 3.5, we can define a linear map $o_{\mathcal{Y}}$ from $W_1 \otimes W_2(0)$ to $W_3(0)$ by sending $w_1 \otimes w_2$ ($w_1 \in W_1(n)$, $w_2 \in W_2(0)$) to $w_1(n-1)w_2$. It can be proved that $w_1(n-1)w_2 = 0$ for $w_1 \in O(W_1)$, and $o_{\mathcal{Y}}$ induces an $A(V)$ -homomorphism

$$\pi(\mathcal{Y}) : A(W_1) \otimes_{A(V)} W_2(0) \rightarrow W_3(0).$$

Thus we get a linear map:

$$\pi : I\left(\begin{smallmatrix} W_3 \\ W_1 W_2 \end{smallmatrix}\right) \rightarrow \text{Hom}_{A(V)}(A(W_1) \otimes_{A(V)} W_2(0), W_3(0)).$$

The Frenkel-Zhu formula (cf. [7]) states that π is an isomorphism if W_i ($i = 1, 2, 3$) are irreducible modules. It was pointed out in [12] that this formula only holds for rational vertex operator algebras, and for more general vertex operator algebras, we have the following proposition (cf. [12]).

Proposition 3.6. *If W_3 is irreducible, then*

$$\pi : I\left(\begin{smallmatrix} W_3 \\ W_1 W_2 \end{smallmatrix}\right) \rightarrow \text{Hom}_{A(V)}(A(W_1) \otimes_{A(V)} W_2(0), W_3(0))$$

is injective.

Now we follow the arguments in [8, §9.3]. First, we consider the three $L(c_{1,q}, 0)$ -modules $L(c_{1,q}, h_{i_n, s_n})$, where $i_n > 0$, $0 < s_n \leq q$ ($n = 1, 2, 3$). We want to compute the dimension of

$$\operatorname{Hom}_{A(L(c_{1,q}, 0))}(A(L(c_{1,q}, h_{i_1, s_1})) \otimes_{A(L(c_{1,q}, 0))} L(c_{1,q}, h_{i_2, s_2})(0), L(c_{1,q}, h_{i_3, s_3})(0)).$$

By Proposition 3.2, its dual space is isomorphic to the simultaneous eigenspace of the left and right actions of $[\omega]$ on $H_0(\mathcal{L}, L(c_{1,q}, h_{i_1, s_1}))^* = H^0(\mathcal{L}, L(c_{1,q}, h_{i_1, s_1}))^*$ with the eigenvalues $-h_{i_3, s_3}$ and $-h_{i_2, s_2}$ respectively; denote this eigenspace by $H^0(\mathcal{L}, L(c_{1,q}, h_{i_1, s_1}))^{*(-h_{i_3, s_3}, -h_{i_2, s_2})}$. Then the surjection

$$M(c_{1,q}, h_{i_1, s_1}) \twoheadrightarrow L(c_{1,q}, h_{i_1, s_1})$$

induces an injection

$$i : H^0(\mathcal{L}, L(c_{1,q}, h_{i_1, s_1}))^{*(-h_{i_3, s_3}, -h_{i_2, s_2})} \hookrightarrow H^0(\mathcal{L}, M(c_{1,q}, h_{i_1, s_1}))^{*(-h_{i_3, s_3}, -h_{i_2, s_2})}.$$

The arguments in [8, §9.3] show that $H^0(\mathcal{L}, M(c_{1,q}, h_{i_1, s_1}))^{*(-h_{i_3, s_3}, -h_{i_2, s_2})}$ is one dimensional, and i is an isomorphism if and only if

$$(3.1) \quad P_{i_1, s_1}(-h_{i_2, s_2}, -h_{i_3, s_3} + h_{i_1, s_1}, q) = 0,$$

where $P_{\alpha, \beta}(a, b; \xi) \in \mathbb{C}[a, b, \xi, \xi^{-1}]$ satisfies

$$P_{\alpha, \beta}(a, b; \xi)^2 = \prod_{k=0}^{\alpha-1} \prod_{l=0}^{\beta-1} Q_{k, l}^{\alpha, \beta}(a, b; \xi),$$

