THE SINGULAR HOMOGENEOUS SOLUTIONS TO ONE PHASE FREE BOUNDARY PROBLEM

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ABSTRACT. We provide some new examples of singular homogeneous of degree one solutions to the well-known one phase free boundary problem. They are critical points of the functional $J(v, B) = \int_B |\nabla v|^2 + \chi_{\{v>0\}}$. We also discuss their stability using a criteria of Caffarelli, Jerison and Kenig.

1. INTRODUCTION

We consider the variational problem of the functional

$$J(v, D) = \int_D |\nabla v|^2 + \chi_{\{v>0\}}$$

in the convex set

$$K := \{v \in L^1_{loc}(D) | \nabla v \in L^2(D) \text{ and } v = u^0 \geq 0 \text{ on a essential part of } \partial D\}.$$

A non-negative critical point u satisfies the following equation in a weak sense:

$$\begin{cases} \Delta u = 0 & \text{in } \{u > 0\} \cap D, \\ \frac{\partial u}{\partial \nu} = -1 & \text{on } \partial\{u > 0\} \cap D. \end{cases}$$

In 1981, Alt and Caffarelli [1] proved that for a local minimum u the reduced free boundary $\partial_{red}\{u > 0\}$ is a regular surface, and in 2 dimensions the singular points cannot occur. They gave an example in 3 dimensions which has an isolated singular point on the free boundary. They showed that this example is a critical point but not a local minimum.

In 1999, Weiss [4] proved that the existence of singularities is equivalent to the existence of an absolute minimum u^* such that u^* is homogeneous of degree 1 with one and only one singularity at 0. This leads to the following partial regularity: there is a maximal dimensions $k^* \geq 3$ such that for $n < k^*$ the free boundary must be smooth, for $n = k^*$ there can be isolated singular points, and for $n > k^*$ the Hausdorff dimensions of singular set is less than $n - k^*$.

In 2004, Caffarelli, Jerison and Kenig [2] proved that in 3 dimensions all singular homogeneous of degree 1 critical points are unstable, and this implied $k^* \geq 4$. They also showed that the simplest axis-symmetric singular solutions (the higher dimensional generalization of the above mentioned Alt-Caffarelli example in 3 dimensions) are unstable in dimensions 3-6. Later, De Silva and Jerison [3] proved

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that this solution is an absolute minimum in dimension 7, and this implied $k^* \leq 7$. One guessed $k^* = 7$, but there has not been any further progress since then.

In this paper, we construct some new singular homogeneous of degree 1 critical points to the one phase free boundary problem, which include the known axis-symmetric solution as a special case. We have 8 new solutions in dimensions 4, 5 and 6, all of them are unstable. In 7 dimension, we have a new solution that satisfies the necessary condition for stability in [2] and is likely to be a global minimizer.

Definition 1.1. We say $u \in C(\mathbb{R}^n)$ is a homogeneous solution to the one phase free boundary problem if: $u \geq 0$; $u(rx) = ru(x)$; u is harmonic in $\Gamma := \{u > 0\}$; $\partial\Gamma \setminus \{0\}$ is smooth and $\frac{\partial u}{\partial \nu} = -1$ on $\partial\Gamma \setminus \{0\}$, where ν is the outer unit normal vector of Γ . We say u is singular if $u(x) \neq x_1^+$ (a module rotation).

Such a u is determined by its values on the unit sphere S^{n-1} as a result of homogeneity. On S^{n-1} , $\Delta u = u_{rr} + \frac{n-1}{r}u_r + \Delta_{S^{n-1}}u = \Delta_{S^{n-1}}u + (n-1)u$. So $u|_{S^{n-1}}$ (we still denote it by u) satisfies the following:

$$(1.1) \quad \begin{cases} \Delta_{S^{n-1}}u + (n-1)u = 0, & u > 0, & \text{in } \Omega := \Gamma \cap S^{n-1}, \\ u = 0, \quad \frac{\partial u}{\partial \nu} = -1, & & \text{on } \partial\Omega, \\ u \equiv 0, & & \text{in } \Omega^c. \end{cases}$$

Therefore, to construct a homogeneous solution to one phase free boundary problem, it is equivalent to find a solution to the overdetermined first eigenvalue problem (1.1) in S^{n-1} .

