THE CONTINUOUS WAVELET TRANSFORM AND WINDOW FUNCTIONS

J. N. PANDEY AND S. K. UPADHYAY

(Communicated by Ken Ono)

ABSTRACT. We define a window function ψ as an element of $L^2(\mathbb{R}^n)$ satisfying certain boundedness properties with respect to the $L^2(\mathbb{R}^n)$ norm and prove that it satisfies the admissibility condition if and only if the integral of $\psi(x_1, x_2, \dots, x_n)$ with respect to each of the variables $x_1, x_2, \dots, x_n$ along the real line is zero. We also prove that each of the window functions is an element of $L^1(\mathbb{R}^n)$. A function $\psi \in L^2(\mathbb{R}^n)$ satisfying the admissibility condition is a wavelet. We define the wavelet transform of $f \in L^2(\mathbb{R}^n)$ (which is a window function) with respect to the wavelet $\psi \in L^2(\mathbb{R}^n)$ and prove an inversion formula interpreting convergence in $L^2(\mathbb{R}^n)$. It is also proved that at a point of continuity of f the convergence of our wavelet inversion formula is in a pointwise sense.

1. INTRODUCTION

The necessity for the inversion formula for the wavelet transform in dimensions higher than 1 has long been felt. Before the inversion formula for a two-dimensional wavelet transform was known, workers in image processing used to separate two-dimensional wavelets into the product of two one-dimensional wavelets, thereby making the job very simple [12]. But even two-dimensional wavelets are sometimes hard to separate and are sometimes impossible. Consider the wavelet $\psi(x_1, x_2)$ in $\mathbb{R}^2$:

$$\psi(x_1, x_2) = x_1 x_2 e^{-(x_1^2+1)(x_2^2+1)} \in S(\mathbb{R}^2).$$

This cannot be separated into the product of two one-dimensional wavelets so the use of the inversion formula for the wavelets in dimensions higher than 1 was needed by workers in image processing. Consider another wavelet in $\mathbb{R}^2$ given by

$$\psi(x_1, x_2) = \begin{cases} x_1 x_2 e^{-\frac{1}{1-(x_1^2+x_2^2)}}, & |x| < 1, \\ 0, & |x| \geq 1, \end{cases}$$

$$|x| = \sqrt{x_1^2 + x_2^2}.$$

Received by the editors April 17, 2014 and, in revised form, July 2, 2014.

2010 *Mathematics Subject Classification.* Primary 46F12; Secondary 46F05, 46F10.

Key words and phrases. Continuous wavelet transform, Fourier inversion theory, inverse wavelet transform.

This wavelet is a wavelet of compact support in $\mathbb{R}^2$ and cannot be separated. Now consider

$$(1.1) \quad \psi(x) = \begin{cases} 1, & 0 \leq x \leq \frac{1}{2}, \\ -1, & \frac{1}{2} < x \leq 1, \\ 0, & \text{elsewhere,} \end{cases}$$

where $\psi(x)$ is a continuous wavelet. Using this wavelet we can define a two-dimensional wavelet of separable type by

$$(1.2) \quad \psi(x_1, x_2) = \psi(x_1)\psi(x_2).$$

So one can work with the one-dimensional wavelet inversion formula in this case but not in the aforesaid previous two cases, as those are not the wavelets of separable type [12, p. 331].

Many authors worked on the inversion formula for the n -dimensional wavelet transform such as Daubechies [5], Meyer [7], Pathak [11], Keinert [6] and others. But the most notable amongst them are the works of Daubechies [5, pp. 33–34] and Meyers [7, pp. 125–126].

We have proved the following wavelet inversion formula for $f \in L^2(\mathbb{R}^n)$. Our wavelet is a window function whose integral along each of the axes is zero, and our inversion formula is as follows:

$$\frac{1}{C_\psi} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |a|^{-1/2} \psi\left(\frac{x-b}{a}\right) W_f(a, b) \frac{da \, db}{|a|^2} = f,$$

where

$$W_f(a, b) = \frac{1}{\sqrt{|a|}} \langle f(t), \psi\left(\frac{t-b}{a}\right) \rangle$$

and $|a| = |a_1 \cdot a_2 \cdots a_n|$. If we take $x = (x_1, x_2, \dots, x_n)$, $\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ and $|\Lambda| = |\lambda_1 \cdot \lambda_2 \cdots \lambda_n|$, then

$$C_\psi = (2\pi)^n \int_{\Lambda^n} \frac{|\hat{\psi}(\Lambda)|^2}{|\Lambda|} d\Lambda < \infty \quad (\text{admissibility condition}),$$

where

$$\hat{\psi}(\lambda_1, \lambda_2, \dots, \lambda_n) = \hat{\psi}(\Lambda), \quad \text{the Fourier transform of } \psi(x) \in L^2(\mathbb{R}^n).$$

The convergence is interpreted in the $L^2(\mathbb{R}^n)$ sense and at a point of continuity of f the convergence is in the pointwise sense. Note that in our derivation as shown later, we take $a = (a_1, a_2, \dots, a_n)$ whereas Daubechies and Meyer take $a = (a, a, \dots, a)$ and we do not choose ψ as spherically symmetric. In our case $a_1, a_2, \dots, a_n$ are all non-zero real numbers, whereas in their cases $a > 0$.

