A GRADIENT ESTIMATE FOR THE HEAT SEMI-GROUP WITHOUT HYPOELLIPTICITY ASSUMPTIONS

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ABSTRACT. We obtain an estimate for the L^p norm of the gradient of the heat semi-group in terms of the L^p norm of the gradient. Our estimates are uniform for small times and $p \in [1, \infty]$. The bounds only require some basic smoothness assumptions on the vector fields defining the diffusion underlying the problem.

1. INTRODUCTION

The heat semi-group is a fundamental object in mathematical analysis. We consider the semi-group $P_t f$ of a diffusion Y_t defined on $\mathbb{R}^N$ by the Stratonovich SDE

$$(1) \quad dY_t^x = \sum_{i=1}^d V_i(Y_t^x) \circ dX_t, \quad Y_0^x = x \in \mathbb{R}^N$$

driven by d dimensional Brownian motion. Under some basic assumptions on the smoothness of the vector fields V_i we obtain gradient bounds of the form

$$(2) \quad \|\nabla P_t f(x)\|_{L^p(dx)} \leq C \|\nabla f(x)\|_{L^p(dx)},$$

where the constant C is independent of $t \in (0, T]$, $p \in [1, \infty]$ and $f \in C^1(\mathbb{R}^N)$.

Pointwise bounds of the form

$$\nabla P_t f(x) \leq C P_t \nabla f(x)$$

have attracted considerable interest and were for example obtained in the deep results of Bakry-Emery et al. (see e.g. [3],[2]). Similarly, probabilistic approaches that rely on coupling the diffusion starting from different initial points and demonstrate that the Jacobian of the flow is bounded (see e.g. [1]), provide pointwise bounds. While pointwise bounds are clearly stronger they require additional geometric assumptions on the semi-group such as hypo-ellipticity. This leads to the natural question if the regularity estimate (2) can be obtained for a general diffusion (1). Surprisingly, we are not aware of a proof of the estimate (2) that does not require hypo-ellipticity type assumptions on the vector fields defining the diffusion, even in the Euclidean case we are presently considering.

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In this note we propose an elementary proof that only relies on fundamental properties of stochastic flows and the uniformity of integrability estimates for the Jacobian of the flow that follow readily from the recent paper by Cass, Litterer, Lyons [5]. The estimates generalise in the sense that we may replace the Brownian motion in (1) with any Gaussian noise satisfying the assumptions of [5]. This class of Gaussian noises includes fractional Brownian motion with Hurst parameter $H > 1/4$, the Ornstein-Uhlenbeck process and the Brownian bridge. It is clear that P_t is no longer a semi-group if the driving noise is not Markov. An important first step towards extending the rough path analysis underlying this paper to more general geometric settings has already been made in [4], where the authors introduce an efficient (and intrinsic) notion of rough differential equations on embedded submanifolds.

2. MAIN ESTIMATE

Let $C^1_{\mathcal{P}}(\mathbb{R}^N)$ be the space of continuously differentiable functions with derivative growing at most polynomially fast and $f \in C^1_{\mathcal{P}}(\mathbb{R}^N)$. Define the heat semi-group $P_t f$ by letting

$$P_t f(x) := E(f(Y_t^x)),$$

where Y_t^x solves the SDE (1). We assume that the vector fields V_i defining the equation (1) are $\text{Lip-}\gamma(\mathbb{R}^N, \mathbb{R}^N)$ for some $\gamma > 2$ in the sense of Stein.¹ Let $WG\Omega_p(\mathbb{R}^d)$ denote the space of weakly geometric p -rough paths (for details on the basic concepts of rough path we refer the reader to the introductions in [11], [10], [9], [7]). For a geometric p -rough path $\mathbf{x} = (1, \mathbf{x}^1, \dots, \mathbf{x}^{[p]})$ we set

$$(3) \quad \|\mathbf{x}\|_{p\text{-var};[0,T]} = \left(\sum_{i=1}^{[p]} \sup_{D=(t_j)} \sum_{j:t_j \in D} \left| \mathbf{x}_{t_j, t_{j+1}}^i \right|_{(\mathbb{R}^d)^{\otimes i}}^{p/i} \right)^{1/p}$$

and note that if (3) is finite, then $\omega(s, t) := \|\mathbf{x}\|_{p\text{-var};[s,t]}^p$ is a control (i.e. it is a continuous, non-negative, super-additive function on the simplex

$$\Delta_T = \{(s, t) : 0 \leq s \leq t \leq T\}$$

which vanishes on the diagonal). Throughout the paper C will denote a positive constant changing from line to line.

