ENDOFUNCTORS OF SINGULARITY CATEGORIES CHARACTERIZING GORENSTEIN RINGS

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ABSTRACT. In this paper, we prove that certain contravariant endofunctors of singularity categories characterize Gorenstein rings.

Let Λ be a noetherian ring. Denote by $D_{\text{sg}}(\Lambda)$ the *singularity category* of Λ , that is, the Verdier quotient of the bounded derived category $D^b(\Lambda)$ of finitely generated (right) Λ -modules by the full subcategory consisting of bounded complexes of finitely generated projective Λ -modules. We are interested in the following question.

Question 1. What contravariant endofunctor of $D_{\text{sg}}(\Lambda)$ characterizes the Iwanaga–Gorenstein property of Λ ?

In this paper we shall consider this question in the case where Λ is commutative and Cohen–Macaulay.

Let R be a commutative Cohen–Macaulay local ring of Krull dimension d . Denote by $\text{CM}(R)$ the category of (maximal) Cohen–Macaulay R -modules and by $\underline{\text{CM}}(R)$ its *stable category*: the objects of $\underline{\text{CM}}(R)$ are the Cohen–Macaulay R -modules, and the hom-set $\text{Hom}_{\underline{\text{CM}}(R)}(M, N)$ is defined as $\underline{\text{Hom}}_R(M, N)$, the quotient module of $\text{Hom}_R(M, N)$ by the submodule consisting of homomorphisms factoring through finitely generated projective (or equivalently, free) R -modules. The natural full embedding functor $\text{CM}(R) \rightarrow D^b(R)$ induces an additive covariant functor

$$\eta : \underline{\text{CM}}(R) \rightarrow D_{\text{sg}}(R).$$

Furthermore, the assignment $M \mapsto \Omega^d \text{Tr} M$, where Ω and Tr stand for the syzygy and transpose functors respectively (see [1, Chapter 2, §1] for details of the functors Ω and Tr), makes an additive contravariant functor

$$\lambda : \underline{\text{CM}}(R) \rightarrow \underline{\text{CM}}(R).$$

The following result gives a partial answer to Question 1.

Theorem 2. *The following are equivalent:*

(1) *The ring R is Gorenstein.*

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- (2) The functor η is an equivalence (i.e. η is full, faithful and dense).
 (3) There exists a functor $\phi : \mathbf{D}_{\text{sg}}(R) \rightarrow \mathbf{D}_{\text{sg}}(R)$ such that the diagram

$$\begin{array}{ccc} \mathbf{D}_{\text{sg}}(R) & \xrightarrow{\phi} & \mathbf{D}_{\text{sg}}(R) \\ \eta \uparrow & & \uparrow \eta \\ \underline{\mathbf{CM}}(R) & \xrightarrow{\lambda} & \underline{\mathbf{CM}}(R) \end{array}$$

of functors commutes up to isomorphism.

Proof. (1) $\Rightarrow$ (2): If R is Gorenstein, then a celebrated theorem of Buchweitz [2, Theorem 4.4.1] implies that the functor η is an equivalence.

(2) $\Rightarrow$ (3): When η is an equivalence, we have a contravariant endofunctor

$$\phi = \eta\lambda\rho : \mathbf{D}_{\text{sg}}(R) \rightarrow \mathbf{D}_{\text{sg}}(R)$$

of $\mathbf{D}_{\text{sg}}(R)$, where ρ stands for a quasi-inverse of η . Condition (3) holds for this functor ϕ .

(3) $\Rightarrow$ (1): In the remainder of the proof, we will omit writing free summands. Let

$$\pi : \text{mod } R \rightarrow \underline{\text{mod}} R$$

be the canonical functor from the category of finitely generated R -modules to its stable category, that is, the objects of $\underline{\text{mod}} R$ are the finitely generated R -modules and the hom-set $\underline{\text{Hom}}_{\underline{\text{mod}} R}(M, N)$ is defined as $\underline{\text{Hom}}_R(M, N)$.

Assume that there are a contravariant functor $\phi : \mathbf{D}_{\text{sg}}(R) \rightarrow \mathbf{D}_{\text{sg}}(R)$ and an isomorphism

$$\Delta : \phi\eta \rightarrow \eta\lambda$$

of functors from $\underline{\mathbf{CM}}(R)$ to $\mathbf{D}_{\text{sg}}(R)$. Take a Cohen–Macaulay R -module M . It follows from [1, Proposition (2.21)] that there exists an exact sequence

$$(2.1) \quad 0 \rightarrow F \rightarrow \text{Tr}\Omega\text{Tr}\Omega M \xrightarrow{f} M \rightarrow 0$$

of finitely generated R -modules with F free. The map f induces a morphism

$$\Theta : \text{Tr}\Omega\text{Tr}\Omega \rightarrow \text{Id}$$

of functors from $\underline{\mathbf{CM}}(R)$ to $\underline{\mathbf{CM}}(R)$, where Id stands for the identity functor. Applying the R -dual functor $(-)^* = \text{Hom}_R(-, R)$ to (2.1) gives an exact sequence