We have used the Fourier inversion theorem to prove the wavelet inversion theorem. The advantage of this method is that it is simple and to prove the pointwise convergence only the continuity of f is required and the continuity of the wavelet ψ at a point $x = x_0$ concerned is not required. To prove the pointwise convergence using the Hilbert space technique, we require the continuity of the function f and the wavelet ψ both at a point $x = x_0$ concerned [4, p. 63]. It sounds quite strange but that is the way it is!

Our inversion formula is valid over $\mathbb{R}^n \times \mathbb{R}^n$ whereas the formulas derived by Daubechies and Meyer are valid only at $\mathbb{R} \times \mathbb{R}^n$. Thus our formula is more general than their inversion formula.

We will give a characterisation of a subclass of functions belonging to $L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ in the proof of the Corollary to Theorem 3.3, which are wavelets. This makes the construction of wavelets useful to applied scientists much easier.

2. DEFINITIONS AND PRELIMINARIES

Definition 2.1. A function $f \in L^2(\mathbb{R}^n)$ is called a window function if it satisfies the following conditions:

- (1) $x_1 f, x_2 f, \dots, x_n f$ all belong to $L^2(\mathbb{R}^n)$.
- (2) $x_i x_j f \in L^2(\mathbb{R}^n)$ for all $i, j = 1, 2, \dots, n$, $i \neq j$.
- (3) $x_i x_j x_k f \in L^2(\mathbb{R}^n)$ for all $i, j, k = 1, 2, \dots, n$, $i \neq j \neq k \neq i$. Note that $i \neq j \neq k \neq i$ implies that i, j, k are all different, $i \neq j \neq k$ may imply that i and k could be equal. Finally we have
- (n) $x_1 x_2 \dots x_n f \in L^2(\mathbb{R}^n)$.

Here, the lower suffixes in a term are all different. Let us illustrate this definition with reference to $n = 2$. So if $f \in L^2(\mathbb{R}^2)$ we must have

$$\begin{aligned} x_1 f, x_2 f &\in L^2(\mathbb{R}^2), \\ x_1 x_2 f &\in L^2(\mathbb{R}^2). \end{aligned}$$

Example 2.2. Define

$$\psi(x) = \begin{cases} 1, & 0 \leq x \leq \frac{1}{2}, \\ -1, & -\frac{1}{2} < x \leq 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Then clearly ψ is a window function in $L^2(\mathbb{R})$, and is a continuous wavelet.

Example 2.3. $\psi(x) = (x_1, x_2, \dots, x_n) e^{-(x_1^2 + x_2^2 + \dots + x_n^2)}$ is a window function in $L^2(\mathbb{R}^n)$, and is a wavelet.

The wavelet ψ that we have chosen satisfies the following conditions:

- (1) $\psi \in L^2(\mathbb{R}^n)$.
- (2) ψ is a window function as defined above.
- (3) $\int_{-\infty}^{\infty} \psi(x) dx_i = 0$, $\forall i = 1, 2, \dots, n$.

We will show that under the set of conditions (1), (2) and (3) the wavelet kernel ψ satisfies the admissibility condition

$$\int_{\Lambda^n} \frac{|\hat{\psi}(\Lambda)|^2}{|\Lambda|} d\Lambda < \infty,$$

which enables us to prove the aforesaid wavelet inversion formula.

3. ORTHANTS AND PSEUDO-ORTHANTS

Let $\mathbb{R}^n$ stand for the n -dimensional Euclidean space. Then $\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \dots \times \mathbb{R}$, n times (Cartesian product):

$$(3.1) \quad \mathbb{R}^n = \prod_{i=1}^n [\mathbb{R}_{|x_i| \geq a} \cup \mathbb{R}_{|x_i| \leq a}], \quad a > 0,$$

$$\mathbb{R}_{|x_i| \geq a} = [x \in \mathbb{R}^n : |x_i| \geq a, x = (x_1, x_2, \dots, x_n)],$$

where $i = 1, 2, \dots, n$.

Let $\Delta_k = [x : |x_{j_1}| \geq a, |x_{j_2}| \geq a, \dots, |x_{j_k}| \geq a; |x_{j_{k+1}}| \leq a, |x_{j_{k+2}}| \leq a, \dots, |x_{j_n}| \leq a]$, where $j_1, j_2, \dots, j_n$ are permutations of $1, 2, 3, \dots, n$; Δ_k is known as a pseudo-orthant and when $a = 0$, Δ_k is an orthant in $\mathbb{R}^n$.

Now $\mathbb{R}^n = \bigcup_{k=0}^n [\bigcup \Delta_k]$. Then the inner union has $\binom{n}{k}$ terms for a fixed $k \in (0, 1, 2, \dots, n)$. Varying k from $0, 1, 2, \dots, n$, we see that there are $\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n} = 2^n$ pseudo-orthants. The total number of pseudo-orthants in $\mathbb{R}^n$ can be easily figured out from the representation (3.1). There are n factors in the representation (3.1) of $\mathbb{R}^n$ and there are two elements in each of the factors so there will be 2^n pseudo-orthants in $\mathbb{R}^n$.

From now on we will choose $a = 1$ as there is no loss of generality in doing so.