and

$$\begin{aligned} & Q_{k, l}^{\alpha, \beta}(a, b; \xi) \\ &= [(b-a) - (k\xi^{\frac{1}{2}} - l\xi^{-\frac{1}{2}})\{(\alpha-k)\xi^{\frac{1}{2}} - (\beta-l)\xi^{-\frac{1}{2}}\}] \\ &\quad \times [(b-a) - \{(k+1)\xi^{\frac{1}{2}} - (l+1)\xi^{-\frac{1}{2}}\}\{(\alpha-k-1)\xi^{\frac{1}{2}} - (\beta-l-1)\xi^{-\frac{1}{2}}\}] \\ &\quad + \{(\alpha-2k-1)\xi^{\frac{1}{2}} - (\beta-l-1)\xi^{-\frac{1}{2}}\}^2 a. \end{aligned}$$

Direct computation shows that equation (3.1) is equivalent to the equation

$$\prod_{k=0}^{i_1-1} \prod_{l=0}^{s_1-1} (h_{i_3, s_3} - h_{i_1+i_2-2k-1, s_1+s_2-2l-1}) = 0.$$

Now combining the symmetry property of fusion rules and Proposition 3.6 yields $\mathcal{N}_{L(c_{1,q}, h_{i_1, s_1}), L(c_{1,q}, h_{i_2, s_2})}^{L(c_{1,q}, h_{i_3, s_3})} \leq 1$, and $\mathcal{N}_{L(c_{1,q}, h_{i_1, s_1}), L(c_{1,q}, h_{i_2, s_2})}^{L(c_{1,q}, h_{i_3, s_3})} = 1$ only if the following two equations hold:

$$\begin{aligned} & \prod_{k=0}^{i_1-1} \prod_{l=0}^{s_1-1} (h_{i_3, s_3} - h_{i_1+i_2-2k-1, s_1+s_2-2l-1}) = 0, \\ & \prod_{k=0}^{i_2-1} \prod_{l=0}^{s_2-1} (h_{i_3, s_3} - h_{i_1+i_2-2k-1, s_1+s_2-2l-1}) = 0. \end{aligned}$$

For $m, n > 0$, set $A_{m, n} = \{m+n-1, m+n-3, \dots, |m-n|+1\}$. Direct computation shows that these two equations are equivalent to the existences of $i \in A_{i_1, i_2}$, $s \in A_{s_1, s_2}$ such that $h_{i_3, s_3} = h_{i, s}$.

To sum up, we have proved that $\mathcal{N}_{L(c_{1,q},h_{i_1,s_1}),L(c_{1,q},h_{i_2,s_2})}^{L(c_{1,q},h_{i_3,s_3})} \leq 1$ if there exists $i \in A_{i_1,i_2}$, $s \in A_{s_1,s_2}$ such that $h_{i_3,s_3} = h_{i,s}$, and $\mathcal{N}_{L(c_{1,q},h_{i_1,s_1}),L(c_{1,q},h_{i_2,s_2})}^{L(c_{1,q},h_{i_3,s_3})} = 0$ otherwise.

4. CONSTRUCTION OF INTERTWINING OPERATORS

We fix three $L(c_{1,q},0)$ -modules $L(c_{1,q},h_{i_n,s_n})$ ($n = 1, 2, 3$), where $i_n > 0$, $0 < s_n \leq q$, and assume that there exist $i \in A_{i_1,i_2}$, $s \in A_{s_1,s_2}$, satisfying $h_{i_3,s_3} = h_{i,s}$. The purpose of this section is to construct a non-zero intertwining operator of type $(\begin{smallmatrix} L(c_{1,q},h_{i_3,s_3}) \\ L(c_{1,q},h_{i_1,s_1}) \end{smallmatrix} L(c_{1,q},h_{i_2,s_2}))$. Set $c_k = c_{k,kq-1}$, and $h_n^k = \frac{(i_n(kq-1)-s_nk)^2-(kq-1-k)^2}{4k(kq-1)}$.

Lemma 4.1. $\lim_{k \rightarrow \infty} c_k = c_{1,q}$, and $\lim_{k \rightarrow \infty} h_n^k = h_{i_n,s_2}$, for $n = 1, 2, 3$.

