SOME DIRECTED SUBSETS OF C^* -ALGEBRAS AND SEMICONTINUITY THEORY

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ABSTRACT. The main result concerns a σ -unital C^* -algebra A , a strongly lower semicontinuous element h of A^{**} , the enveloping von Neumann algebra, and the set of self-adjoint elements a of A such that $a \leq h - \delta \mathbf{1}$ for some $\delta > 0$, where $\mathbf{1}$ is the identity of A^{**} . The theorem is that this set is directed upward. It follows that if this set is non-empty, then h is the limit of an increasing net of self-adjoint elements of A . A complement to the main result, which may be new even if $h = \mathbf{1}$, is that if a and b are self-adjoint in A , $a \leq h$, and $b \leq h - \delta \mathbf{1}$ for $\delta > 0$, then there is a self-adjoint c in A such that $c \leq h$, $a \leq c$, and $b \leq c$.

Let A be a C^* -algebra and denote by A^{**} its bidual, the enveloping von Neumann algebra. In [1], C. Akemann and G. Pedersen discussed several classes of self-adjoint elements of A^{**} as analogues of semicontinuous functions on topological spaces.

For a subset S of A^{**} , S_{sa} denotes $\{h \in S : h^* = h\}$, S_+ denotes $\{h \in S; h \geq 0\}$, and S^- denotes the norm closure of S . And for $S \subset A_{\text{sa}}^{**}$, S^{m} denotes the set of $(\sigma$ -weak) limits of bounded increasing nets in S , S_{m} denotes the set of limits of bounded decreasing nets in S , and S^σ denotes the set of limits of bounded increasing sequences in S . Also $\tilde{A}$ denotes $A + \mathbb{C}\mathbf{1}$, where $\mathbf{1}$ is the identity of A^{**} , $LM(A) = \{T \in A^{**} : TA \subset A\}$, $QM(A) = \{T \in A^{**} : ATA \subset A\}$, $\mathcal{K}$ is the C^* -algebra of compact operators on a separable infinite dimensional Hilbert space, and M_2 is the C^* -algebra of 2×2 matrices. Finally, $S(A)$ denotes the state space of A , and states of A are also considered as (normal) states of A^{**} .

Three of the classes considered in [1] are $(A_{\text{sa}}^{\text{m}})^-$, whose elements are called strongly lsc; $(\tilde{A}_{\text{sa}})^{\text{m}}$, whose elements are called middle lsc; and $((\tilde{A}_{\text{sa}})^{\text{m}})^-$, whose elements are called weakly lsc. Also h is called usc in any sense if $-h$ is lsc in that sense. For example, $((\tilde{A}_{\text{sa}})_{\text{m}})^-$ is the set of weakly usc elements. Akemann and Pedersen showed in [1] that these three kinds of semicontinuity can all be different. But they also considered the class A_{sa}^{m} , and left open the question whether A_{sa}^{m} always equals $(A_{\text{sa}}^{\text{m}})^-$.

In [2, Corollary 3.25] I showed that A_{sa}^{m} does equal $(A_{\text{sa}}^{\text{m}})^-$ if A is separable. Whether this is so for general A is still open, and I have no conjecture concerning this. I think the main obstacle to proving the affirmative answer is that it is difficult to construct increasing nets in non-commutative C^* -algebras. I know two ways of doing this. In cases where sequences suffice, one can use recursive constructions. Or one can find a subset of A_{sa} which is a directed set in the natural ordering. The first method was used in the proof of [2, Corollary 3.25], and the second method has

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been used many times. For example, it is used in the construction of approximate identities of C^* -algebras, and it is used several times in [1]. It seems to me that the best hope for proving $A_{sa}^m = (A_{sa}^m)^-$ when A is non-separable is to prove that a suitable subset of A_{sa} is directed upward.

For h in $(A_{sa}^m)^-$ consider the following sets:

Let $\mathcal{A}_1 = \{a \in A_{sa} : a \leq h\}$.

Let $\mathcal{A}_2 = \{a \in A_{sa} : \varphi(a) < \varphi(h), \forall \varphi \in S(A)\}$.

If $h \geq 0$, let $\mathcal{A}_3 = \{a \in A_+ : a \leq \theta h \text{ for some } \theta \text{ in } (0, 1)\}$.