Theorem 3.1. *Let $f \in L^2(\mathbb{R}^n)$ be a window function on $\mathbb{R}^n$. Then $f \in L^1(\mathbb{R}^n)$.*

Proof. By using Holder's inequality one can see that

$$\begin{aligned}
 & \int_{\Delta_k} |f(x)| dx \\
 &= \int_{\Delta_k} |x_{j_1} x_{j_2} \cdots x_{j_k} f(x)| \frac{1}{|x_{j_1} x_{j_2} \cdots x_{j_k}|} dx \\
 &\leq \left(\int_{\Delta_k} |x_{j_1} x_{j_2} \cdots x_{j_k} f(x)|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Delta_k} \frac{1}{|x_{j_1} x_{j_2} \cdots x_{j_k}|^2} dx \right)^{\frac{1}{2}} \\
 &\leq \left(\int_{\mathbb{R}^n} |x_{j_1} x_{j_2} \cdots x_{j_k} f(x)|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Delta_k} \frac{1}{|x_{j_1} x_{j_2} \cdots x_{j_k}|^2} dx \right)^{\frac{1}{2}} \\
 &\leq \|x_{j_1} x_{j_2} \cdots x_{j_k} f(x)\|_2 \left(\int_{\Delta_k} \frac{1}{x_{j_1}^2 x_{j_2}^2 \cdots x_{j_k}^2} dx \right)^{\frac{1}{2}} \\
 &\leq \|x_{j_1} x_{j_2} \cdots x_{j_k} f\|_2 \left(\left(\int_{-\infty}^{-1} + \int_1^{+\infty} \right) \frac{1}{x_{j_1}^2} dx_{j_1} \right)^{\frac{1}{2}} \\
 &\times \left(\int_{-\infty}^{-1} + \int_1^{+\infty} \right) \frac{1}{x_{j_2}^2} dx_{j_2}^{\frac{1}{2}} \cdots \left(\int_{-\infty}^{-1} + \int_1^{+\infty} \right) \frac{1}{x_{j_k}^2} dx_{j_k}^{\frac{1}{2}} \\
 &\times \left(\int_{-1}^1 dx_{j_{k+1}} \int_{-1}^1 dx_{j_{k+2}} \cdots \int_{-1}^1 dx_{j_{k+n-k}} \right)^{\frac{1}{2}} \\
 &\leq \|x_{j_1} x_{j_2} \cdots x_{j_k} f\|_2 2^{\frac{k}{2}} \cdot 2^{\frac{n-k}{2}} \\
 &\leq 2^{\frac{n}{2}} \|x_{j_1} x_{j_2} \cdots x_{j_k} f\|_2,
 \end{aligned}$$

and

$$\int_{\bigcup \Delta_k} |f(x)| dx \leq 2^{\frac{n}{2}} \sum \|x_{j_1} x_{j_2} \cdots x_{j_i} \cdots x_{j_k} f\|_2,$$

where the number of terms in the R.H.S. summation is $\binom{n}{k}$.

Therefore,

$$\begin{aligned}
 \int_{\mathbb{R}^n} |f(x)| dx &= \sum_{k=0}^n \int_{\bigcup \Delta_k} |f(x)| dx \\
 (3.2) \quad &\leq 2^{\frac{n}{2}} [\|f\|_2 + \sum_{i=1}^n \|x_i f\|_2 + \sum_{i,j=1, i \neq j}^n \|x_i x_j f\|_2 + \cdots + \|x_1 x_2 \cdots x_n f\|_2].
 \end{aligned}$$

Since f is a window function, each of the terms in (3.2) is bounded. Therefore $\int_{\mathbb{R}^n} |f(x)| dx$ is bounded; i.e. $f \in L^1(\mathbb{R}^n)$. For the details of the proof one can see [10]. $\square$

Corollary. *Let $f \in L^2(\mathbb{R}^n)$ be a window function. Then $\hat{f}(\lambda_1, \lambda_2, \dots, \lambda_n)$ is a continuous function of $\lambda_1, \lambda_2, \dots, \lambda_n$, where $\hat{f}$ is the Fourier transform of f .*

Note that Theorem 3.1 cannot be proved by breaking $\mathbb{R}^n$ into the unit sphere with the centre at the origin and a complement of it with respect to $\mathbb{R}^n$, hence there lies the importance of our window function technique.

Theorem 3.2. *Let $f : \mathbb{R}^n \rightarrow C$ be a $L^2(\mathbb{R}^n)$ window function. Let $\hat{f}(\lambda_1, \lambda_2, \dots, \lambda_n)$ be the Fourier transform of f defined by*

$$\begin{aligned} \hat{f}(\lambda_1, \lambda_2, \dots, \lambda_n) \\ = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x_1, x_2, \dots, x_n) e^{-i(x_1\lambda_1 + x_2\lambda_2 + \dots + x_n\lambda_n)} dx_1 dx_2 \dots dx_n. \end{aligned}$$

Then the following two statements are equivalent.

- (a) $\hat{f}(\lambda_1, \lambda_2, \dots, \lambda_n) \Big|_{\lambda_j=0} = 0.$
- (b) $\int_{-\infty}^{\infty} f(x_1, x_2, \dots, x_j, \dots, x_n) dx_j = 0, \quad j = 1, 2, 3, \dots$

Proof. The proof is very simple and so is omitted. $\square$

Plancherel formula ([3, p. 107] and [1, p. 75]). Suppose that f and $g \in L^2$. Then $\langle Ff, g \rangle = \langle f, Fg \rangle$, $\langle Ff, Fg \rangle_{L^2} = \langle f, g \rangle_{L^2}$ and $\langle F^{-1}f, F^{-1}g \rangle_{L^2} = \langle f, g \rangle_{L^2}$. In particular

$$\|Ff\|_{L^2} = \|f\|_{L^2}.$$

Here, $\langle u, v \rangle_{L^2} = \int_{-\infty}^{\infty} u(t) \overline{v(t)} dt$. F, F^{-1} are defined below.