By the fusion rules of $L(c_k,0)$ when k is large enough, there exists a non-trivial intertwining operator $\mathcal{Y}_k(\cdot, x)$ of type $(\begin{smallmatrix} L(c_k,h_3^k) \\ L(c_k,h_1^k) \end{smallmatrix} L(c_k,h_2^k))$. Our method is to get the desired intertwining operator from the limit of $\mathcal{Y}_k(\cdot, x)$ as k approaches infinity. Hence, from now on, we always assume that k is large enough when needed. We say that a sequence of monomials $\{a_k x^{n_k}\}$ converges to the limit ax^n

$$\lim_{k \rightarrow \infty} a_k x^{n_k} = ax^n$$

if $\{a_k\}$ and $\{n_k\}$ converge to the limits a and n respectively. The following proposition is crucial for our construction.

Proposition 4.2. As a left $A(L(c_k,0))$ -module, $A(L(c_k,h_n^k))$ is generated by

$$[v_{c_k,h_n^k}], [L_{-1}v_{c_k,h_n^k}], \dots, [L_{-1}^{i_n s_n - 1}v_{c_k,h_n^k}].$$

Proof. Combining Propositions 3.2, 3.3 and 3.4 implies

$$[L_{-n}v] = (-1)^n (ny - x + wt(v))[v]$$

in $A(M(c_k,h_n^k))$ for each homogenous $v \in L(c_k,h_n^k)$ (recall the identification $A(M(c_k,h_n^k)) = \mathbb{C}[x,y]$ in Proposition 3.3). By this formula and Proposition 2.1, we have

$$[L_{-1}^m v_{c_k,h_n^k}] = (x-y)^m + \text{lower powers of } x \text{ and } y$$

and

$$[S_{i_n s_n} v_{c_k,h_n^k}] = [v_{i_n,s_n}] = (x-y)^{i_n s_n} + \text{lower powers of } x \text{ and } y$$

in $A(M(c_k,h_n^k)) \cong \mathbb{C}[x,y]$. Now by Proposition 3.1, $[S_{i_n s_n} v_{c_k,h_n^k}]$ lies in the kernel of the surjective morphism $A(M(c_k,h_n^k)) \rightarrow A(L(c_k,h_n^k))$, hence $A(L(c_k,h_n^k))$ can be generated, as a left $A(L(c_k,0))$ -module by

$$[v_{c_k,h_n^k}], [L_{-1}v_{c_k,h_n^k}], \dots, [L_{-1}^{i_n s_n - 1}v_{c_k,h_n^k}].$$

□

Now we are well prepared for the construction. It suffices to construct a bilinear pair $[\cdot, \cdot]$ (taking values in $\mathbb{C}\{x\}$) between $(L(c_{1,q},h_{i_3,s_3}))^*$ and $L(c_{1,q},h_{i_1,s_1}) \otimes L(c_{1,q},h_{i_2,s_2})$ that satisfies the corresponding properties. The construction is divided into several steps. Throughout the construction, we adopt the following convention:

We always identify $M(c_{1,q},h_{i_n,s_n})$ ($n = 1, 2, 3$) with $M(c_k,h_n^k)$, for $k \geq 1$, by isomorphisms which send $v_{c_{1,q},h_{i_n,s_n}}$ to v_{c_k,h_n^k} and commute with the action of Vir^- .

For a Vir -module M of lowest weight h , we always use $M(n)$ to denote the subspace of vectors of weight $h + n$.

Step 1. Let $v' \in L(c_{1,q}, h_{i_3, s_3})^*(0) = M(c_{1,q}, h_{i_3, s_3})^*(0) = M(c_k, h_3^k)^*(0)$ be defined by $v'(v_{c_{1,q}, h_{i_3, s_3}}) = 1$. For each homogenous $v_1 \in M(c_{1,q}, h_{i_1, s_1}) = M(c_k, h_1^k)$ and $v_2 = v_{c_{1,q}, h_{i_2, s_2}} = v_{c_k, h_2^k}$, $[v', v_1 \otimes v_2]$ is defined as follows:

