ON COUNTING POLYNOMIALS OVER FINITE FIELDS

CHIH-YUN CHUANG, YEN-LIANG KUAN, AND JING YU

(Communicated by Matthew A. Papanikolas)

ABSTRACT. We give an asymptotic formula for the proportion of polynomials of a given degree over a finite field which do not have odd degree irreducible factors. In the case of odd characteristic, this leads to an asymptotic formula for certain weighted partition function which describes the major proportion of the fundamental discriminants where the “negative” Pell equation cannot be solved. We also extend the results to counting positive divisors over an arbitrary global function field.

1. INTRODUCTION

From the perspective of arithmetic statistics, the family of quadratic fields is of particular interest. This family is parametrized naturally by the fundamental discriminants. Simple properties of these discriminants can tell information of the corresponding quadratic fields. Thus it is often useful to collect data about densities of special subsets of discriminants.

One such special set $\mathcal{D}$ consists of discriminants D of real quadratic fields that are not divisible by any prime congruent to 3 mod 4. For $X > 1$, Rieger [6] has proved that the cardinality of $\mathcal{D} \cap [0, X] := \mathcal{D}_{\leq X}$ is asymptotically equal to $cX/\sqrt{\log X}$, where c has the value

$$c = \frac{9}{8\pi} \prod_{\substack{p:\text{prime} \\ p \equiv 1 \pmod{4}}} (1 - p^{-2})^{1/2}.$$

For a quadratic field $\mathbb{Q}(\sqrt{D})$ the condition that $D \notin \mathcal{D}$ implies its fundamental unit ϵ_D must have norm 1. If d denotes the squarefree part of D , then this implication also amounts to the curious fact that the corresponding ‘negative’ Pell equation

$$x^2 - dy^2 = -1 \quad \text{with } x, y \in \mathbb{Z}$$

is not solvable for $D \notin \mathcal{D}$.

The solvability of the negative Pell equation for squarefree d turns out to be a subtle phenomenon. Let $\mathcal{D}^- \subset \mathcal{D}$ be the set of discriminants D of real quadratic fields for which the fundamental unit ϵ_D has norm -1 . Steinhagen [8] conjectured that the density of $\mathcal{D}^-$ inside $\mathcal{D}$ exists and is equal to $1 - \alpha$, where α is the constant

$$\alpha := \prod_{\substack{j \geq 1 \\ j \text{ odd}}} (1 - 2^{-j}) = 0.5805775582 \dots$$

Received by the editors February 17, 2014 and, in revised form, July 21, 2014.

2010 *Mathematics Subject Classification.* Primary 11N45, 11R11, 14H05, 11P81.

Key words and phrases. Asymptotic results, function fields, quadratic extensions, partitions.

The authors were supported by NSC grant 102-2119-M-002-187.

Presently it is known from Fouvry and Klüners [3], that as $X \rightarrow \infty$,

$$(\alpha - o(1)) \#\mathcal{D}_{\leq X} \leq \#\mathcal{D}_{\leq X}^- \leq (2/3 + o(1)) \#\mathcal{D}_{\leq X}.$$

In this note we study analogues of Rieger's results [6] for global function fields. Let $A = \mathbb{F}_q[t]$ be the polynomial ring in one variable over the finite field $\mathbb{F}_q$ and $k = \mathbb{F}_q(t)$ the rational function field. For each positive integer n , let $\mathfrak{D}_{2n}$ be the set of monic polynomials D of degree $2n$ such that the degree of P is even for any irreducible factor P dividing D . Let a_{2n} be the cardinality of $\mathfrak{D}_{2n}$. The number a_{2n} can be thought of as function field analogues of the number $S(X)$ of integers $x \leq X$ which are a sum of two squares and every prime divisor of x is congruent to 1 mod 4. In the classical case, Landau [5, § 183] showed an asymptotic formula for the number of integers which are a sum of two squares. For the number $S(X)$, Rieger [6, Theorem 2] proved that

$$S(X) \sim \frac{\sqrt{2}}{4} \prod_{\substack{p: \text{prime} \\ p \equiv 3 \pmod{4}}} (1 - p^{-2})^{1/2} \frac{X}{\sqrt{\log X}} \quad \text{as } X \rightarrow \infty.$$

A function field analogue of Landau's result has been studied by Hoffstein, Merrill and Walling [4]. The following theorem is an analogue of Rieger's result for the rational function fields.

Theorem 1. *For positive integer N , we have*

$$a_{2N} = \sqrt{\frac{2}{\pi}} \prod_{\deg P: \text{odd}} \left(1 - \frac{1}{q^{2 \deg P}}\right)^{1/2} \frac{q^{2N}}{\sqrt{2N}} + O\left(\frac{q^{2N}}{2N\sqrt{2N}}\right),$$

where in the product P runs through all monic irreducible polynomials of odd degree.