Fourier inversion theorem. *Let $x = (x_1, x_2, \dots, x_n)$ and $\omega = (\omega_1, \omega_2, \dots, \omega_n)$ be elements of $\mathbb{R}^n$ and $f \in L^2(\mathbb{R}^n)$. Then*

$$F^{-1}(\hat{f})(x) = \lim_{N \rightarrow \infty} \frac{1}{(2\pi)^{n/2}} \int_{-N}^N \hat{f}(\omega) e^{i\omega \cdot x} d\omega = f(x).$$

This defines convergence in $L^2(\mathbb{R}^n)$ and is called the limit in the mean (l.i.m.) [1, p. 75]. Here $\hat{f}(\omega)$ is the Fourier transform of f defined by

$$\hat{f}(\omega) = \lim_{N \rightarrow \infty} \frac{1}{(2\pi)^{n/2}} \int_{-N}^N f(x) e^{-i\omega \cdot x} dx,$$

where $\omega \cdot x = \omega_1 x_1 + \omega_2 x_2 + \dots + \omega_n x_n$, $N = (N_1, N_2, \dots, N_n)$ and $N \rightarrow \infty$ implies that each of the components of N tend to ∞ independently of each other. This is a well-known result, which was proved in several books such as [1, 9]. At the point of continuity of f the convergence in the above inversion formula is proved in the pointwise sense. The proof is given using the Plancherel formula

$$\langle Ff, Fg \rangle = \langle f, g \rangle, \quad \forall f, g \in L^2(\mathbb{R}^n),$$

and replacing g by the elements of the sequence $\{g_m(x)\}_{m=1}^\infty$ and letting $m \rightarrow \infty$; elements g_m are defined as follows:

$$g_m(x) = \begin{cases} m^n & \text{if } \|x_m - x_0\|_1 \leq \frac{1}{2m}, \\ 0 & \text{if } \|x_m - x_0\|_1 > \frac{1}{2m}, \end{cases}$$

$$x_0 = (x_0^1, x_0^2, \dots, x_0^n),$$

$$\|x\|_1 = \max(|x_1|, |x_2|, \dots, |x_n|),$$

$$x_m = \left(x_0^1 + \frac{1}{2m}, x_0^2 + \frac{1}{2m}, \dots, x_0^n + \frac{1}{2m}\right).$$

The norm $\|\cdot\|_1$ and $|x| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2} = \|x\|$ generate the same topology over $\mathbb{R}^n$ as

$$\|x\|_1 \leq \|x\| \leq n\|x\|_1.$$

The Fourier inversion formula $\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} F(\omega) e^{i\omega x_0} d\omega = f(x_0)$ with pointwise convergence at $x = x_0$ follows letting $m \rightarrow \infty$ in $\langle Ff, Fg_m \rangle = \langle f, g_m \rangle$. We assume that f is continuous at $x_0 \in \mathbb{R}^n$. Details of the proof are omitted and the proof is left as a simple and interesting exercise for the readers.

Admissibility condition. Let $f \in L^2(\mathbb{R}^n)$. We say that f satisfies the admissibility condition if its Fourier transform $\hat{f}(\lambda_1, \lambda_2, \dots, \lambda_n)$ satisfies the condition

$$(3.3) \quad \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|\hat{f}(\lambda_1, \lambda_2, \dots, \lambda_n)|^2}{|\lambda_1 \lambda_2 \dots \lambda_n|} d\lambda_1 d\lambda_2 \dots d\lambda_n < \infty,$$

and then the function f is said to be a basic wavelet.

We know that the Fourier transform $\hat{f}(\lambda_1, \lambda_2, \dots, \lambda_n)$ of a window function f is a continuous function of n variables $\lambda_1, \lambda_2, \dots, \lambda_n$. A window function belonging to $L^2(\mathbb{R}^n)$ is a regular (basic) wavelet function if it satisfies the admissibility condition

$$(3.4) \quad \int_{\Lambda^n} \frac{|\hat{f}(\lambda_1, \lambda_2, \dots, \lambda_n)|^2}{|\lambda_1 \lambda_2 \dots \lambda_n|} d\lambda_1 d\lambda_2 \dots d\lambda_n < \infty.$$

So we see that $\hat{f}$ should be zero at each of the points $(0, \lambda_1, \lambda_2, \dots, \lambda_n)$, $(\lambda_1, 0, \lambda_2, \dots, \lambda_n)$, $\dots$, $(\lambda_1, \lambda_2, \dots, \lambda_{n-1}, 0)$, i.e.

$$(3.5) \quad \hat{f}(0, \lambda_2, \lambda_3, \dots, \lambda_n), \hat{f}(\lambda_1, 0, \lambda_3, \lambda_4, \dots, \lambda_n) \dots$$

and

$$\hat{f}(\lambda_1, \lambda_2, \dots, \lambda_{n-1}, 0) \text{ are all zero.}$$

By Theorem 3.2

$$(3.6) \quad \begin{aligned} \hat{f}(0, \lambda_2, \lambda_3, \dots, \lambda_n) &= 0 \Leftrightarrow \int_{-\infty}^{\infty} f(x_1, x_2, \dots, x_n) dx_1 = 0, \\ \hat{f}(\lambda_2, 0, \lambda_3, \dots, \lambda_n) &= 0 \Leftrightarrow \int_{-\infty}^{\infty} f(x_1, x_2, \dots, x_n) dx_2 = 0, \\ \hat{f}(\lambda_2, \lambda_3, \dots, \lambda_n, 0) &= 0 \Leftrightarrow \int_{-\infty}^{\infty} f(x_1, x_2, \dots, x_n) dx_n = 0. \end{aligned}$$

If the window function f is a regular (basic) wavelet, then it satisfies $(3.4) \Rightarrow (3.5) \Rightarrow (3.6)$. So we propose the following theorem.