From now on, we consider u as a function in S^{n-1} . Denote $Ridge(u) := \{x \in \Omega : u(x) = \max_{y \in \Omega} u(y)\}$. We will construct some special solutions u whose values $u(x)$ only depend on $dist_{S^{n-1}}(x, Ridge(u))$, the intrinsic distance from x to the set $Ridge(u)$ in S^{n-1} . The ridges in our examples are

$$S^p(1) := \{x \in \mathbb{R}^n : x_1^2 + \cdots + x_{p+1}^2 = 1, x_{p+2} = \cdots = x_n = 0\}$$

for $p = 1, \dots, n-2$. Note that when $p = n-2$, u is the known axis-symmetric solution. If $Ridge(u) = \{(1, 0, \dots, 0)\}$, u is the regular half space solution. In the case that n is even, we also have a solution u that has the Clifford torus

$$S^{\frac{n-2}{2}}(\frac{1}{\sqrt{2}}) \times S^{\frac{n-2}{2}}(\frac{1}{\sqrt{2}}) := \{x \in \mathbb{R}^n : x_1^2 + \cdots + x_{\frac{n}{2}}^2 = x_{\frac{n}{2}+1}^2 + \cdots + x_n^2 = \frac{1}{2}\}$$

as its ridge.

2. CONSTRUCTION OF HOMOGENEOUS SOLUTIONS

We first prove a lemma.

Lemma 2.1. Assume that $h(t) \in C^2(0, T) \cap C^1[0, T)$ solves the problem

$$\begin{cases} (a(t)h'(t))' + b(t)h(t) = 0, & t \in (0, T), \\ h(0) = 1, h'(0) = 0, \end{cases}$$

where $a(t) > 0, b(t) > 0$ for $t \in (0, T)$. If $\limsup_{t \rightarrow 0} |a(t)| < +\infty$ and for some $N \geq 1$ $\lim_{t \rightarrow T} \frac{a(t)}{(T-t)^N} = k > 0$, then there exists $t_0 \in (0, T)$ such that $h(t_0) = 0$, $h'(t_0) < 0$ and $h(t) > 0$ for $t \in (0, t_0)$.

Proof. Suppose $h(t) > 0$ for $t \in [0, T)$. Integrating the equation from ϵ' to $T - \epsilon$ and letting ϵ' go to 0, we get

$$a(T - \epsilon)h'(T - \epsilon) = - \int_0^{T-\epsilon} b(t)h(t)dt \leq -c_1 < 0.$$

There exists $\epsilon_0 > 0$ such that

$$1.1k^N\epsilon \leq a(T - \epsilon) \leq 0.9k^N\epsilon \text{ for } \epsilon \in (0, \epsilon_0).$$

So

$$h'(T - \epsilon) \leq \frac{-c_1}{1.1k\epsilon^N} = \frac{-c_2}{\epsilon^N} \text{ for } \epsilon \in (0, \epsilon_0).$$

Integrating $h'(t)$ from $T - \epsilon_0$ to $T - \epsilon$, we get

$$\int_{T-\epsilon_0}^{T-\epsilon} h'(t)dt \leq \int_{T-\epsilon_0}^{T-\epsilon} \frac{-c_2}{(T-t)^N} dt = \frac{c_2}{1-N} (\epsilon^{1-N} - \epsilon_0^{1-N}) \text{ (or } c_2(\ln \epsilon - \ln \epsilon_0) \text{ if } N=1).$$

So

$$h(T - \epsilon) \leq h(T - \epsilon_0) + \frac{c_2}{1-N} (\epsilon^{1-N} - \epsilon_0^{1-N}) \text{ (or } c_2(\ln \epsilon - \ln \epsilon_0)) \rightarrow -\infty \text{ as } \epsilon \rightarrow 0.$$