When q is odd, squarefree monic polynomials in A parametrize the quadratic function fields over the rational function field k . We are interested in data about densities of subsets of these polynomials. In particular, if b_{2n} denotes the cardinality of the set of squarefree polynomials in $\mathfrak{D}_{2n}$, then we have the following theorem.

Theorem 2. *Let $\zeta_A(s)$ be the zeta function of A . For positive integer N , we have*

$$b_{2N} = \sqrt{\frac{2}{\zeta_A(2)\pi}} \prod_{\deg P: \text{even}} \left(1 - \frac{1}{q^{2 \deg P}}\right)^{1/2} \frac{q^{2N}}{\sqrt{2N}} + O\left(\frac{q^{2N}}{2N\sqrt{2N}}\right),$$

where in the product P runs through all monic irreducible polynomials of even degree.

In his thesis [1] E. Artin initiated the study of Pell's equations over the polynomial ring $A = \mathbb{F}_q[t]$ with q odd. In this case, let D be a fundamental discriminant for "real" quadratic function field, i.e., a squarefree monic polynomial in A , the 'negative' Pell equation asks for the solvability of

$$x^2 - Dy^2 = \gamma_q \quad \text{with } x, y \in A,$$

where γ_q is a generator of $\mathbb{F}_q^\star$. As in the classical story, the solvability of such an equation is equivalent to the assertion that a fundamental unit ϵ_D of the quadratic function field $k(\sqrt{D})$ has norm equal to γ_q . If D has an irreducible factor of odd degree, such units cannot exist. Thus, the negative Pell equation has no solution for a given fundamental discriminant D of degree $2N$ unless D is among those b_{2N} polynomials counted above.

One is certainly interested in the totality of discriminants in $\mathfrak{D}_{2n}$ for which the fundamental units ϵ_D do have norm γ_q . Let $\mathfrak{D}_{2n}^-$ be the subset consisting of these discriminants. We expect that for any fixed odd q , there is an analogue of Stevenhagen's conjecture: the proportion $\#\mathfrak{D}_{2n}^-/b_{2n}$ is asymptotically equal to $1 - \alpha$. In this direction, Bae and Jung [2] also prove an analogue of the result of Fouvry-Klüners for the case of rational function fields $\mathbb{F}_q(t)$ with $q \equiv 3 \pmod{4}$:

$$(\alpha - o(1))b_{2n} \leq \#\mathfrak{D}_{2n}^- \leq (2/3 + o(1))b_{2n},$$

holds as n goes to infinity.

More generally, consider a global function field k with finite constant field $\mathbb{F}_q$, i.e., k is a finitely generated field extension of transcendence degree one over $\mathbb{F}_q$ and $\mathbb{F}_q$ is algebraically closed in k . For each place v of k , the residue field at v is denoted by $\mathbb{F}_v$ which has cardinality q_v . The degree $\deg v$ of v is $[\mathbb{F}_v : \mathbb{F}_q]$. Let K_r be the fixed constant field extension over k with degree r .

Let $\text{Div}(k)$ be the divisor group of k . We adopt the multiplicative notation so that every element $\mathfrak{m}$ in $\text{Div}(k)$ is written as

$$\mathfrak{m} = \prod_v v^{\text{ord}_v(\mathfrak{m})}.$$

Given $\mathfrak{m} \in \text{Div}(k)$, we define

$$\|\mathfrak{m}\| := \prod_v q_v^{\text{ord}_v(\mathfrak{m})} = q^{\deg \mathfrak{m}},$$

where

$$\deg \mathfrak{m} := \sum_v \deg v \cdot \text{ord}_v(\mathfrak{m})$$

is the degree of $\mathfrak{m}$. A divisor $\mathfrak{m}$ is said to be *positive* if $\text{ord}_v(\mathfrak{m}) \geq 0$ for every place v of k . The set of positive divisors is denoted by $\text{Div}_{\geq 0}(k)$.

Given positive integers $n, l, r \geq 2$, we are interested in counting the cardinality a_{rn} of:

$$\{g \in \text{Div}_{\geq 0}(k) \text{ with } \deg(g) = rn \mid g = \prod_{i=1}^j v_i^{e_i} \text{ with } r \mid \deg v_i \text{ and } e_i \in \mathbb{N} \text{ for all } i\},$$

as well as the cardinality $b_{rn,l}$ of the set:

$$\{g \in \text{Div}_{\geq 0}(k) \text{ with } \deg(g) = rn \mid g = \prod_{i=1}^j v_i^{e_i} \text{ with } r \mid \deg v_i \text{ and } e_i \leq l \text{ for all } i\}.$$