Set $a_1 = v_{c_{1,q}, h_{i_1, s_1}}, a_2 = L_{-1}v_{c_{1,q}, h_{i_1, s_1}}, \dots, a_{i_1 s_1} = L_{-1}^{i_1 s_1 - 1}v_{c_{1,q}, h_{i_1, s_1}}$ in $M(c_{1,q}, h_{i_1, s_1}) = M_{c_k, h_1^k}$. Then for each k , there is some i such that $\langle v', \mathcal{Y}_k(a_i, x)v_2 \rangle \neq 0$. Otherwise, by Propositions 3.6 and 4.2, $\mathcal{Y}_k$ will be zero. Since $\langle v', \mathcal{Y}_k(a_{i+1}, x)v_2 \rangle$ is the derivation of $\langle v', \mathcal{Y}_k(a_i, x)v_2 \rangle$, we have $\langle v', \mathcal{Y}_k(a_1, x)v_2 \rangle \neq 0$ for each k . By Proposition 3.5 and Lemma 4.1, when replacing each $\mathcal{Y}_k$ by a suitable non-zero multiple,

$$\langle v', \mathcal{Y}_1(a_1, x)v_2 \rangle, \langle v', \mathcal{Y}_2(a_1, x)v_2 \rangle, \dots, \langle v', \mathcal{Y}_k(a_1, x)v_2 \rangle, \dots$$

converges to a non-zero monomial. Now assume that the sequence

$$\langle v', \mathcal{Y}_1(v, x)v_2 \rangle, \langle v', \mathcal{Y}_2(v, x)v_2 \rangle, \dots, \langle v', \mathcal{Y}_k(v, x)v_2 \rangle, \dots$$

converges for a homogeneous $v \in M(c_{1,q}, h_{i_1, s_1})$. Then the same result holds for $L_0 v$ and $L_{-1} v$. As the left action of $A(V_c)$ on $A(M(c, h))$ is given by

$$[\omega][v] = [(L_{-2} + 2L_{-1} + L_0)v],$$

by the construction of the linear map π in Section 3, we see that

$$\langle v', \mathcal{Y}_1(L_{-2}v, x)v_2 \rangle, \langle v', \mathcal{Y}_2(L_{-2}v, x)v_2 \rangle, \dots, \langle v', \mathcal{Y}_k(L_{-2}v, x)v_2 \rangle, \dots$$

also converges. By induction and the equality $[L_{-n}, L_{-1}] = (1-n)L_{-n-1}$, the same is true for $L_{-n}v$ ($n > 2$). Since $M(c_{1,q}, h_{i_1, s_1})$ is generated by $a_1 = v_{c_{1,q}, h_{i_1, s_1}}$, we conclude by induction that for each homogeneous $v_1 \in M(c_{1,q}, h_{i_1, s_1})$, the sequence

$$\langle v', \mathcal{Y}_1(v_1, x)v_2 \rangle, \langle v', \mathcal{Y}_2(v_1, x)v_2 \rangle, \dots, \langle v', \mathcal{Y}_k(v_1, x)v_2 \rangle, \dots$$

converges. Let $[v', v_1 \otimes v_2]$ be the limit and Step 1 is complete.

Step 2. For any $v_1 \in M(c_{1,q}, h_{i_1, s_1})$ and $v_2 \in M(c_{1,q}, h_{i_2, s_2})$ we want to define $[v', v_1 \otimes v_2]$ as the limit of the sequence

$$(4.1) \quad \langle v', \mathcal{Y}_1(v_1, x)v_2 \rangle, \langle v', \mathcal{Y}_2(v_1, x)v_2 \rangle, \dots, \langle v', \mathcal{Y}_k(v_1, x)v_2 \rangle, \dots$$

Thus we need to show that the limit of the sequence (4.1) exists for each $v_1 \in M(c_{1,q}, h_{i_1, s_1})$ and $v_2 \in M(c_{1,q}, h_{i_2, s_2})$. Step 1 shows that when $v_2 = v_{c_{1,q}, h_{i_2, s_2}}$, the limit exists. Thus by induction, it suffices to prove that if the limit of the sequence (4.1) exists for any $v_1 \in M(c_{1,q}, h_{i_1, s_1})$ and a fixed $v_2 \in M(c_{1,q}, h_{i_2, s_2})$, then the same is true with v_2 replaced by $L_n v_2$ ($n \leq -1$). But this follows directly from the following identity:

$$\begin{aligned} \langle v', \mathcal{Y}_k(v_1, x)L_n v_2 \rangle &= \langle v', L_n \mathcal{Y}_k(v_1, x)v_2 \rangle \\ &= - \sum_{i=0}^{\infty} \binom{n+1}{i} x^{n+1-i} \langle v', \mathcal{Y}_k(L_{i-1}v_1, x)v_2 \rangle \\ &= - \sum_{i=0}^{\infty} \binom{n+1}{i} x^{n+1-i} \langle v', \mathcal{Y}_k(L_{i-1}v_1, x)v_2 \rangle. \end{aligned}$$

Hence we conclude that the limit of the sequence (4.1) exists for any homogeneous $v_2 \in M(c_{1,q}, h_{i_2, s_2})$. Set

$$[v', v_1 \otimes v_2] = \lim_{k \rightarrow \infty} \langle v', \mathcal{Y}_k(v_1, x)v_2 \rangle,$$

and Step 2 is complete.