This contradicts $h(t) > 0$ for $t \in [0, T)$. So there must exist $t_0 \in (0, T)$ such that $h(t_0) = 0$ and $h(t) > 0$ for $t \in (0, t_0)$. Integrating the equation from ϵ' to t_0 and letting ϵ' go to 0, we get

$$a(t_0)h'(t_0) = - \int_0^{t_0} b(t)h(t)dt := c_3 < 0.$$

So $h'(t_0) = \frac{-c_3}{a(t_0)} < 0$. □

Example 1 ($n=4$, $\text{Ridge}(u) = S^1$). In $S^3 \subset \mathbb{R}^4$, we use coordinates

$$\begin{cases} x_1 = \cos \theta \cos \varphi, & x_2 = \cos \theta \sin \varphi, \\ x_3 = \sin \theta \cos \psi, & x_4 = \sin \theta \sin \psi, \end{cases}$$

where $\theta \in [0, \frac{\pi}{2}]$ and $\varphi, \psi \in [0, 2\pi]$. Note that $\{\theta = 0\} = S^1(1)$. The Riemannian metric g of S^3 under this coordinates system reads

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos^2 \theta & 0 \\ 0 & 0 & \sin^2 \theta \end{pmatrix}.$$

Assume $u(\theta, \varphi, \psi) = \tau f(\theta)$ for some real number $\tau > 0$. Then

$$\Delta_{S^3} u = \frac{1}{\sqrt{\det g}} \partial_i (\sqrt{\det g} g^{ij} \partial_j u) = \frac{\tau}{\sin \theta \cos \theta} (\sin \theta \cos \theta f')'.$$

The equation in (1.1) becomes an ODE

$$(\sin \theta \cos \theta f')' + 3 \sin \theta \cos \theta f = 0.$$

Now we consider the initial value problem of the second order linear ODE,

$$(2.1) \quad \begin{cases} (\sin \theta \cos \theta f')' + 3 \sin \theta \cos \theta f = 0, & \theta \in (0, \frac{\pi}{2}), \\ f(0) = 1, f'(0) = 0. \end{cases}$$

From Lemma 2.1, we know that there exists $\theta_0 \in (0, \frac{\pi}{2})$ such that $f(\theta_0) = 0$, $f'(\theta_0) < 0$ and $f(\theta) > 0$ for $\theta \in (0, \theta_0)$.

Let $u = \frac{-1}{f'(\theta_0)} f(\theta)$ for $\theta \in [0, \theta_0)$ and $u \equiv 0$ for $\theta \in [\theta_0, \frac{\pi}{2}]$. We finished the construction of Example 1.

Example 2 ($n=4$, $\text{Ridge}(u) = S^1(\frac{1}{\sqrt{2}}) \times S^1(\frac{1}{\sqrt{2}})$). We use the same coordinates as in Example 1 and solve the ODE problem

$$\begin{cases} (\sin \theta \cos \theta f')' + 3 \sin \theta \cos \theta f = 0, & \theta \in (0, \frac{\pi}{2}), \\ f(\frac{\pi}{4}) = 1, f'(\frac{\pi}{4}) = 0. \end{cases}$$

It is easy to verify that $f(\theta) = f(\frac{\pi}{2} - \theta)$. From Lemma 2.1, we know there exists $\theta_0 \in (0, \frac{\pi}{4})$ such that $f(\theta_0) = 0$, $f'(\theta_0) > 0$ and $f(\theta) > 0$ for $\theta \in (\theta_0, \frac{\pi}{2} - \theta_0)$.

Let $u = \frac{1}{f'(\theta_0)} f(\theta)$ for $\theta \in (\theta_0, \frac{\pi}{2} - \theta_0)$ and $u \equiv 0$ elsewhere.

Example 3 ($n=5$, $\text{Ridge}(u) = S^1$). In $S^4 \subset \mathbb{R}^5$, we use coordinates

$$\begin{cases} x_1 = \cos \theta \cos \varphi, & x_2 = \cos \theta \sin \varphi, \\ x_3 = \sin \theta \sin \psi \cos \xi, & x_4 = \sin \theta \sin \psi \sin \xi, & x_5 = \sin \theta \cos \psi, \end{cases}$$

where $\theta \in [0, \frac{\pi}{2}]$, $\psi \in [0, \pi]$ and $\varphi, \xi \in [0, 2\pi]$. The Riemannian metric g_{ij} of S^{n-1} under this coordinates system $(\theta, \varphi, \psi, \xi)$ reads