Let g_k be the genus of k and h_{K_r} denote the number of divisor classes of degree zero for the constant field extension K_r . Then Theorems 1 and 2 are generalized to:

Theorem 3. *Given a function field k with constant field $\mathbb{F}_q$ and a positive integer $r \geq 2$, we have for all integers n the following formula:*

$$\begin{aligned} a_{rn} = & \frac{q^{1-g_k}}{\Gamma\left(\frac{1}{r}\right)} \cdot \sqrt[r]{\frac{h_{K_r} \cdot r^{r-1}}{(q^r - 1)}} \cdot \left[\prod_{\substack{d \mid r \\ d \neq r}} \prod_{(r, \deg v)=d} (1 - q_v^{-\frac{r}{d}})^{\frac{d}{r}} \right] \\ & \cdot \left(\frac{q^{rn}}{\sqrt[r]{(rn)^{r-1}}} \right) + O\left(\frac{q^{rn}}{rn \sqrt[r]{(rn)^{r-1}}} \right). \end{aligned}$$

Here the constant in the big O -term depends only on the field k and the integer r .

Theorem 4. *Given a function field k with constant field $\mathbb{F}_q$ and positive integers $l, r \geq 2$, we have*

$$b_{rn,l} = \frac{q^{1-g_k}}{\Gamma\left(\frac{1}{r}\right)} \sqrt[r]{\frac{h_{K_r} \cdot r^{r-1}}{(q^r - 1)}} \left[\prod_{\substack{d|r \\ d \neq r}} \prod_{(r, \deg v)=d} (1 - q_v^{-\frac{r}{d}})^{\frac{d}{r}} \right] \\ \cdot \left[\prod_{r|\deg v} (1 - q_v^{-l-1}) \right] \left(\frac{q^{rn}}{\sqrt[r]{(rn)^{r-1}}} \right) + O\left(\frac{q^{rn}}{rn \sqrt[r]{(rn)^{r-1}}} \right),$$

with the constant in the big O -term depending only on the field k and the integers r, l .

Remark. When $k = \mathbb{F}_q(t)$ and $r = 2, l = 1$, even degree polynomials in A without odd degree irreducible factors are identified with positive ‘finite’ divisors, Theorem 3 reduces to Theorem 1, and Theorem 4 reduces to Theorem 2. Just note that for the infinite place $v = \infty$:

$$\prod_{(r, \deg v)=d=1} (1 - q_v^{-\frac{r}{d}})^{\frac{d}{r}} = \frac{\sqrt[r]{(q^2 - 1)}}{q}.$$

2. SOME PARTIAL EULER PRODUCTS

We need some basic facts about zeta functions of function fields.

Definition. The zeta function of k , $\zeta_k(s)$, is defined by

$$\zeta_k(s) = \sum_{\mathfrak{m} \in \text{Div}_{\geq 0}(k)} \frac{1}{\|\mathfrak{m}\|^s},$$

which converges absolutely for $\text{Re}(s) > 1$.

Theorem 5 ([7, Theorem 5.9]). *Let k be a global function field in one variable with a finite constant field $\mathbb{F}_q$. Suppose that the genus of k is g_k . Then there is a polynomial $L_k(u) \in \mathbb{Z}[u]$ of degree $2g_k$ such that*

$$\zeta_k(s) = \frac{L_k(q^{-s})}{(1 - q^{-s})(1 - q^{1-s})}.$$

This holds for all s such that $\text{Re}(s) > 1$ and the right-hand side provides an analytic continuation of $\zeta_k(s)$ to all of $\mathbb{C}$. $\zeta_k(s)$ has simple poles at $s = 0$ and $s = 1$. One has $L_k(0) = 1$ and $L_k(1) = h_k$. Here h_k is the number of divisor classes of degree zero. Finally, set $\xi_k(s) = q^{(g_k-1)s} \zeta_k(s)$. Then for all s one has $\xi_k(1-s) = \xi_k(s)$.

Corollary 6. *We have*

$$\text{Res}_{s=1} \zeta_k(s) = q^{1-g_k} \frac{h_k}{\log q \cdot (q-1)}.$$

Theorem 7 ([7, Theorem 5.10] (The Riemann Hypothesis)). *Let k be a global function field whose constant field is $\mathbb{F}_q$. All the roots of $\zeta_k(s)$ lie on the line $\text{Re}(s) = \frac{1}{2}$.*

We introduce “partial Euler products”. As v is a place of k , v splits into $d := (r, \deg v)$ places in K_r . We consider the following:

$$\mathcal{L}_d(s) := \prod_{v: (r, \deg v) = d} (1 - q_v^{-\frac{rs}{d}})^{-d} \text{ for } \operatorname{Re}(s) > \frac{d}{r}.$$