Theorem 3.3. Let $f \in L^2(\mathbb{R}^n)$ be a window function. Assume also that

$$\int_{-\infty}^{\infty} f(x_1, x_2, \dots, x_i, x_n) dx_i = 0 \quad \forall i = 1, 2, \dots, n.$$

Then f satisfies the admissibility condition

$$(3.7) \quad \int_{\Lambda^n} \frac{|\hat{f}(\lambda_1, \lambda_2, \dots, \lambda_n)|^2}{|\lambda_1 \lambda_2 \dots \lambda_n|} d\lambda_1 d\lambda_2 \dots d\lambda_n < \infty.$$

More precisely we have

$$(3.8) \quad \begin{aligned} & \int_{\Lambda^n} \frac{|\hat{f}(\lambda_1, \lambda_2, \dots, \lambda_n)|^2}{|\lambda_1 \lambda_2 \dots \lambda_n|} d\lambda_1 d\lambda_2 \dots d\lambda_n \\ & \leq \left[\|f\|_2^2 + 2^1 \sum_{i=1}^n \|x_i f\|_2^2 + 2^2 \sum_{i,j=1, i \neq j}^n \|x_i x_j f\|_2^2 \right. \\ & \quad \left. + 2^3 \sum_{\substack{i,j,k=1 \\ i \neq j \neq k \neq i}}^{\infty} \|x_i x_j x_k f\|_2^2 + \dots + 2^n \|x_1 x_2 \dots x_n f\|_2^2 \right]. \end{aligned}$$

Note that all the terms in the R.H.S. of (3.8) are bounded by virtue of the fact that f is a window function.

Let us recall some notation as follows:

$$\Lambda = (\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n) \quad \text{and} \quad |\Lambda| = |\lambda_1 \lambda_2 \dots \lambda_n|.$$

$$\hat{f}(\lambda_1, \lambda_2, \dots, \lambda_n) = \hat{f}(\Lambda), \quad \text{the Fourier transform of } f(x) \in L^2(\mathbb{R}^n).$$

Here $x = (x_1, x_2, \dots, x_n)$ and Λ^n will stand for the n -dimensional Euclidean space where $(\lambda_1, \lambda_2, \dots, \lambda_n)$ represents the coordinates of a point of it in general.

We also introduce the symbols $\Lambda_0, \Lambda_1, \Lambda_2, \dots, \Lambda_k, \dots, \Lambda_n$ as follows:

$$\begin{aligned} \Lambda_k &= [(\lambda_1, \lambda_2, \dots, \lambda_n) : |\lambda_{j_i}| \leq 1, \quad i = 1, 2, \dots, k; \quad |\lambda_{j_i}| \geq 1, \quad i = k+1, k+2, \dots, n], \\ \Lambda_0 &= [(\lambda_1, \lambda_2, \dots, \lambda_n) : |\lambda_{j_i}| \geq 1, \quad i = 1, 2, 3, \dots, k, \dots, n], \\ \Lambda_n &= [(\lambda_1, \lambda_2, \dots, \lambda_n) : |\lambda_{j_i}| \leq 1, \quad i = 1, 2, 3, \dots, n]. \end{aligned}$$

Taking all variations of suffixes of λ_{j_i} we can see that Λ_k can represent $\binom{n}{k}$ pseudo-orthants. The regions (pseudo-orthants) $\Lambda_0, \Lambda_1, \Lambda_2, \dots, \Lambda_k, \dots, \Lambda_n$ are all contained in Λ^n .

It is easy to show that

$$\Lambda^n = \Lambda_0 \cup \left(\bigcup \Lambda_1 \right) \cup \left(\bigcup \Lambda_2 \right) \cup \dots \cup \left(\bigcup \Lambda_k \right) \cup \dots \cup \Lambda_n.$$

The symbol $d\Lambda$ stands for $d\lambda_1 d\lambda_2 \dots d\lambda_k \dots d\lambda_n$. Now

$$(3.9) \quad \begin{aligned} \int_{\Lambda^n} \frac{|\hat{f}(\Lambda)|^2 d\Lambda}{|\Lambda|} &= \int_{\Lambda_0} \frac{|\hat{f}(\Lambda)|^2}{|\Lambda|} d\Lambda + \int_{\bigcup \Lambda_1} \frac{|\hat{f}(\Lambda)|^2 d\Lambda}{|\Lambda|} \\ &\quad + \int_{\bigcup \Lambda_2} \frac{|\hat{f}(\Lambda)|^2}{|\Lambda|} d\Lambda + \dots + \int_{\Lambda_n} \frac{|\hat{f}(\Lambda)|^2}{|\Lambda|} d\Lambda \\ &= \int_{\substack{\Lambda_0 \\ \binom{n}{0} \text{ term}}} \frac{|\hat{f}(\Lambda)|^2}{|\Lambda|} d\Lambda + \sum \int_{\substack{\Lambda_1 \\ \binom{n}{1} \text{ terms}}} \frac{|\hat{f}(\Lambda)|^2}{|\Lambda|} d\Lambda \end{aligned}$$