Consider the flow $U_{t \leftarrow 0}^{\mathbf{X}}(y_0)$ of the RDE (rough differential equation)

$$dU_{t \leftarrow 0}^{\mathbf{X}}(y_0) = V(U_{t \leftarrow 0}^{\mathbf{X}}(y_0)) d\mathbf{X}_t, \quad U_{0 \leftarrow 0}^{\mathbf{X}}(y_0) = y_0.$$

Then, under sufficient regularity on V , the map $U_{t \leftarrow 0}^{\mathbf{X}}(\cdot)$ is a differentiable function (see, for example, [9]) and its derivative (“the Jacobian”):

$$J_{t \leftarrow 0}^{\mathbf{X}}(y_0) \equiv DU_{t \leftarrow 0}^{\mathbf{X}}(\cdot)|_{\cdot=y_0}$$

satisfies

$$dJ_{t \leftarrow 0}^{\mathbf{X}}(y_0) = DV(U_{t \leftarrow 0}^{\mathbf{X}}(y_0)) J_{t \leftarrow 0}^{\mathbf{X}}(y_0) d\mathbf{X}_t, \quad J_{0 \leftarrow 0}^{\mathbf{X}}(y_0) = y_0.$$

In the following we will also write $J_{t \leftarrow 0}(y_0)$ for $J_{t \leftarrow 0}^{\mathbf{X}}(y_0)$ where no ambiguity arises and denote by $J_{0 \leftarrow t}(x)$ the inverse of the Jacobian.

¹See e.g. [11] and note the contrast with classical Lipschitzness. A simple sufficient condition is bounded vector fields with two bounded, Hölder continuous derivatives.

Let $\alpha > 0$. Following [5] we introduce a particular choice of partition of time for $\mathbf{x} \in W\Omega_p(\mathbb{R}^d)$ by letting $(\tau_i(\alpha, p, \mathbf{x}))_{i=0}^\infty = (\tau_i(\alpha))_{i=0}^\infty$ in $[0, T]$ be given by $\tau_0(\alpha) = 0$,

$$(4) \quad \tau_{i+1}(\alpha) = \inf \left\{ t : \|\mathbf{x}\|_{p-\text{var}[\tau_i, t]}^p \geq \alpha, \tau_i(\alpha) < t \leq T \right\} \wedge T,$$

with the convention that $\inf \emptyset = +\infty$. The function $N_{\alpha, [0, T], p} : W\Omega_p(\mathbb{R}^d) \rightarrow \mathbb{R}_+$ given by

$$(5) \quad N_{\alpha, [0, T], p}(\mathbf{x}) := \sup \{ n \in \mathbb{N} \cup \{0\} : \tau_n(\alpha) < T \}$$

describes the size of the non-trivial part of the sequence $(\tau_i(\alpha))_{i=0}^\infty$. We first observe a slight refinement of the integrability estimates for the Jacobian and its inverse obtained in [5]. Let $M_{\mathbf{X}(\cdot)} : \mathcal{W} \rightarrow \mathbb{R}_+$ denote the function

$$M_{\mathbf{X}(\cdot)}(\omega) := \sup_{y_0 \in \mathbb{R}^N} \sup_{t \in [0, T]} \left| J_{t \leftarrow 0}^{\mathbf{X}(\omega)}(y_0) \right|.$$

We first derive a deterministic bound for $M_{\mathbf{X}(\cdot)}$ in terms of $N_{\alpha, [0, T], p}$ that holds for almost every ω .

Proposition 1. *Let $(X_t)_{t \in [0, T]} = (X_t^1, \dots, X_t^d)_{t \in [0, T]}$ be a standard Brownian motion on the abstract Wiener space $(\mathcal{W}, \mathcal{H}, \mu)$ and $p > 2$. Then for a.e. ω $X(\omega)$ lifts to a geometric p -rough path $\mathbf{X}$ and we have*

$$M_{\mathbf{X}(\cdot)}(\omega) \leq C \exp(C\alpha N_{\alpha, [0, T], p}).$$

Proof. There are a number of ways to obtain such a deterministic bound. We follow the approach outlined in [5] Remark 4.12 (see also [8]). From Friz, Victoir [9] Theorem 10.16 we have a uniform Lipschitz estimate of the form