In particular, if $d = r$, we have

$$\mathcal{L}_r(s) = \prod_{\substack{v \\ v \text{ splits completely} \\ \text{in } K_r}} (1 - q_v^{-s})^{-r}.$$

And the following identity holds:

$$\mathcal{L}_r(s) = \zeta_{K_r}(s) \cdot \prod_{\substack{d|r, \\ d \neq r}} \mathcal{L}_d^{-1}(s).$$

Here $\mathcal{L}_d^{-1}(s)$ is well defined for $\operatorname{Re}(s) > \frac{1}{2}$. Since $\zeta_{K_r}(s)$ is a rational function of q^{-s} with poles $s = 0$ and 1 , $\mathcal{L}_r(s)$ has meromorphic continuation to $\operatorname{Re}(s) > \frac{1}{2}$ with a simple pole at $s = 1$. On the other hand, it is not hard to see that

$$\mathcal{L}_r(s) = \left(\sum_{n=0}^{\infty} \frac{a_{rn}}{q^{rns}} \right)^r.$$

Now, we rewrite $\mathcal{L}_r(s)$ as a power series of variable $u := q^{-rs}$ and set

$$F(u) := \sum_{n=0}^{\infty} a_{rn} u^n.$$

Then $F^r(u)$ is a meromorphic function on the

$$\{u \in \mathbb{C} : |u| < q^{-\frac{r}{2}}\}$$

with a simple pole at $u = q^{-r}$ and

$$\begin{aligned} \lim_{u \rightarrow q^{-r}} (u - q^{-r}) \cdot F^r(u) &= \lim_{s \rightarrow 1} \frac{u - q^{-r}}{s - 1} \cdot (s - 1) \cdot (\mathcal{L}_r(s))^r \\ &= q^{r(1-g_k)} \cdot \frac{-h_{K_r}}{q^r \cdot (q^r - 1)} \cdot \left[\prod_{\substack{d|r, \\ d \neq r}} \prod_{(r, \deg v) = d} (1 - q_v^{-\frac{r}{d}})^d \right]. \end{aligned}$$

Let $\sqrt[r]{u} := e^{\frac{\log u}{r}}$, analytic branch of $\log u$ fixed with $0 < \operatorname{Arg} u < 2\pi$. Applying Theorem 7 to the field K_r , $F^r(u)$ is non-vanishing on $\{u \in \mathbb{C} : |u| < q^{-\frac{r}{2}}\}$. Therefore $F(u) = \exp(\frac{1}{r} \ln \mathcal{L}_r(u))$ can be defined on the simply connected domain

$$\{u \in \mathbb{C} - [q^{-r}, \infty) : |u| < q^{-\frac{r}{2}}\}.$$

Corollary 8. (i)

$$F(u) = \sqrt[r]{\zeta_{K_r}(u)} \prod_{\substack{d|r, \\ d \neq r}} \left(1 - u^{\frac{\deg v}{d}}\right)^{\frac{d}{r}}$$

is a meromorphic function on

$$\{u \in \mathbb{C} - [q^{-r}, \infty) : |u| < q^{-\frac{r}{2}}\}$$

FIGURE 1

with a pole $u = q^{-r}$.

(ii)

$$\lim_{u \rightarrow q^{-r}} \sqrt[r]{u - q^{-r}} F(u) = q^{1-g_k} \cdot \sqrt[r]{\frac{-h_{K_r}}{q^r \cdot (q^r - 1)}} \cdot \left[\prod_{\substack{d|r \\ d \neq r}} \prod_{(r, \deg v) = d} (1 - q_v^{-\frac{r}{d}})^{\frac{d}{r}} \right].$$

3. PROOF OF THEOREM 3

Choose $0 < \delta < \frac{1}{2}$ such that $F(u)$ is holomorphic on the domain $\{u \in \mathbb{C} - [q^{-r}, \infty) \mid |u| \leq q^{-r+r\delta}\}$. Fix $\epsilon, \epsilon' > 0$ such that $\epsilon < \min\{q^{-r}, q^{-r+r\delta} - q^{-r}\}$ and $\epsilon' < q^{-r} - \epsilon$. Let $C_0 = \{u \in \mathbb{C} \mid |u| = \epsilon'\}$ be a small circle oriented clockwise. Consider the keyhole contour starting from a small clockwise oriented circle C_ϵ about q^{-r} of radius ϵ , extending to a line segment γ_1 parallel, close to and above the branch cut, then continued to the circle C_1 counterclockwise oriented around the origin of radius $q^{-r+r\delta}$, returning to a line segment γ_2 parallel, close to and below the branch cut in the negative sense, finally returning to the original small circle centered at q^{-r} as in Figure 1.