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos^2 \theta & 0 & 0 \\ 0 & 0 & \sin^2 \theta & 0 \\ 0 & 0 & 0 & \sin^2 \theta \sin^2 \psi \end{pmatrix}.$$

Assume $u(\theta, \varphi, \psi) = \tau f(\theta)$. Then

$$\Delta_{S^3} u = \frac{\tau}{\sin^2 \theta \cos \theta} (\sin^2 \theta \cos \theta f')'.$$

We solve the ODE problem

$$\begin{cases} (\sin^2 \theta \cos \theta f')' + 4 \sin^2 \theta \cos \theta f = 0, & \theta \in (0, \frac{\pi}{2}), \\ f(0) = 1, f'(\frac{\pi}{2}) = 0. \end{cases}$$

From Lemma 2.1, there exists $\theta_0 \in (0, \frac{\pi}{2})$ such that $f(\theta_0) = 0$, $f'(\theta_0) < 0$ and $f(\theta) > 0$ for $\theta \in (0, \theta_0)$.

Let $u = \frac{-1}{f'(\theta_0)} f(\theta)$ for $\theta \in [0, \theta_0]$ and $u \equiv 0$ elsewhere.

Example 4 ($n=5$, $\text{Ridge}(u) = S^2$). We use the same coordinates as in Example 3 and solve the ODE problem

$$\begin{cases} (\sin^2 \theta \cos \theta f')' + 4 \sin^2 \theta \cos \theta f = 0, & \theta \in (0, \frac{\pi}{2}), \\ f(\frac{\pi}{2}) = 1, f'(\frac{\pi}{2}) = 0. \end{cases}$$

From Lemma 2.1, there exists $\theta_0 \in (0, \frac{\pi}{2})$ such that $f(\theta_0) = 0$, $f'(\theta_0) > 0$ and $f(\theta) > 0$ for $\theta \in (\theta_0, \frac{\pi}{2})$.

Let $u = \frac{1}{f'(\theta_0)} f(\theta)$ for $\theta \in (\theta_0, \frac{\pi}{2}]$ and $u \equiv 0$ elsewhere.

Example 5 ($n=6$, $\text{Ridge}(u) = S^1$). Choose coordinates such that $x_1^2 + x_2^2 = \cos^2 \theta$ and $x_3^2 + \dots + x_6^2 = \sin^2 \theta$ with $\theta \in [0, \frac{\pi}{2}]$. We solve the ODE problem

$$\begin{cases} (\sin^3 \theta \cos \theta f')' + 5 \sin^3 \theta \cos \theta f = 0, & \theta \in (0, \frac{\pi}{2}), \\ f(0) = 1, f'(\frac{\pi}{2}) = 0. \end{cases}$$

The rest is the same as Example 1.

Example 6 ($n=6$, $\text{Ridge}(u) = S^3$). With the same coordinates as Example 5, we solve

$$\begin{cases} (\sin^3 \theta \cos \theta f')' + 5 \sin^3 \theta \cos \theta f = 0, & \theta \in (0, \frac{\pi}{2}), \\ f(\frac{\pi}{2}) = 1, f'(\frac{\pi}{2}) = 0. \end{cases}$$

The rest is the same as Example 4.

Example 7 ($n=6$, $\text{Ridge}(u) = S^2$). Choose coordinates such that $x_1^2 + x_2^2 + x_3^2 = \cos^2 \theta$ and $x_4^2 + x_5^2 + x_6^2 = \sin^2 \theta$ with $\theta \in [0, \frac{\pi}{2}]$. We solve the ODE problem

$$\begin{cases} (\sin^2 \theta \cos^2 \theta f')' + 5 \sin^2 \theta \cos^2 \theta f = 0, & \theta \in (0, \frac{\pi}{2}), \\ f(0) = 1, f'(\frac{\pi}{2}) = 0. \end{cases}$$

The rest is the same as Example 1.

Example 8 ($n=6$, $\text{Ridge}(u) = S^2(\frac{1}{\sqrt{2}}) \times S^2(\frac{1}{\sqrt{2}})$). With the same coordinates as Example 5, we solve

$$\begin{cases} (\sin^2 \theta \cos^2 \theta f')' + 5 \sin^2 \theta \cos^2 \theta f = 0, & \theta \in (0, \frac{\pi}{2}), \\ f(\frac{\pi}{4}) = 1, f'(\frac{\pi}{4}) = 0. \end{cases}$$

The rest is the same as Example 2.