Step 3. Now we want to define $[v'_3, v_1 \otimes v_2]$ for any homogeneous $v_1 \in M(c_{1,q}, h_{i_1, s_1})$, $v_2 \in M(c_{1,q}, h_{i_2, s_2})$ and $v'_3 \in L(c_{1,q}, h_{i_3, s_3})^* \subset M(c_{1,q}, h_{i_3, s_3})^*$. By Lemma 4.1, we have

Lemma 4.3. *Given a positive integer m and a homogeneous $v \in M(c_{1,q}, h_{i_n, s_n}) = M(c_k, h_n^k)$, $L_m v$ as a vector in $M(c_k, h_n^k)$, converges to $L_m v$ as a vector in $M(c_{1,q}, h_{i_n, s_n})$, when k approaches infinity.*

Lemma 4.4. *For any $\mathbb{I} \in \mathcal{P}_n$, $\langle e_{\mathbb{I}} v', \mathcal{Y}_k(v_1, x)v_2 \rangle$ converges to a finite limit as k approaches infinity, where $e_{\mathbb{I}} v' \in M(c_k, h_3^k)$.*

Proof. By induction on the length of $\mathbb{I}$, this lemma follows directly from Lemma 4.3 and the formula

$$\langle L_n w', w \rangle = \langle w', L_{-n} w \rangle$$

for any $w' \in L(c_k, h_3^k)^*$ and $w \in L(c_k, h_3^k)$. $\square$

Since $L(c_{1,q}, h_{i_3, s_3})$ is irreducible, $L(c_{1,q}, h_{i_3, s_3})^*$ is generated by $L(c_{1,q}, h_{i_3, s_3})^*(0) = \mathbb{C}v'$ as a module over Vir^- . Hence we can choose a subset $\{\mathbb{I}_1, \dots, \mathbb{I}_s\}$ of $\mathcal{P}_n$ such that $e_{\mathbb{I}_1} v', \dots, e_{\mathbb{I}_s} v'$ form a basis of $L(c_{1,q}, h_{i_3, s_3})^*(n)$. By our convention, both $L(c_{1,q}, h_{i_3, s_3})^*(n)$ and $L(c_k, h_3^k)^*(n)$ are subspaces of $M(c_{1,q}, h_{i_3, s_3})^*(n) = M(c_k, h_3^k)^*(n)$. Moreover, by Proposition 2.1 and Lemma 4.1,

$$\lim_{k \rightarrow \infty} L(c_k, h_3^k)^*(n) = L(c_{1,q}, h_{i_3, s_3})^*(n),$$

where dimensions of $L(c_k, h_3^k)^*(n)$ and $L(c_{1,q}, h_{i_3, s_3})^*(n)$ are equal for large enough k and the limit is taken with respect to the topology of complex Grassmanian. Hence it is easy to see that $e_{\mathbb{I}_1} v', \dots, e_{\mathbb{I}_s} v'$, as elements of $M(c_k, h_3^k)^*$, form a basis of $L(c_k, h_3^k)^*(n)$, and converge to $e_{\mathbb{I}_1} v', \dots, e_{\mathbb{I}_s} v'$ respectively in $M(c_{1,q}, h_{i_3, s_3})^*$ as k approaches infinity.

Now for any homogeneous $v'_3 \in L(c_{1,q}, h_{i_3, s_3})^*(n)$ we can choose a

$$v'_{3,k} \in L(c_k, h_3^k)^*(n)$$

for each k , such that the sequence $\{v'_{3,k}\}$ converges to v'_3 . If we write

$$v'_3 = a_1 e_{\mathbb{I}_1} v' + \dots + a_s e_{\mathbb{I}_s} v'$$

and

$$v'_{3,k} = a_{1,k} e_{\mathbb{I}_1} v' + \dots + a_{s,k} e_{\mathbb{I}_s} v',$$

then for each i , the sequence

$$\dots a_{i,k}, a_{i,k+1}, a_{i,k+2} \dots$$

converges to a_i . By Lemma 4.4 we can set

$$[v'_3, v_1 \otimes v_2] = \lim_{k \rightarrow \infty} \langle v'_{3,k}, \mathcal{Y}_k(v_1, x)v_2 \rangle.$$

One checks that it is independent of the choice of the sequence $\{v'_{3,k}\}$.