Lemma 9. For each $n \in \mathbb{N}$, we have

$$a_{rn} = \frac{1}{2\pi i} \int_{C_1 + \gamma_1 + C_\epsilon + \gamma_2} \frac{F(u)}{u^{n+1}} du.$$

Proof. Since $F(u)/u^{n+1}$ is analytic inside the contour $C_1 + \gamma_1 + C_\epsilon + \gamma_2$ except at the point $u = 0$,

$$\int_{C_0 + C_1 + \gamma_1 + C_\epsilon + \gamma_2} \frac{F(u)}{u^{n+1}} du = 0.$$

From the power series expansion of $F(u)$ about $u = 0$, we obtain

$$\frac{1}{2\pi i} \int_{C_0} \frac{F(u)}{u^{n+1}} du = -a_{rn}.$$

This completes the proof of the lemma. $\square$

Before we start to prove the main theorem, we also establish

Lemma 10. *Let $r \geq 2$ be an integer, and $\delta, \epsilon, \epsilon'$ be as above; then*

$$\int_{q^{-r}}^{q^{r(-1+\delta)}} \frac{du}{u^{n+1} \cdot \sqrt[r]{u - q^{-r}}} = \left(\frac{q}{\Gamma\left(\frac{1}{r}\right)} \cdot \frac{\pi}{\sin \frac{\pi}{r}} \right) \cdot \frac{q^{rn}}{\sqrt[r]{n^{r-1}}} + O\left(\frac{q^{rn}}{n \sqrt[r]{n^{r-1}}}\right) \text{ as } n \rightarrow \infty.$$

Proof. Substituting $q^{-r}u$ for u , the integral changes to

$$\begin{aligned} q^{rn+1} \cdot \int_1^{q^{r\delta}} \frac{du}{u^{n+1} \cdot \sqrt[r]{u-1}} &= q^{rn+1} \cdot \int_0^{q^{r\delta}-1} \frac{du}{(u+1)^{n+1} \cdot \sqrt[r]{u}} \\ &= q^{rn+1} \cdot \left[\int_0^\infty \frac{du}{(u+1)^{n+1} \cdot \sqrt[r]{u}} - \int_{q^{r\delta}-1}^\infty \frac{du}{(u+1)^{n+1} \cdot \sqrt[r]{u}} \right] := q^{rn+1}(I_1 + I_2). \end{aligned}$$

The integral I_2 is simply equal to $O\left(\frac{q^{-r\delta n}}{n}\right)$. The integral I_1 can be checked to be

$$\frac{2\pi i}{1 - e^{-\frac{2\pi i}{r}}} \cdot \text{Res} \left(\frac{u^{-\frac{1}{r}}}{(u+1)^{n+1}}, -1 \right)$$

by the residue theorem. Furthermore, we have

$$\begin{aligned} &\text{Res} \left(\frac{u^{-\frac{1}{r}}}{(u+1)^{n+1}}, -1 \right) \\ &= \left[\frac{1}{n! \cdot n^{\frac{1}{r}}} \left(\prod_{i=0}^{n-1} \left(-\frac{1}{r} - i \right) \right) \left(\frac{1}{r} + n \right) \cdot (-1)^{-\frac{1}{r}-n} \right] \frac{n^{\frac{1}{r}}}{n + \frac{1}{r}} \\ &= \frac{e^{-\frac{\pi i}{r}}}{\Gamma\left(\frac{1}{r}\right)} n^{\frac{1}{r}-1} \left(1 + O\left(\frac{1}{n}\right) \right). \end{aligned}$$

This gives the desired identity. $\square$

We are now ready to prove Theorem 3 by establishing the following Proposition 11. From Theorem 5 and Corollary 8, one has

$$F(u) = \sqrt[r]{\frac{L_k(u)}{(1-u)(1-q^r u)}} \prod_{\substack{d|r, \\ d \neq r}} \left(1 - u^{\frac{\deg v}{d}} \right)^{\frac{d}{r}}$$

on $\{u \in \mathbb{C} - [q^{-r}, \infty) : |u| < q^{-\frac{r}{2}}\}$. Let

$$\beta_r := q^{1-g_k} \cdot \sqrt[r]{\frac{-h_{K_r}}{q^r \cdot (q^r - 1)}} \cdot \left[\prod_{\substack{d|r, \\ d \neq r}} \prod_{(r, \deg v)=d} (1 - q^{-\frac{r}{d}})^{\frac{d}{r}} \right].$$

Corollary 8 says that

$$F(u) = \frac{\beta_r}{\sqrt[r]{u - q^{-r}}} (1 + O(u - q^{-r})) \text{ as } u \rightarrow q^{-r},$$

with the constant in the big O -term depending only on the field k and the integer r .