Example 9 ($n=7$, $\text{Ridge}(u) = S^1$). Solve the ODE

$$\begin{cases} (\sin^4 \theta \cos \theta f')' + 6 \sin^4 \theta \cos \theta f = 0, & \theta \in (0, \frac{\pi}{2}), \\ f(0) = 1, f'(\frac{\pi}{2}) = 0. \end{cases}$$

The rest is the same as Example 1.

General case I: $\text{Ridge}(u) = S^p$ 1.

We should solve ODE

$$\begin{cases} (\sin^{n-2-p} \theta \cos^p \theta f')' + (n-1) \sin^{n-2-p} \theta \cos^p \theta f = 0, & \theta \in (0, \frac{\pi}{2}), \\ f(0) = 1, f'(\frac{\pi}{2}) = 0. \end{cases}$$

General case II: $\text{Ridge}(u) = S^{\frac{n-2}{2}}(\frac{1}{\sqrt{2}}) \times S^{\frac{n-2}{2}}(\frac{1}{\sqrt{2}})$ 1.

We should solve ODE

$$\begin{cases} (\sin^{\frac{n-2}{2}} \theta \cos^{\frac{n-2}{2}} \theta f')' + (n-1) \sin^{\frac{n-2}{2}} \theta \cos^{\frac{n-2}{2}} \theta f = 0, & \theta \in (0, \frac{\pi}{2}), \\ f(\frac{\pi}{4}) = 1, f'(\frac{\pi}{4}) = 0. \end{cases}$$

In both cases, for all n and p Lemma 2.1 guarantees that we can get the expected homogeneous solutions u from the ODE's solutions f .

3. STABILITY

Definition 3.1. Let $u \in C(\mathbb{R}^n)$ be a homogeneous solution as in Definition 1.1. We say u is a global minimizer (or an absolute minimum) if for any ball $B \subset \mathbb{R}^n$ and any continuous function v defined on $\bar{B}$ such that $v = u$ on ∂B , $J(u, B) \leq J(v, B)$. We say u is stable if for all non-negative $\eta \in C_0^\infty(\mathbb{R}^n)$, $\frac{d^2}{dt^2}|_{t=0} J((u - t\eta)^+, B) \geq 0$, where B is any ball that contains the supporting set of η .

In [2], Caffarelli, Jerison and Kenig deduced a necessary condition for stability for the axis-symmetric solutions. However, this result also obviously fits our examples.

Theorem 3.2. Let u be a homogeneous solution as in Definition 1.1. Assume that $u|_{S^{n-1}}(x) = f(\theta)$, where $\theta = \text{dist}_{S^{n-1}}(x, \text{Ridge}(u|_{S^{n-1}}))$, $\partial\Omega = \{\theta = \theta_0\}$ has constant mean curvature H_1 , and $W(\theta) \in C^2(0, \theta_0) \cap C^1[0, \theta_0]$ is a function such that $G(x) = r^{-\frac{n-2}{2}} W(\theta)$ is harmonic in Γ . If u is stable, then

$$H_1 \leq |W'(\theta_0)/W(\theta_0)|.$$

Proof. See Lemma 1 and Section 3 in [2]. □

For our type I examples, $W(\theta)$ solves the ODE problem

$$\begin{cases} (\sin^{n-2-p} \theta \cos^p \theta W')' - (\frac{n-2}{2})^2 \sin^{n-2-p} \theta \cos^p \theta W = 0, & \theta \in (0, \frac{\pi}{2}), \\ W(0) = 1, W'(0) = 0, \end{cases}$$

while

$$H_1 = p \tan \theta_0 - (n - 2 - p) \cot \theta_0.$$

For the type II examples, $W(\theta)$ solves the ODE problem

$$\begin{cases} (\sin^{\frac{n-2}{2}} \theta \cos^{\frac{n-2}{2}} \theta W')' - (\frac{n-2}{2})^2 \sin^{\frac{n-2}{2}} \theta \cos^{\frac{n-2}{2}} \theta W = 0, & \theta \in (0, \frac{\pi}{2}), \\ W(\frac{\pi}{4}) = 1, W'(\frac{\pi}{4}) = 0, \end{cases}$$