$$(6) \quad |U_{t \leftarrow 0}^{\mathbf{X}}(y_0^1) - U_{t \leftarrow 0}^{\mathbf{X}}(y_0^2)| \leq C |y_0^1 - y_0^2| \exp[C\omega(0, t)],$$

where the constant C depends on $|V|_{\text{Lip}-\gamma}$. Letting $y_0^1 = y_0$ and $y_0^2 = y_0 + \varepsilon h$, dividing by ε , letting ε tend to zero and taking the supremum over $|h| = 1$ we deduce

$$(7) \quad \sup_{s \in [0, t]} \left| J_{s \leftarrow 0}^{\mathbf{X}(\omega)}(y_0) \right| \leq C \exp(C\omega(0, t)).$$

It is important to note that this estimate is independent of the starting point of the underlying diffusion y_0 . The flow satisfies $U_{t \leftarrow s}^{\mathbf{X}}(U_{s \leftarrow 0}^{\mathbf{X}}(y_0)) = U_{t \leftarrow 0}^{\mathbf{X}}(y_0)$ and for the Jacobian the cocycle property

$$J_{t \leftarrow 0}^{\mathbf{X}(\omega)}(y_0) = J_{t \leftarrow s}^{\mathbf{X}(\omega)}(U_{s \leftarrow 0}^{\mathbf{X}}(y_0)) J_{s \leftarrow 0}^{\mathbf{X}(\omega)}(y_0)$$

holds. Thus if we consider the partition $(\tau_i(\alpha))_{i=0}^{N_{\alpha, [0, T], p}+1}$ defined in (4) and we may use the estimate in (7) iteratively and get

$$M_{\mathbf{X}(\cdot)}(\omega) \leq C \exp \left(C \sum_i \omega(\tau_i(\alpha), \tau_{i+1}(\alpha)) \right) \leq C \exp(C\alpha(N_{\alpha, [0, T], p} + 1)).$$

□

Corollary 2. *The random variable $\sup_{y_0 \in \mathbb{R}^N} \sup_{t \in [0, T]} \left| J_{t \leftarrow 0}^{\mathbf{X}(\omega)}(y_0) \right|$ has moments of all orders.*

Proof. It follows from Remark 6.5 and Theorem 6.6 of [5] that $N_{\alpha, [0, T], p}$ has Gaussian tails and the claim immediately follows from Proposition 1. □

Note that it is straightforward to obtain the same results when replacing the Jacobian by its inverse $J_{0 \leftarrow t}^{\mathbf{X}^{(\omega)}}(y_0)$ (see [5]). Armed with these uniform moment estimates we proceed to prove our main result.

Theorem 3. *Let $\gamma > 2$ and P_t be the heat semi-group defined by vector fields $V_i \in \text{Lip} - \gamma(\mathbb{R}^N, \mathbb{R}^N)$, $i = 1, \dots, d$. There exists a constant $C > 0$ such that*

$$(8) \quad \|\nabla P_t f(x)\|_{L^p(dx)} \leq C \|\nabla f(x)\|_{L^p(dx)}$$

for all $f \in C_p^1(\mathbb{R}^N)$, $t \in (0, 1]$ and $p \in [1, \infty]$.

Remark 4. For the proof we will obtain two - essentially dual - estimates for bounds of the form (8). The constants $C(p)$ in these bounds however will depend on p and diverge for p tending to infinity and one respectively. Considering a suitable mixture of our two bounds we can find a finite constant that works uniformly for all $p \in [1, \infty)$. Finally, the case $p = \infty$, which is straightforward, is treated separately to complete the proof.

Proof. On the one hand differentiating under the expectation (see e.g. Friedmann Theorem 5.5 [6]) and exchanging the order of integration we get

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla E(f(Y_t(x)))|^p dx &\leq \int_{\mathbb{R}^N} E |\nabla [f(Y_t(x))]|^p dx \\ &= E \left(\int_{\mathbb{R}^N} |(\nabla f)(Y_t(x)) J_{t \leftarrow 0}(x)|^p dx \right). \end{aligned}$$