Proposition 11. (i)

$$\int_{C_1} \frac{F(u)}{u^{n+1}} du = o\left(\frac{q^{rn}}{rn \sqrt{(rn)^{r-1}}}\right) \quad \text{as } n \rightarrow \infty.$$

(ii) For each $n \in \mathbb{N}$, we have

$$\int_{C_\epsilon} \frac{F(u)}{u^{n+1}} du \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0.$$

(iii)

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \frac{1}{2\pi i} \cdot \int_{\gamma_1 + \gamma_2} \frac{F(u)}{u^{n+1}} du &= \frac{q^{1-g_k}}{\Gamma\left(\frac{1}{r}\right)} \cdot \sqrt[r]{h_{K_r} \cdot r^{r-1}} \cdot \left[\prod_{\substack{d|r \\ d \neq r}} \prod_{(r, \deg v)=d} (1 - q_v^{-\frac{r}{d}})^{\frac{d}{r}} \right] \\ &\cdot \left(\frac{q^{rn}}{\sqrt{(rn)^{r-1}}} \right) + O\left(\frac{q^{rn}}{rn \sqrt{(rn)^{r-1}}}\right) \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Proof. (i) Since $q^{-r+r\delta} < q^{-r/2}$,

$$L_k(u) \left(\prod_{\substack{d|r, \\ d \neq r}} \left(1 - u^{\frac{\deg v}{d}} \right)^{\frac{d}{r}} \right)$$

is bounded on C_1 , say,

$$\left| L_k(u) \cdot \left(\prod_{\substack{d|r, \\ d \neq r}} \left(1 - u^{\frac{\deg v}{d}} \right)^{\frac{d}{r}} \right) \right| < M_2, \quad \forall u \in C_1.$$

Then we have

$$\begin{aligned} \left| \int_{C_1} \frac{F(u)}{u^{n+1}} du \right| &\leq M_2 \int_0^{2\pi} \left| \frac{1}{\sqrt[r]{(1 - q^r q^{-r+r\delta} e^{i\theta})(1 - q^{-r+r\delta} e^{i\theta})(q^{-r+r\delta} e^{i\theta})^n}} \right| d\theta \\ &\leq M_2 \cdot q^{rn(1-\delta)} \int_0^{2\pi} \left| \frac{1}{\sqrt[r]{q^{r\delta} - 1}} \right| d\theta \\ &= O\left(\frac{q^{rn(1-\delta)}}{\delta}\right). \end{aligned}$$

If n is large enough, we may take $\delta = \frac{1}{\sqrt[r]{rn}}$. Then

$$\frac{q^{rn(1-\delta)}}{\delta} = o\left(\frac{q^{rn}}{rn \sqrt{(rn)^{r-1}}}\right).$$

(ii) Note that

$$\int_{C_\epsilon} \frac{F(u)}{u^{n+1}} du = \int_{C_\epsilon} \frac{L_k(u) \cdot \left(\prod_{\substack{d|r, \\ d \neq r}} \left(1 - u^{\frac{\deg v}{d}} \right)^{\frac{d}{r}} \right)}{u^{n+1} \cdot \sqrt[r]{(1 - q^r u)(1 - u)}} du.$$

Since $0 < \delta < 1/2$ and $q^{-r} + \epsilon < q^{-r+r\delta}$,

$$L_k(u) \cdot \left(\prod_{\substack{d|r, \\ d \neq r}} \left(1 - u^{\frac{\deg v}{d}} \right)^{\frac{d}{r}} \right)$$

is bounded on C_ϵ , say,

$$\left| L_k(u) \cdot \prod_{\substack{d|r, \\ d \neq r}} \left(1 - u^{\frac{\deg v}{d}} \right)^{\frac{d}{r}} \right| < M_1, \quad \forall u \in C_\epsilon.$$

Thus we have

$$\begin{aligned} & \left| \int_{C_\epsilon} \frac{F(u)}{u^{n+1}} du \right| \\ & \leq M_1 \int_0^{2\pi} \left| \frac{\epsilon}{\sqrt[r]{(1-q^r(q^{-1} + \epsilon e^{i\theta})^r)(1-q^{-1} - \epsilon e^{i\theta})}(q^{-1} + \epsilon e^{i\theta})^{n+1}} \right| d\theta \\ & = 0, \text{ as } \epsilon \rightarrow 0. \end{aligned}$$