$$0 \rightarrow M^* \xrightarrow{f^*} (\text{Tr}\Omega\text{Tr}\Omega M)^* \xrightarrow{g} F^* \rightarrow \text{Tr}M \xrightarrow{h} \text{Tr}(\text{Tr}\Omega\text{Tr}\Omega M) \rightarrow \text{Tr}F \rightarrow 0$$

with $\pi(h) = \text{Tr}\pi(f)$; see [1, Lemma (3.9)]. Note that there is also an exact sequence

$$0 \rightarrow M^* \xrightarrow{f^*} (\text{Tr}\Omega\text{Tr}\Omega M)^* \xrightarrow{g} F^* \rightarrow \text{Ext}_R^1(M, R) \rightarrow \text{Ext}_R^1(\text{Tr}\Omega\text{Tr}\Omega M, R).$$

Since $\text{Ext}_R^1(\text{Tr}\Omega\text{Tr}\Omega M, R) = 0$ by [1, Theorem (2.17)] and since $\text{Tr}F$ is free, we obtain an exact sequence

$$0 \rightarrow \text{Ext}_R^1(M, R) \rightarrow \text{Tr}M \xrightarrow{h'} \text{Tr}(\text{Tr}\Omega\text{Tr}\Omega M) \rightarrow 0$$

such that $\pi(h') = \pi(h)$. Taking the d -th syzygies of $\text{Ext}_R^1(M, R)$ and $\text{Tr}(\text{Tr}\Omega\text{Tr}\Omega M)$ and using the horseshoe lemma, we get an exact sequence of Cohen–Macaulay R -modules

$$(2.2) \quad 0 \rightarrow \Omega^d \text{Ext}_R^1(M, R) \rightarrow \lambda M \xrightarrow{\ell} \lambda(\text{Tr}\Omega\text{Tr}\Omega M) \rightarrow 0$$

with $\pi(\ell) = \lambda\pi(f)$. Note that for each short exact sequence $\sigma : 0 \rightarrow X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \rightarrow 0$ of Cohen–Macaulay R -modules, the image of σ by the canonical functor π is sent by η to an exact triangle $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \rightsquigarrow$ in $\mathbf{D}_{\text{sg}}(R)$. Hence, η sends (2.2) to an exact triangle

$$\eta\Omega^d\text{Ext}_R^1(M, R) \rightarrow \eta\lambda M \xrightarrow{\eta\lambda(\Theta M)} \eta\lambda(\text{Tr}\Omega\text{Tr}\Omega M) \rightsquigarrow$$

in $\mathbf{D}_{\text{sg}}(R)$. We have a commutative diagram

$$\begin{array}{ccc} \eta\lambda M & \xrightarrow{\eta\lambda(\Theta M)} & \eta\lambda(\text{Tr}\Omega\text{Tr}\Omega M) \\ \Delta M \uparrow \cong & & \cong \uparrow \Delta(\text{Tr}\Omega\text{Tr}\Omega M) \\ \phi\eta M & \xrightarrow[\phi\eta(\Theta M)]{} & \phi\eta(\text{Tr}\Omega\text{Tr}\Omega M) \end{array}$$

of morphisms in $\mathbf{D}_{\text{sg}}(R)$, and the exact sequence (2.1) induces an isomorphism

$$\eta(\Theta M) : \eta(\text{Tr}\Omega\text{Tr}\Omega M) \rightarrow \eta M$$

in $\mathbf{D}_{\text{sg}}(R)$. Therefore $\eta\Omega^d\text{Ext}_R^1(M, R)$ is isomorphic to 0 in $\mathbf{D}_{\text{sg}}(R)$, which means that the R -module $\text{Ext}_R^1(M, R)$ has finite projective dimension. Thus, letting $M := \Omega^d k$, where k denotes the residue field of R , shows that $\text{Ext}_R^{d+1}(k, R)$ has finite projective dimension. If $\text{Ext}_R^{d+1}(k, R) = 0$, then R is Gorenstein. If $\text{Ext}_R^{d+1}(k, R) \neq 0$, then the R -module k has finite projective dimension, which implies that R is regular, so that $\text{Ext}_R^{d+1}(k, R) = 0$, a contradiction. Consequently, in either case R is a Gorenstein ring.

Now the proof of the theorem is completed. $\square$

Remark 3. In the proof of the theorem, the assumption that the ring R is commutative is used to deduce the Gorensteinness of R from the fact that the R -module $\text{Ext}_R^{d+1}(k, R)$ has finite projective dimension. For a noncommutative ring Λ with Jacobson radical J and an integer n , the n -th Ext group $\text{Ext}_\Lambda^n(\Lambda/J, \Lambda)$ of the right Λ -modules Λ/J and Λ is not necessarily semisimple as a left Λ -module.

We end this paper by stating a direct consequence of the theorem.

Corollary 4. *Suppose that R is artinian. Then R is Gorenstein if and only if the transpose functor $\text{Tr} : \underline{\text{mod}} R \rightarrow \underline{\text{mod}} R$ extends to the singularity category $\mathbf{D}_{\text{sg}}(R)$.*

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