If $\varphi(h) > 0, \forall \varphi \in S(A)$, let $\mathcal{A}_4 = \{a \in A_+ : \exists \theta \text{ in } (0, 1) \text{ such that } \varphi(a) < \varphi(\theta h), \forall \varphi \in S(A)\}$.

Let $\mathcal{A}_5 = \{a \in A_{sa} : a \leq h - \delta \mathbf{1} \text{ for some } \delta > 0\}$.

Examples will be given at the end of this note to show that, even if A is separable, none of $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4$ need be directed upward. Counterexamples for $\mathcal{A}_1$ were already given in [2, 3.23 and 5.6]. The main result of this note is that $\mathcal{A}_5$ is directed upward if A is σ -unital. However, this does not imply that $A_{sa}^m = (A_{sa}^m)^-$ for A σ -unital, since there are some choices of h for which $\mathcal{A}_5$ is empty.

Proposition 1. *Let A be a C^* -algebra and $a, b \in A_+$ such that $\|a\| \leq 1$ and $\|b\| < 1$. Then there is c in A such that $a \leq c, b \leq c$, and $\|c\| \leq 1$.*

Proof. Let $c = b + (1 - b)^{\frac{1}{2}}[(1 - b)^{-\frac{1}{2}}(a - b)(1 - b)^{-\frac{1}{2}}]_+(1 - b)^{\frac{1}{2}}$. Since $(1 - b)^{-\frac{1}{2}}(a - b)(1 - b)^{-\frac{1}{2}} \leq (1 - b)^{-\frac{1}{2}}(1 - b)(1 - b)^{-\frac{1}{2}} = \mathbf{1}$, then $0 \leq t \leq \mathbf{1}$ if $t = [(1 - b)^{-\frac{1}{2}}(a - b)(1 - b)^{-\frac{1}{2}}]_+$. Therefore $0 \leq (1 - b)^{\frac{1}{2}}t(1 - b)^{\frac{1}{2}} \leq (1 - b)$, and hence $b \leq c \leq \mathbf{1}$. Also since $t \geq (1 - b)^{-\frac{1}{2}}(a - b)(1 - b)^{-\frac{1}{2}}$, then $c \geq b + (a - b) \geq a$. $\square$

Lemma 2. *Let A be a separable stable C^* -algebra, $h \in (A_{sa}^m)^-$, and let $\mathcal{A} = \{a \in A_{sa} : h - a \geq \delta \mathbf{1} \text{ for some } \delta > 0\}$. Then A is directed upward. Also if $a \in A_{sa}, a \leq h$, and $b \in \mathcal{A}$, then there is c in A_{sa} such that $c \leq h, a \leq c$, and $b \leq c$.*

Proof. Assume $\mathcal{A} \neq \emptyset$. Since the hypothesis and conclusion are unaffected if h is replaced by $h - a, a \in A_{sa}$, we may assume $h \geq \epsilon \mathbf{1}$ for some $\epsilon > 0$. By [2, Theorem 4.4(a)] $h = TT^*$ for some T in $LM(A)$. Therefore $T^*T \in QM(A)$ and $\sigma(T^*T)$ omits the interval $(0, \epsilon)$. Let p denote the range projection of T^*T . Then p is closed (cf. [2, Proposition 2.44(b)]). Let $T' = (T^*T)^{-1}T^*$, where the inverse is computed in $pA^{**}p$. Then $TT' = \mathbf{1}$ and $T'T = p$.

We claim that $TA = A$. To see this, note first that $T = Tp$ and hence $TA = (Tp)(pA)$. Since pA is norm closed by [3] and $(Tp)^*(Tp) \geq \epsilon p$, it follows that TA is norm closed in A . Therefore it is sufficient to show that TA is σ -weakly dense in A^{**} . But the σ -weak closure of TA includes TA^{**} , and $TA^{**} \supset TT'A^{**} = A^{**}$.