$$\begin{aligned}
& + \sum \int_{\substack{\Lambda_2 \\ \binom{n}{2} \text{ terms}}} \frac{|\hat{f}(\Lambda)|^2}{|\Lambda|} d\Lambda + \cdots + \sum \int_{\substack{\Lambda_k \\ \binom{n}{k} \text{ terms}}} \frac{|\hat{f}(\Lambda)|^2}{|\Lambda|} d\Lambda \\
& + \cdots + \int_{\substack{\Lambda_n \\ \binom{n}{n}=1 \text{ term}}} \frac{|\hat{f}(\Lambda)|^2}{|\Lambda|} d\Lambda, \\
\int_{\Lambda_0} \frac{|\hat{f}(\Lambda)|^2}{|\Lambda|} d\Lambda & = \int_{\substack{|\lambda_i| \geq 1 \\ i=1,2,\dots,n}} \frac{|\hat{f}(\Lambda)|^2}{|\Lambda|} d\Lambda \leq \int_{\Lambda_n} |\hat{f}(\Lambda)|^2 d\Lambda \\
& = \int_{\mathbb{R}^n} |f(x)|^2 dx \\
& = \|f\|_2^2 \quad \quad \quad [\text{Plancherel's Theorem}].
\end{aligned}$$

So

$$(3.10) \quad \int_{\Lambda_0} \frac{|\hat{f}(\Lambda)|^2}{|\Lambda|} d\Lambda \leq \|f\|_2^2.$$

Next we can find a bound for $\int_{\Lambda_n} \frac{|\hat{f}(\Lambda)|^2}{|\Lambda|} d\Lambda$ as follows. The function $\hat{f}(\lambda_1, \lambda_2, \dots, \lambda_{i-1}, \lambda_i, \lambda_{i+1}, \dots, \lambda_n)$ vanishes at points $\lambda_i = 0$, $i = 1, 2, 3, \dots, n$. Therefore all the first partials of this function with respect to $\lambda_1, \lambda_2, \dots, \lambda_{i-1}, \lambda_{i+1}, \dots, \lambda_n$ vanish. Therefore,

$$\hat{f}(\Lambda) = \int_0^{\lambda_n} \cdots \int_0^{\lambda_2} \int_0^{\lambda_1} \frac{\partial^n f(\Lambda)}{\partial \lambda_1 \partial \lambda_2 \cdots \partial \lambda_n} d\Lambda.$$

Now by Holder's inequality we have

$$|\hat{f}(\Lambda)|^2 \leq \int_0^{\lambda_n} \cdots \int_0^{\lambda_2} \int_0^{\lambda_1} \left| \frac{\partial^n \hat{f}(\Lambda)}{\partial \lambda_1 \partial \lambda_2 \cdots \partial \lambda_n} \right|^2 d\Lambda \left| \int_0^{\lambda_n} \cdots \int_0^{\lambda_2} \int_0^{\lambda_1} 1^2 d\Lambda \right|,$$

$$\begin{aligned}
\frac{|\hat{f}(\Lambda)|^2}{|\Lambda|} & \leq \int_0^{\lambda_n} \cdots \int_0^{\lambda_2} \int_0^{\lambda_1} \left| \frac{\partial^n \hat{f}(\Lambda)}{\partial \lambda_1 \partial \lambda_2 \cdots \partial \lambda_n} \right|^2 d\Lambda \\
& \leq \int_{\mathbb{R}^n} \left| \frac{\partial^n \hat{f}(\Lambda)}{\partial \lambda_1 \partial \lambda_2 \cdots \partial \lambda_n} \right|^2 d\Lambda \\
& \leq \|x_1 x_2 \cdots x_n f(x)\|_2^2 \quad \quad \quad [\text{Plancherel's Theorem}],
\end{aligned}$$

(3.11)

$$\int_{\Lambda_n} \frac{|\hat{f}(\Lambda)|^2}{|\Lambda|} d\Lambda \leq \|x_1 x_2 \cdots x_n f(x)\|_2^2 \times \int_{\Lambda_n} d\Lambda = \|x f(x)\|_2^2 2^n.$$

We now proceed to find a bound for

$$\begin{aligned}
& \int_{\Lambda_k} \frac{|\hat{f}(\Lambda)|^2}{|\Lambda|} d\Lambda, \quad 1 < k < n, \\
\hat{f}(\Lambda) & = \int_0^{\lambda_{j_k}} \int_0^{\lambda_{j_{k-1}}} \cdots \int_0^{\lambda_{j_1}} \frac{\partial^k \hat{f}(\Lambda)}{\partial \lambda_{j_1} \partial \lambda_{j_2} \cdots \partial \lambda_{j_k}} d\lambda_{j_1} d\lambda_{j_2} \cdots d\lambda_{j_k}.
\end{aligned}$$

Using Holder's inequality we get

$$\begin{aligned} |\hat{f}(\Lambda)|^2 &\leq \int_0^{\lambda_{j_k}} \cdots \int_0^{\lambda_{j_2}} \int_0^{\lambda_{j_1}} \left| \frac{\partial^k \hat{f}(\Lambda)}{\partial \lambda_{j_1} \partial \lambda_{j_2} \cdots \partial \lambda_{j_k}} \right|^2 d\lambda_{j_1} d\lambda_{j_2} \cdots d\lambda_{j_k} \\ &\quad \times \left| \int_0^{\lambda_{j_k}} \cdots \int_0^{\lambda_{j_2}} \int_0^{\lambda_{j_1}} 1^2 d\lambda_{j_1} d\lambda_{j_2} \cdots d\lambda_{j_k} \right|. \end{aligned}$$