Under the Lip- γ assumption on the vector fields V it is a well-known fact that the stochastic flow Y_t is almost surely a C^1 diffeomorphism (see e.g. Proposition 11.11 of [9]). More precisely for almost all ω , $t > 0$ the map $Y_t(\cdot, \omega) : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a C^1 -diffeomorphism and the Jacobian thus invertible. Hence, we may change variables and have

$$\begin{aligned} &E \left(\int_{\mathbb{R}^N} |\nabla f(Y_t(x)) J_{t \leftarrow 0}(x)|^p dx \right) \\ &= E \left(\int_{Y_t(\mathbb{R}^N)} |\nabla f(z) J_{t \leftarrow 0}(Y_t^{-1}(z))|^p |\det(J_{0 \leftarrow t}(Y_t^{-1}(z)))| dz \right) \\ &\leq \|\nabla f(x)\|_{L^p(dx)}^p E \sup_{z \in \mathbb{R}^N} |J_{0 \leftarrow t}(z)|^p |\det(J_{0 \leftarrow t}(z))| \\ (9) \quad &\leq \|\nabla f(x)\|_{L^p(dx)}^p \left\| \left(\sup_{z \in \mathbb{R}^N} |J_{t \leftarrow 0}(z)| \right)^p \right\|_2 \left\| \sup_{z \in \mathbb{R}^N} |\det J_{0 \leftarrow t}(z)| \right\|_2. \end{aligned}$$

Noting that the determinant is a polynomial of degree at most N in the entries of the inverse Jacobian matrix we deduce from Corollary 2 that both

$$\left\| \sup_{z \in \mathbb{R}^N} |J_{t \leftarrow 0}(z)| \right\|_{2p}^p < \infty \quad \text{and} \quad \left\| \sup_{z \in \mathbb{R}^N} |\det J_{0 \leftarrow t}(z)| \right\|_2 < \infty$$

for $p \in [1, \infty)$. Let

$$C_1 := \max_{p \in [1, 2]} \left\| \sup_{z \in \mathbb{R}^N} |J_{t \leftarrow 0}(z)| \right\|_{2p} \left\| \sup_{z \in \mathbb{R}^N} |\det J_{0 \leftarrow t}(z)| \right\|_2^{1/p},$$

providing us with the desired bound if $p \in [1, 2]$. On the other hand, let $p \in (1, \infty)$ and let q denote the Hölder conjugate of p . Once again differentiating under the

expectation we have

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla E(f(Y_t(x)))|^p dx &\leq \int_{\mathbb{R}^N} |E[\nabla f(Y_t(x))] J_{t \leftarrow 0}(x)|^p dx \\ &\leq \int_{\mathbb{R}^N} \left| [E|\nabla f(Y_t(x))|^p]^{1/p} [E|J_{t \leftarrow 0}(x)|^q]^{1/q} \right|^p dx \\ &\leq \sup_{z \in \mathbb{R}^N} [E|J_{t \leftarrow 0}(z)|^q]^{p/q} \int_{\mathbb{R}^N} E|\nabla f(Y_t(x))|^p dx. \end{aligned}$$

Furthermore, by an argument analogous to (9) we estimate the L^p operator norm of P_t by

$$\begin{aligned} E \int_{\mathbb{R}^N} |\nabla f(Y_t(z))|^p dx &\leq E \left(\int_{Y_t(\mathbb{R}^N)} |\nabla f(z)|^p |\det(J_{0 \leftarrow t}(Y_t^{-1}(z)))| dz \right) \\ (10) \qquad \qquad \qquad &\leq \|\nabla f(x)\|_{L^p(dx)}^p E \left| \sup_{z \in \mathbb{R}^N} \det J_{0 \leftarrow t}(z) \right|. \end{aligned}$$

We define the constant C_2 by setting

$$C_2 := \max_{q \in [1, 2]} \sup_{z \in \mathbb{R}^N} [E|J_{t \leftarrow 0}(z)|^q]^{1/q} E \left| \sup_{z \in \mathbb{R}^N} \det J_{0 \leftarrow t}(z) \right|^{1-1/q}.$$

Note that $q \in (1, 2]$ corresponds to $p \in [2, \infty)$. Finally, we have for $p = \infty$

$$\begin{aligned} \sup_{x \in \mathbb{R}^N} |\nabla P_t f(x)| &\leq \sup_{x \in \mathbb{R}^N} E |(\nabla f)(Y_t(x)) J_{t \leftarrow 0}(x)| \\ &\leq \|\nabla f\|_{\infty} \sup_{x \in \mathbb{R}^N} E |J_{t \leftarrow 0}(x)|. \end{aligned}$$

The claim now follows by letting

$$C := \max \left(C_1, C_2, \sup_{x \in \mathbb{R}^N} E |J_{t \leftarrow 0}(x)| \right).$$

□

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