Step 4. Now we check that the pairing $[\cdot, \cdot]$ induces an intertwining operator of type $\left(\begin{smallmatrix} L(c_{1,q}, h_{i_3, s_3}) \\ M(c_{1,q}, h_{i_1, s_1}) \quad M(c_{1,q}, h_{i_2, s_2}) \end{smallmatrix} \right)$.

It suffices to verify the L_{-1} -derivative formula

$$\frac{d}{dx} [v'_3, v_1 \otimes v_2] = [v'_3, L_{-1} v_1 \otimes v_2]$$

and the Jacobi identity

$$\begin{aligned} x_0^{-1} \delta\left(\frac{x_1 - x}{x_0}\right) [Y(e^{x_1 L_1} (-x_1^{-2})^{L_0} v, x_1^{-1}) v'_3, v_1 \otimes v_2] \\ - x_0^{-1} \delta\left(\frac{x - x_1}{-x_0}\right) [v'_3, v_1 \otimes Y(v, x_1) v_2] \\ = x^{-1} \delta\left(\frac{x_1 - x_0}{x}\right) [v'_3, Y(v, x_1) v_1 \otimes v_2], \end{aligned}$$

where $v \in V_{c_{1,q}}$ (this form of Jacobi identity follows from the graded dual module structure of $L(c_{1,q}, h_{i_3, s_3})^*$ defined in §5.2 of [6]), and the truncation property follows as a consequence.

The L_{-1} -derivative formula follows directly from our definition of $[\cdot, \cdot]$ and the fact that the derivation $\frac{d}{dx}$ commutes with the limiting operation. In order to prove the Jacobi identity, we identify $V_{c_{1,q}}$ with V_{c_k} by linear isomorphism $V_{c_{1,q}} \rightarrow V_{c_k}$ which sends $\mathbf{1}$ to $\mathbf{1}$ and commutes with the action of Vir^- . Then by Lemma 4.1 the coefficients of $Y(v, x_0) v_1$ (resp. $Y(v, x_1) v_2$), as elements of $M(c_k, h_1^k)$ (resp. $M(c_k, h_2^k)$), converge to the corresponding coefficients of $Y(v, x_0) v_1$ (resp. $Y(v, x_1) v_2$), as elements of $M(c_{1,q}, h_{i_1, s_1})$ (resp. $M(c_{1,q}, h_{i_2, s_2})$). If we choose, as in Step 3, a $v'_{3,k} \in L(c_k, h_3^k)^*$ for each k , such that the sequence $\{v'_{3,k}\}$ converges to v'_3 , then the coefficients of $Y(e^{x_1 L_1} (-x_1^{-2})^{L_0} v, x_1^{-1}) v'_{3,k}$ converge to the corresponding coefficients of $Y(e^{x_1 L_1} (-x_1^{-2})^{L_0} v, x_1^{-1}) v'_3$. Now the Jacobi identity of $[\cdot, \cdot]$ follows from the Jacobi identities of $\mathcal{Y}_k$ and Step 4 is complete.

Step 5. Show that $[v'_3, v_1 \otimes v_2] = 0$ if v_1 lies in the maximal proper submodule $M'(c_{1,q}, h_{i_1, s_1})$ of $M(c_{1,q}, h_{i_1, s_1})$. Let $M'(c_k, h_1^k)$ be the maximal submodule of $M(c_k, h_1^k)$; then it is easy to see that $M'(c_k, h_1^k)(n)$ converges to $M'(c_{1,q}, h_{i_1, s_1})(n)$ for each n as k approaches infinity. Thus there exists a $v_{1,k} \in M'(c_k, h_1^k)$ for each k , such that the sequence $\{v_{1,k}\}$ converges to v_1 . Using the arguments of Step 3, we see that

$$[v'_3, v_1 \otimes v_2] = \lim_{k \rightarrow \infty} \langle v'_{3,k}, \mathcal{Y}_k(v_{1,k}, x) v_2 \rangle$$

where $\{v'_{3,k}\}$ is the sequence converging to v'_3 in Step 3. Since $\mathcal{Y}_k(\cdot, x)$ is an intertwining operator of type $\left(\begin{smallmatrix} L(c_k, h_3^k) \\ L(c_k, h_1^k) \quad L(c_k, h_2^k) \end{smallmatrix} \right)$, this forces $[v'_3, v_1 \otimes v_2] = 0$.