So

$$\begin{aligned} &\frac{|\hat{f}(\Lambda)|^2}{|\lambda_{j_1} \lambda_{j_2} \cdots \lambda_{j_k}|} \\ &\leq \int_0^{\lambda_{j_k}} \int_0^{\lambda_{j_{k-1}}} \cdots \int_0^{\lambda_{j_2}} \int_0^{\lambda_{j_1}} \left| \frac{\partial^k \hat{f}(\Lambda)}{\partial \lambda_{j_1} \partial \lambda_{j_2} \cdots \partial \lambda_{j_k}} \right|^2 d\lambda_{j_1} d\lambda_{j_2} \cdots d\lambda_{j_k}, \\ &\frac{|\hat{f}(\Lambda)|^2}{|\lambda_{j_1} \lambda_{j_2} \cdots \lambda_k \lambda_{j_{k+1}} \cdots \lambda_{j_n}|} \\ &\leq \int_0^{\lambda_{j_k}} \cdots \int_0^{\lambda_{j_2}} \int_0^{\lambda_{j_1}} \left| \frac{\partial^k \hat{f}(\Lambda)}{\partial \lambda_{j_1} \partial \lambda_{j_2} \cdots \partial \lambda_{j_k}} \right|^2 d\lambda_{j_1} d\lambda_{j_2} \cdots d\lambda_{j_k} \end{aligned}$$

as $|\lambda_{j_i}| \geq 1$ for $i = k+1, k+2, \dots, n$,

$$\begin{aligned} &\int \cdots \int_{\substack{|\lambda_{j_i}| \geq 1 \\ i=k+1, k+2, \dots, n}} \frac{|\hat{f}(\Lambda)|^2}{|\Lambda|} d\lambda_{j_{k+1}} d\lambda_{j_{k+2}} \cdots d\lambda_{j_n} \\ &\leq \int_0^{\lambda_{j_k}} \int_0^{\lambda_{j_{k-1}}} \cdots \int_0^{\lambda_{j_2}} \int_0^{\lambda_{j_1}} \int \cdots \int_{\substack{|\lambda_{j_i}| \geq 1 \\ i=k+1, k+2, \dots, n}} \left| \frac{\partial^k \hat{f}(\Lambda)}{\partial \lambda_{j_1} \partial \lambda_{j_2} \cdots \partial \lambda_{j_k}} \right|^2 d\Lambda \\ &\leq \left\| \frac{\partial^k \hat{f}(\Lambda)}{\partial \lambda_{j_1} \partial \lambda_{j_2} \cdots \partial \lambda_{j_k}} \right\|_2^2 \\ (3.12) \quad &\leq \left\| x_{j_1} x_{j_2} \cdots x_{j_k} f(x) \right\|_2^2, \end{aligned}$$

$$\begin{aligned} &\int \cdots \int_{\substack{|\lambda_{j_i}| \leq 1 \\ i=1, 2, 3, \dots, k}} \int \cdots \int_{\substack{|\lambda_{j_i}| \geq 1 \\ i=k+1, k+2, \dots, n}} \frac{|\hat{f}(\Lambda)|^2}{|\Lambda|} d\lambda_{j_1} d\lambda_{j_2} \cdots d\lambda_{j_k} d\lambda_{j_{k+1}} d\lambda_{j_{k+2}} \cdots d\lambda_{j_n} \\ &\leq \left\| x_{j_1} x_{j_2} \cdots x_{j_k} f(x) \right\|_2^2 \int \cdots \int_{\substack{|\lambda_i| \leq 1 \\ i=1, 2, \dots, k}} d\lambda_{j_1} d\lambda_{j_2} \cdots d\lambda_{j_k}, \end{aligned}$$

$$\int_{\Lambda_k} \frac{|\hat{f}(\Lambda)|^2}{|\Lambda|} d\Lambda \leq 2^k \left\| x_{j_1} x_{j_2} \cdots x_{j_k} f(x) \right\|_2^2.$$

Now using (3.9) and the other foregoing results, we get

$$(3.13) \quad \int_{\Lambda^n} \frac{|\hat{f}(\Lambda)|^2}{|\Lambda|} d\Lambda \leq \|f(x)\|_2^2 + 2 \sum_{i=1}^n \|x_i f(x)\|_2^2 + 2^2 \sum_{\substack{i,j=1 \\ i \neq j}}^n \|x_i x_j f(x)\|_2^2 \\ + \cdots + 2^n \|x_1 x_2 \cdots x_n f(x)\|_2^2.$$

Corollary. *Let $f \in L^2(\mathbb{R}^n)$ be a window function. Then f satisfies the admissibility condition if and only if $\int_{-\infty}^{\infty} f(x_1, x_2, \dots, x_i, \dots, x_n) dx_i = 0$, $i = 1, 2, \dots, n$.*

Proof. Sufficiency follows from Theorem 3.3. The necessity follows by virtue of the fact that if f is a window function, then its Fourier transform is a continuous function of $\lambda_1, \lambda_2, \dots, \lambda_n$. Therefore $\hat{f}$ must vanish at any point in λ -coordinates whose one component is zero to maintain the admissibility condition (3.7).

$$\int_{-\infty}^{\infty} f(x_1, x_2, \dots, x_i, \dots, x_n) dx_i = 0 \\ \Leftrightarrow \hat{f}(\lambda_1, \lambda_2, \dots, \lambda_{i-1}, 0, \lambda_{i+1}, \dots, \lambda_n) = 0, \lambda_i = 0 \quad \forall i = 1, 2, \dots, n.$$