Now it is clear that $TAT^* = (TA)(AT^*) = (TA)(TA)^* = A \cdot A = A$. If $TaT^* \in \mathcal{A}$, then since $TT^* \geq TaT^* + \delta \mathbf{1}$, we have $T'(TT^*)T'^* \geq (T'T)a(T'T)^* + \delta T'T'^* = pap + \delta(T^*T)^{-1}$. Since $T'(TT^*)T'^* = p$ and $(T^*T)^{-1} \geq \eta p$ for some $\eta > 0$, this implies that $p \geq pap + \delta' p$ for some $\delta' > 0$. Conversely, if $p \geq pap + \delta' p$ for $\delta' > 0$, then

$$\begin{aligned} h = TT^* &= TpT^* \geq T(pap)T^* + \delta'(TT^*) = TaT^* + \delta'h, \\ &\geq TaT^* + \delta'\epsilon \mathbf{1}. \end{aligned}$$

Therefore $TaT^* \in \mathcal{A}$. By [2, Corollary 3.4] $\{x \in pA_{sa}p : x \leq (1 - \delta')p \text{ for some } \delta' > 0\} = \{pyp : y \in A_{sa} \text{ and } y \leq (1 - \delta')\mathbf{1} \text{ for some } \delta' > 0\}$. It is known that the

last set is directed upward (cf. [5, Theorem 1.4.2]), so it follows that $\mathcal{A}$ is directed upward. The last sentence is proved similarly using Proposition 1. $\square$

Lemma 3. *Let A be a separable C^* -algebra, $h \in (A_{sa}^m)^-$, and let $\mathcal{A} = \{a \in A_{sa} : h - a \geq \delta \mathbf{1} \text{ for some } \delta > 0\}$. Then $\mathcal{A}$ is directed upward. Also if $a \in A_{sa}$, $a \leq h$, and $b \in \mathcal{A}$, then there is c in A_{sa} such that $c \leq h$, $a \leq c$, and $b \leq c$.*

Proof. As in the proof of Lemma 2, we may assume $h \geq \epsilon \mathbf{1}$ for $\epsilon > 0$. Consider $B = A \otimes \mathcal{K}$ and $\tilde{h} = h \otimes \mathbf{1}$ in B^{**} . Obviously $\tilde{h} \in (B_{sa}^m)^-$. Let $\tilde{\mathcal{A}} = \{b \in B_{sa} : \tilde{h} - b \geq \delta \mathbf{1} \text{ for some } \delta > 0\}$, and let e be a rank one projection in $\mathcal{K}$. If $a_1, a_2 \in \mathcal{A}$, then $a_1 \otimes e, a_2 \otimes e \in \tilde{\mathcal{A}}$. By Lemma 2, $\exists b \in B_{sa}$ and $\delta > 0$ such that $a_i \otimes e \leq b \leq \tilde{h} - \delta \mathbf{1}$ for $i = 1, 2$. Therefore $a_i \otimes e \leq (\mathbf{1} \otimes e)b(\mathbf{1} \otimes e) \leq (\mathbf{1} \otimes e)\tilde{h}(\mathbf{1} \otimes e) - \delta(\mathbf{1} \otimes e)$. Under the canonical isomorphism of A with $(\mathbf{1} \otimes e)B(\mathbf{1} \otimes e)$, this is the desired conclusion. The last sentence is proved similarly. $\square$

The following is implicit in the proof of [2, Theorem 3.24], but we provide a proof here.

Lemma 4. *If A is a C^* -algebra, $h \in (A_{sa}^m)^-$, and if $\{a_n\}$ is a countable subset of A_{sa} such that $a_n \leq h, \forall n$, then there is $h_1 \in A_{sa}^\sigma$ such that $h_1 \leq h$ and $a_n \leq h_1, \forall n$.*

Proof. We construct recursively an increasing sequence (b_n) in A_{sa} such that $a_k \leq b_n + \frac{1}{n}\mathbf{1}$ for $1 \leq k \leq n$ and $b_n \leq h, \forall n$. Then let $h_1 = \lim b_n$. Let $b_1 = a_1$. If $n > 1$ and $b_1, \dots, b_{n-1}$ have been constructed, then by [2, Corollary 3.17] there is a function f such that $\lim_{\epsilon \rightarrow 0^+} f(\epsilon) = 0$ and the following is true: If $\epsilon > 0, x \in A_{sa}$, and $b_{n-1} - \epsilon \mathbf{1} \leq x \leq h + \epsilon \mathbf{1}$, then there is y in A_{sa} such that $b_{n-1} \leq y \leq h$ and $\|y - x\| < f(\epsilon)$. Choose ϵ in $(0, 1/2n)$ such that $f(\epsilon) < 1/2n$. By [2, Theorem 3.24(b)] there is x in A_{sa} such that $a_k - \epsilon \mathbf{1} \leq x \leq h$ for $i = 1, \dots, k$ and $b_{n-1} - \epsilon \mathbf{1} \leq x$. Choose b_n to be the y indicated above, then $a_k \leq x + \frac{1}{2n}\mathbf{1} \leq y + \frac{1}{n}\mathbf{1}$ for $k = 1, \dots, n$. $\square$