(iii) Applying Corollary 8 and Lemma 10, one has

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \frac{1}{2\pi i} \int_{\gamma_1 + \gamma_2} \frac{F(u)}{u^{n+1}} du \\ & = \lim_{\epsilon \rightarrow 0} \frac{1}{2\pi i} \left[\int_{\gamma_1 + \gamma_2} \frac{\beta_r}{\sqrt[r]{u - q^{-r}} u^{n+1}} du + O \left(\int_{\gamma_1 + \gamma_2} \frac{\sqrt[r]{(u - q^{-r})^{r-1}}}{u^{n+1}} du \right) \right] \\ & = \frac{q^{1-g_k}}{\Gamma\left(\frac{1}{r}\right)} \cdot \sqrt[r]{\frac{h_{K_r} \cdot r^{r-1}}{(q^r - 1)}} \cdot \left[\prod_{\substack{d|r, \\ d \neq r}} \prod_{(r, \deg v) = d} (1 - q_v^{-\frac{r}{d}})^{\frac{d}{r}} \right] \cdot \left(\frac{q^{rn}}{\sqrt[r]{(rn)^{r-1}}} \right) + O(I_2). \end{aligned}$$

For the error term I_2 , since $s = -\frac{\log u}{r \log q}$ and $u - q^{-r} = O(s - 1)$ as $u \rightarrow q^{-r}$,

$$\begin{aligned} I_2 & = O \left(\int_{q^{-r}}^{q^{-r(1-\delta)}} \left| \frac{\sqrt[r]{-\left(\frac{\log u}{r \log q} + 1\right)^{r-1}}}{u^{n+1}} \right| du \right) \\ & = O \left(\int_{q^{-r}}^{q^{-r+r\delta}} \left| \frac{\sqrt[r]{(\log u + r \log q)^{r-1}}}{u^{n+1}} \right| du \right) \\ & = O \left(q^{rn} \int_1^{q^{r\delta}} \frac{\sqrt[r]{(\log u')^{r-1}}}{u'^{n+1}} du' \right) \text{ (changing the variable } u \text{ by } q^{-r}u') \\ & = O \left(\frac{q^{rn}}{rn \sqrt[r]{(rn)^{r-1}}} \int_0^\infty \sqrt[r]{x} e^{-x} dx \right) \text{ (changing the variable } u' \text{ by } e^{x/n}) \\ & = O \left(\frac{q^{rn}}{rn \sqrt[r]{(rn)^{r-1}}} \right). \end{aligned}$$

□

Combining Lemma 9 and Proposition 11, we obtain Theorem 3.

4. THEOREM 4 AND A WEIGHTED PARTITION FUNCTION

For positive integers n, r and l , let $b_{rn,l}$ be the cardinality of the following set:

$$\{g \in \text{Div}_{\geq 0}(k) \text{ with } \deg(g) = rn \mid g = \prod_{i=1}^j v_i^{e_i} \text{ with } r \mid \deg v_i \text{ and } e_i \leq l \text{ for all } i\}.$$

We introduce $\tilde{\mathcal{L}}(s)$ by the series

$$\tilde{\mathcal{L}}(s) := \left(\sum_{n=0}^{\infty} \frac{b_{rn,l}}{q^{rns}} \right)^r.$$

From the definition of $\mathcal{L}_r(s)$ in Section 2, we immediately obtain the following lemma.

Lemma 12. *We have*

$$\tilde{\mathcal{L}}(s) = \left(\prod_{r \mid \deg v} \left(\sum_{m=0}^l q_v^{-ms} \right) \right)^r = \frac{\mathcal{L}_r(s)}{\mathcal{L}_r((l+1)s)}, \quad \text{for } \text{Re}(s) > \frac{1}{2}.$$

Set $\tilde{F}(u) := \sum \frac{b_{rn,l}}{u^n}$. Then $\tilde{F}(u)^r$ has meromorphic continuation to the set $\{u \in \mathbb{C} : |u| < q^{-\frac{r}{2}}\}$ with simple pole at q^{-r} . Thus

$$\lim_{u \rightarrow q^{-r}} \sqrt[r]{u - q^{-r}} \tilde{F}(u) = \lim_{u \rightarrow q^{-r}} \sqrt[r]{u - q^{-r}} F(u) \cdot \frac{1}{F(q^{-r(l+1)})} = \beta_r \cdot F(q^{-r(l+1)}).$$

Using the same method as in the proof of Theorem 3 we derive that

$$b_{rn,l} = \frac{q^{1-g_k}}{\Gamma\left(\frac{1}{r}\right)} \cdot \sqrt[r]{\frac{h_{K_r} \cdot r^{r-1}}{(q^r - 1)}} \cdot \left[\prod_{\substack{d \mid r \\ d \neq r}} \prod_{(r, \deg v) = d} (1 - q_v^{-\frac{r}{d}})^{\frac{d}{r}} \right] \\ \cdot \left[\prod_{r \mid \deg v} (1 - q_v^{-l-1}) \right] \frac{q^{rn}}{\sqrt[r]{(rn)^{r-1}}} + O\left(\frac{q^{rn}}{rn \sqrt[r]{(rn)^{r-1}}}\right).$$

This completes the proof of Theorem 4.