and

$$H_1 = \frac{n-2}{2} \cot \theta_0 - \frac{n-2}{2} \tan \theta_0.$$

All the ODE problems for f and W can be solved by hypergeometric functions using Mathematica. For example, the solution for (2.1) is

$$f(\theta) = \text{LegendreP}[\frac{1}{2}, \cos 2\theta].^1$$

However, many others have much more complicated expressions. We do not present them in this paper. We computed the values of θ_0 , H_1 and $|W'(\theta_0)/W(\theta_0)|$ numerically using Mathematica:

Example 1: $\theta_0 = 1.14066$, $H_1 = 1.72088$, $|W'(\theta_0)/W(\theta_0)| = 0.86044$, $n = 4$.

Example 2: $\theta_0 = 0.11103$, $H_1 = 8.85835$, $|W'(\theta_0)/W(\theta_0)| = 1.69548$, $n = 4$.

Example 3: $\theta_0 = 1.21570$, $H_1 = 1.95514$, $|W'(\theta_0)/W(\theta_0)| = 1.42123$, $n = 5$.

Example 4: $\theta_0 = 0.61548$, $H_1 = 2.12132$, $|W'(\theta_0)/W(\theta_0)| = 1.53979$, $n = 5$.

Example 5: $\theta_0 = 1.26177$, $H_1 = 2.17450$, $|W'(\theta_0)/W(\theta_0)| = 1.99301$, $n = 6$.

Example 6: $\theta_0 = 0.73083$, $H_1 = 2.45029$, $|W'(\theta_0)/W(\theta_0)| = 2.23113$, $n = 6$.

Example 7: $\theta_0 = 1.04720$, $H_1 = 2.30940$, $|W'(\theta_0)/W(\theta_0)| = 2.10964$, $n = 6$.

Example 8: $\theta_0 = 0.26180$, $H_1 = 6.92820$, $|W'(\theta_0)/W(\theta_0)| = 3.46410$, $n = 6$.

Example 9: $\theta_0 = 1.29365$, $H_1 = 2.37747$, $|W'(\theta_0)/W(\theta_0)| = 2.56697$, $n = 7$.

So the first 8 examples are unstable. Example 9 satisfies the necessary condition for stability. We will check whether it is a global minimizer by the method of De Silva-Jerison [3] in another work. The first type II example that satisfies the condition $H_1 \leq |W'(\theta_0)/W(\theta_0)|$ appears at 14 dimensions, where $\theta_0 = 0.46189$, $H_1 = 9.06622$ and $|W'(\theta_0)/W(\theta_0)| = 9.78353$.

4. FINAL REMARK

The classical result of Serrin [5] confirms that if for some smooth bounded domain $\Omega \subset \mathbb{R}^n$, the overdetermined problem

$$\begin{cases} \Delta u + \lambda u = 0, & \text{in } \Omega, \\ u = 0, \quad \frac{\partial u}{\partial \nu} = -1, & \text{on } \partial\Omega \end{cases}$$

has a positive solution, then Ω must be a ball and u is radial symmetric. So for the similar problem in a sphere we conjecture that u must be ridge symmetric, which means $u(x)$ only depend on $\text{dist}(x, \text{Ridge}(u))$ (or maybe a signed distance function in the case that $\text{Ridge}(u)$ is a hypersurface), and all the level sets of u are constant mean curvature surfaces (or even better, for example, isoparametric surfaces).

¹ $\text{LegendreP}[\frac{1}{2}, x] = P_{\frac{1}{2}}^0(x)$ is the fractional order Legendre function; see http://en.wikipedia.org/wiki/Associated_Legendre_polynomial.

The proof in [2] of the fact that in 3 dimensions all singular homogeneous solutions are unstable depended heavily on the simple geometry of S^2 and cannot be extended to higher dimensions definitely. Our thought is that one has to investigate sufficiently the symmetry of homogeneous solutions before finding a way to prove the full regularity in dimensions 4-6.

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