Theorem 5. *Let A be a σ -unital C^* -algebra, $h \in (A_{sa}^m)^-$, and let $\mathcal{A} = \{a \in A_{sa} : h - a \geq \delta \mathbf{1} \text{ for some } \delta > 0\}$. Then $\mathcal{A}$ is directed upward. Also if $a \in A_{sa}$, $a \leq h$, and $b \in \mathcal{A}$, then there is c in A_{sa} such that $c \leq h$, $a \leq c$, and $b \leq c$.*

Proof. As in the proof of Lemma 2, we may assume $h \geq \epsilon \mathbf{1}$ for some $\epsilon > 0$. Consider $a_1, a_2 \in A_{sa}$ such that $a_i \leq h - \delta_i \mathbf{1}$ for $i = 1, 2$ and $\delta_i > 0$. Let (e_n) be a sequential approximate identity of A . Then $\{a_i + \delta_i e_n : i = 1, 2, n = 1, 2, \dots\}$ is a countable subset of $\{a \in A_{sa} : a \leq h\}$. Let h_1 in A_{sa}^σ be as in Lemma 4. If $h_1 = \lim b_m, b_m \in A_{sa}$, let A_1 be the separable C^* -algebra generated by a_1, a_2 , the e_n 's, and the b_m 's. Thus A_1^{**} and A^{**} have the same identity element. Since $a_i + \delta_i e_n \leq h_1, n = 1, 2, \dots$, then $a_i + \delta_i \mathbf{1} \leq h_1$. Therefore by Lemma 3, there is c in $(A_1)_{sa} \subset A_{sa}$ such that $a_i \leq c$ for $i = 1, 2$ and $c \leq h_1 - \delta \mathbf{1} \leq h - \delta \mathbf{1}$ for some $\delta > 0$. The last sentence is proved similarly. $\square$

Corollary 6. *Let A be a σ -unital C^* -algebra and h an element of $(A_{sa}^m)^-$ such that $h \geq a + \delta \mathbf{1}$ for some a in A_{sa} and some $\delta > 0$. Then $h \in A_{sa}^m$.*

Proof. The theorem produces an increasing net in A_{sa} whose limit is the supremum of $\mathcal{A}$ in A_{sa}^{**} . To show that this supremum is h , we may assume $h \geq \epsilon \mathbf{1}$ for some $\epsilon > 0$, as in the proof of Lemma 2. Then by [1, Proposition 3.5], $h^{-1} \in ((\tilde{A}_{sa})_m)^-$. Next [1, Proposition 3.1] shows that there is a net (h_i) in $\tilde{A}_{sa}$ such that $h_i \geq h^{-1}, \forall i$, and $h_i \rightarrow h^{-1}$ σ -weakly. Finally [4, Lemma 3.2] implies that $h_i^{-1} \rightarrow h$ σ -strongly.

(So far we have just used the argument for [1, Theorem 3.3] in a slightly different context.) Now for each i , $h_i^{-1} = \delta_i \mathbf{1} + a_i$ for some $\delta_i > 0$ and $a_i \in A_{sa}$. If $e \in A_+$ and $\|e\| < 1$, then $\delta_i e + a_i \in \mathcal{A}$, and the supremum of these elements, for fixed i , is h_i^{-1} . So the supremum of $\mathcal{A}$ is indeed equal to h , whenever $\mathcal{A} \neq \emptyset$. $\square$

Examples and Remarks. (i) There are a separable unital C^* -algebra A , a positive $h \in (A_{sa}^m)^-$, $a_+, a_- \in A_+$, and $\theta \in (0, 1)$ such that $a_i \leq \theta h$ and there does not exist b in A_{sa} with $a_{\pm} \leq b \leq h$. This shows that neither $\mathcal{A}_1$ nor $\mathcal{A}_3$ is directed upward.