As an application of Theorem 4 we consider partitions of n into parts which are unequal. For each non-negative integer n , let $q(n)$ be the number of partitions of n into distinct parts. Then the generating function of $q(n)$ is given by

$$\sum_{n=0}^{\infty} q(n)x^n = \prod_{m=1}^{\infty} (1 + x^m).$$

We now give a weight p_m to each m where p_m is the number of irreducible polynomials in $\mathbb{F}_q[t]$ of degree m and consider the generating function, for positive integers l and $r \geq 2$,

$$\prod_{m=1}^{\infty} (1 + x^{rm} + x^{2rm} + \cdots + x^{lrm})^{p_{rm}} := \sum_{n=0}^{\infty} c_{rn,l} x^{rn}.$$

q=2; r=2; l=1; $\beta = 0.542323 \dots$		q=3; r=2; l=1; $\beta = 0.638491 \dots$	
n	$b_{rn,l} \cdot \frac{\sqrt{2n}}{r^n}$	n	$b_{rn,l} \cdot \frac{\sqrt{2n}}{r^n}$
6	0.407280	1	0.471405
7	0.506302	2	0.510519
8	0.510498	3	0.574572
9	0.514809	4	0.588447
10	0.517630	5	0.601030
11	0.520129	6	0.607475
12	0.522063	7	0.612341
13	0.523732	8	0.615765
14	0.525122	9	0.618424
15	0.526328	10	0.620511
16	0.527370	11	0.622206
17	0.528286	12	0.623606
18	0.529095	13	0.624785
19	0.529816	14	0.625790
20	0.530461	15	0.626657
21	0.531044	q=2; r=4; l=1; $\beta = 0.721565 \dots$	
22	0.531572		
23	0.532052	n	$b_{rn,l} \cdot \frac{\sqrt{(4n)^l}}{r^{4n}}$
24	0.532491	10	0.708991
25	0.532894	11	0.710193
26	0.533266	12	0.711184
27	0.533609	13	0.712015
28	0.533927	14	0.712723
29	0.534223	15	0.713333
30	0.534498		

FIGURE 2

Then observe that $c_{rn,l} = b_{rn,l}$ for each n . Applying Theorem 4, we have

$$c_{rn,l} = \frac{r^{\sqrt{r^{r-1}}}}{\Gamma\left(\frac{1}{r}\right)} \left[\prod_{\substack{d|r \\ d \neq r}} \prod_{(r, \deg P)=d} \left(1 - q^{-\frac{r \deg P}{d}}\right)^{\frac{d}{r}} \right] \cdot \left[\prod_{r|\deg P} (1 - q^{-\deg P(l+1)}) \right] \frac{q^{rn}}{r^{\sqrt{(rn)^{r-1}}}} + O\left(\frac{q^{rn}}{rn^{\sqrt{(rn)^{r-1}}}}\right).$$

In particular,

$$c_{2n,1} \sim \sqrt{\frac{2(q-1)}{\pi q}} \prod_{\deg P: \text{even}} (1 - q^{-2 \deg P})^{\frac{1}{2}} \frac{q^{2n}}{\sqrt{2n}},$$

where $c_{2n,1}$ counts squarefree monic polynomials f of degree $2n$ with the degree of P is even for any irreducible factor P dividing f .

The tables in Figure 2 show Theorem 4 in the case of $\mathbb{F}_q(t)$. Let β be the constant of the main term as the following:

$$\beta := \frac{r^{\sqrt{r^{r-1}}}}{\Gamma\left(\frac{1}{r}\right)} \left[\prod_{\substack{d|r \\ d \neq r}} \prod_{(r, \deg P)=d} \left(1 - q^{-\frac{r \deg P}{d}}\right)^{\frac{d}{r}} \right] \left[\prod_{r|\deg P} (1 - q^{-\deg P(l+1)}) \right].$$

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DEPARTMENT OF MATHEMATICS, NATIONAL TSING HUA UNIVERSITY, HSINCHU CITY, TAIWAN 300

Current address: Taida Institute for Mathematical Sciences, National Taiwan University, Taipei City, Taiwan 10617

E-mail address: cychuang@ntu.edu.tw

DEPARTMENT OF MATHEMATICS, NATIONAL TAIWAN UNIVERSITY, TAIPEI CITY, TAIWAN 10617

Current address: Taida Institute for Mathematical Sciences, National Taiwan University, Taipei City, Taiwan 10617

E-mail address: ylkuan@ntu.edu.tw

DEPARTMENT OF MATHEMATICS, NATIONAL TAIWAN UNIVERSITY, TAIPEI CITY, TAIWAN 10617

E-mail address: yu@math.ntu.edu.tw