Step 6. Show that $[v'_3, v_1 \otimes v_2] = 0$ if v_2 lies in the maximal proper submodule $M'(c_{1,q}, h_{i_2, s_2})$ of $M(c_{1,q}, h_{i_2, s_2})$. It suffices to repeat the arguments of Step 5 and we omit the details.

From the above construction we see that the pairing $[\cdot, \cdot]$ induces a non-zero intertwining operator $\mathcal{Y}(\cdot, x)$ of type $\left(\begin{smallmatrix} L(c_{1,q}, h_{i_3, s_3}) \\ L(c_{1,q}, h_{i_1, s_1}) \quad L(c_{1,q}, h_{i_2, s_2}) \end{smallmatrix} \right)$ such that $[v'_3, v_1 \otimes v_2] = \langle v'_3, \mathcal{Y}(v_1, x) v_2 \rangle$. This finishes our construction. Hence the proof of Theorem 2.4 is complete.

Remark 4.5. The limit method is quite necessary, for we cannot construct these intertwining operators by lattice vertex operator algebras as in [3, 10]. We hope to formalize this method and find more applications in future work.

5. FURTHER EXTENSION

From Section 2, we see that the Verma module $M(c_{1,q}, h)$ is irreducible if and only if $h \neq h_{i,s} = \frac{(iq-s)^2 - (q-1)^2}{4q}$ for any $i > 0$, $0 < s \leq q$. In this section, we consider the fusion rules of the type

$$\left(\begin{matrix} M(c_{1,q}, h') \\ L(c_{1,q}, h_{i_1, s_1}) M(c_{1,q}, h) \end{matrix} \right) \quad \text{and} \quad \left(\begin{matrix} M(c_{1,q}, h) \\ L(c_{1,q}, h_{i_1, s_1}) L(c_{1,q}, h_{i_2, s_2}) \end{matrix} \right)$$

where $M(c_{1,q}, h)$ and $M(c_{1,q}, h')$ are irreducible Verma modules and i_1, s_1, i_2, s_2 are as before. By Theorem 2.11 in [12], Frenkel-Zhu's formula holds in the first case. Hence the argument of Section 3 directly implies

Theorem 5.1.

$$\mathcal{N}_{L(c_{1,q}, h_{i,s}), M(c_{1,q}, h)}^{M(c_{1,q}, h')} \leq 1$$

where $i > 0$, $0 < s \leq q$, and $M(c_{1,q}, h)$ and $M(c_{1,q}, h')$ are irreducible Verma modules. Set $h = \frac{s'^2 - (q-1)^2}{4q}$ for some complex number s' . Then the equality holds if and only if $h' = \frac{(jq-s'-t)^2 - (q-1)^2}{4q}$ for some $j \in \{-i+1, -i+3, \dots, i-1\}$ and $t \in \{-s+1, -s+3, \dots, s-1\}$.

Remark 5.2. As in [10], Theorem 5.1 can also be proved by considering $M(c_{1,q}, h)$ and $M(c_{1,q}, h')$ as modules over lattice vertex operator algebra.

Similarly, we have

Theorem 5.3.

$$\mathcal{N}_{L(c_{1,q}, h_{i_1, s_1}), L(c_{1,q}, h_{i_2, s_2})}^{M(c_{1,q}, h)} = \mathcal{N}_{L(c_{1,q}, h_{i_1, s_1}), M(c_{1,q}, h)}^{L(c_{1,q}, h_{i_2, s_2})} = 0$$

where $i_1, i_2 > 0$, $0 < s_1, s_2 \leq q$ and $M(c_{1,q}, h)$ is an irreducible Verma module.

6. ACKNOWLEDGMENTS

The author would like to express his deep gratitude to A. Milas for valuable comments and suggestions about this paper.

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