Let A be the algebra of convergent sequences in $\mathbb{M}_2$. Then A^{**} can be identified with the set of bounded collections, $h = \{h_n\}_{1 \leq n \leq \infty}$ with each $h_n \in \mathbb{M}_2$. Here $h \in A$ if and only if $h_n \rightarrow h_{\infty}$. Define h in $(A_{sa}^m)^-$ by $h_n = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{n} \end{pmatrix}$ for $n < \infty$ and $h_{\infty} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix}$. Let $(a_{\pm})_n = \begin{pmatrix} \theta/2 & \pm n^{-\frac{1}{2}}\theta/2 \\ \pm n^{-\frac{1}{2}}\theta/2 & n^{-1}\theta/2 \end{pmatrix} = \theta h_n^{\frac{1}{2}} \begin{pmatrix} 1/2 & \pm 1/2 \\ \pm 1/2 & 1/2 \end{pmatrix} h_n^{\frac{1}{2}}$ for $n < \infty$ and $(a_{\pm})_{\infty} = \begin{pmatrix} \theta/2 & 0 \\ 0 & 0 \end{pmatrix}$, where θ is to be determined. If $a_{\pm} \leq b \leq h$, then $b_n = h_n^{\frac{1}{2}} \begin{pmatrix} \alpha_n & \beta_n \\ \bar{\beta}_n & \gamma_n \end{pmatrix} h_n^{\frac{1}{2}}$ where

$$\theta \begin{pmatrix} 1/2 & \pm 1/2 \\ \pm 1/2 & 1/2 \end{pmatrix} \leq \begin{pmatrix} \alpha_n & \beta_n \\ \bar{\beta}_n & \gamma_n \end{pmatrix} \leq \mathbf{1}.$$

Note that the larger of $|\beta_n \pm \theta/2|$ is at least $\theta/2$. Therefore $(\theta/2)^2 \leq (\alpha_n - \theta/2)(\gamma_n - \theta/2)$. Since $\gamma_n \leq 1$, then $\alpha_n - \theta/2 \geq \frac{\theta^2/4}{1-\theta/2}$. For suitable θ , this implies that $\alpha_n \geq 3/4, \forall n$. Since $\lim \alpha_n \leq \frac{1}{2}$, this is a contradiction.

(ii) It is impossible to have a separable unital counterexample for $\mathcal{A}_2$ or $\mathcal{A}_4$, but a slight variation of the above gives a separable non-unital counterexample. Keeping the above notation, let B be the hereditary C^* -subalgebra of A consisting of sequences with limit of the form $\begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix}$. Then $a_{\pm} \in B$ and $h \in B^{**} \subset A^{**}$. Also $\varphi(h) > 0, \forall \varphi \in S(B)$. If $\theta < \theta' < 1$, then $\varphi(\theta' h - a_{\pm}) > 0, \forall \varphi \in S(B)$. Therefore neither $\mathcal{A}_2$ nor $\mathcal{A}_4$ is directed upward.

(iii) For an arbitrary C^* -algebra A , if $h \in (A_{sa}^m)^-$ and $h \geq 0$, then q , the range projection of h , is open (cf. [2, Proposition 2.44(a)]). Let B be the hereditary C^* -subalgebra supported by q . Then $h \in B^{**} \subset A^{**}$, $\varphi(h) > 0, \forall \varphi \in S(B)$, and $h \in (B_{sa}^m)^-$ ([2, Proposition 2.14]). Clearly if $a \in A_+$ and $a \leq h$, then $a \in B$.

If A is separable, then it follows from [2, Theorem 3.24(a)] that $h \geq e$ for some strictly positive element e of B . Therefore if $a \in \mathcal{A}_4$, then $\theta h - a \geq f$ for some strictly positive element f of B .

There are some variants on $\mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4$ that I could have included on the list near the beginning of this paper, and this remark shows why I didn't bother.

(iv) The arguments for Lemmas 2 and 3 apply more generally and can prove the following:

Let A be a separable C^* -algebra and h a positive strongly lsc element of A^{**} whose spectrum omits $(0, \epsilon)$ for some $\epsilon > 0$. Then for B , as in the previous remark, there are a closed projection p in $(A \otimes \mathcal{K})^{**}$ and a complete order isomorphism $\varphi : B \rightarrow p(A \otimes \mathcal{K})p$ such that $\varphi^{**}(h) = p$. Here φ^{**} is the natural extension of φ to the biduals, and in particular for b in B_{sa} , $b \leq h$ if and only if $\varphi(b) \leq p$.

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