□

Examples of window functions which are wavelets. $te^{-|t|}$, $\frac{t}{1+|t|^3}$, $\frac{t}{1+t^4}$ in

$$L^2(\mathbb{R}), \prod_{i=1}^n t_i e^{-t_i^2}, \prod_{i=1}^n \frac{t_i}{1+t_i^4}, \text{ in } L^2(\mathbb{R}^n).$$

4. WAVELET INVERSION FORMULA FOR FUNCTIONS IN $L^2(\mathbb{R}^n)$

Let $\psi \in L^2(\mathbb{R}^n)$ be a window function satisfying the additional conditions $\int_{-\infty}^{\infty} \psi(x_1, x_2, x_3, \dots, x_i, \dots, x_n) dx_i = 0$, $i = 1, 2, 3, \dots, n$. Then ψ is called a regular or basic wavelet function since it then also satisfies the admissibility condition

$$(4.1) \quad \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n)|^2}{|\lambda_1 \lambda_2 \cdots \lambda_n|} d\lambda_1 d\lambda_2 \cdots d\lambda_n < \infty.$$

Let $\hat{\psi}$ be the Fourier transform of $\psi \in L^2(\mathbb{R}^n)$ as described above and let $C_\psi = (2\pi)^n \int_{\mathbb{R}^n} \frac{|\hat{\psi}(\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n)|^2}{|\lambda_1 \lambda_2 \cdots \lambda_n|} d\lambda_1 d\lambda_2 \cdots d\lambda_n$. The integral in one dimension of a function f on $\mathbb{R}$ can be written as

$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\mathbf{N}}^{\mathbf{M}} f(x) dx + \int_{\mathbf{M}}^{\infty} f(x) dx + \int_{-\infty}^{-\mathbf{N}} f(x) dx$$

provided the integral of $f(x)$ on the real line is convergent. So

$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\mathbf{N}}^{\mathbf{M}} f(x) dx + R(M, -N),$$

where $R(M, -N)$ is the remainder term. If $f(x, y)$ is integrable over the XY -plane, then

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = \int_{-N_2}^{M_2} \int_{-N_1}^{M_1} f(x, y) dx dy + R(M_1, M_2, -N_1, -N_2).$$

The remainder term $R(M_1, M_2, -N_1, -N_2)$ can be represented by $R(\mathbf{M}, -\mathbf{N})$ for simplicity and it can be worked out that $R(\mathbf{M}, -\mathbf{N}) =$ the sum of eight integrals and if $f(x, y) \in L(\mathbb{R}^2)$, then $R(\mathbf{M}, -\mathbf{N}) \rightarrow 0$ as $\mathbf{M}, \mathbf{N} \rightarrow \infty$ independently of each other. If $f \in L(\mathbb{R}^n)$ the integral

$$\int_{\mathbb{R}^n} f(x) dx = \int_{-N_n}^{M_n} \cdots \int_{-N_2}^{M_2} \int_{-N_1}^{M_1} f(x) dx + R(\mathbf{M}, -\mathbf{N}).$$

The remainder term $R(\mathbf{M}, -\mathbf{N})$ consists of finitely many, i.e. $(3^n - 1)$, integrals and the remainder $R(\mathbf{M}, -\mathbf{N}) \rightarrow 0$ as $M_1, M_2, \dots, M_n, N_1, N_2, \dots, N_n$ all go to infinity independently of each other. We will make use of this fact in proving the following.

Theorem 4.1. *Let $f \in L^2(\mathbb{R}^n)$ and define the wavelet transform of f by*

$$(4.2) \quad W_f(b_1, b_2, \dots, b_n, a_1, a_2, \dots, a_n) \\ = \left\langle f(x_1, x_2, \dots, x_n), \frac{1}{\sqrt{|a_1 a_2 \dots a_n|}} \psi \left(\frac{x_1 - b_1}{a_1}, \frac{x_2 - b_2}{a_2}, \dots, \frac{x_n - b_n}{a_n} \right) \right\rangle,$$

where ψ is a basic wavelet as defined above in Section 2, i.e. ψ is a window function satisfying

$$\int_{-\infty}^{\infty} \psi(x) dx_i = 0 \quad \forall i = 1, 2, \dots, n.$$

Let $F_{b_1, b_2, \dots, b_n}$ be the Fourier transform operator with respect to variables $b_1, b_2, \dots, b_n$. Then

$$F_{b_1, b_2, \dots, b_n} W_f(b_1, b_2, \dots, b_n, a_1, a_2, \dots, a_n) \\ = \left\langle f(x_1, x_2, \dots, x_n), \bar{F}_{b_1, b_2, \dots, b_n} \frac{\psi \left(\frac{x_1 - b_1}{a_1}, \frac{x_2 - b_2}{a_2}, \dots, \frac{x_n - b_n}{a_n} \right)}{\sqrt{|a_1 a_2 \dots a_n|}} \right\rangle.$$

Proof.

$$F_{x_1, x_2, \dots, x_n} \psi \left(\frac{x_1 - b_1}{a_1}, \frac{x_2 - b_2}{a_2}, \dots, \frac{x_n - b_n}{a_n} \right) (\lambda_1, \lambda_2, \dots, \lambda_n) \\ = |a_1 a_2 \dots a_n| \hat{\psi}(a_1 \lambda_1, a_2 \lambda_2, \dots, a_n \lambda_n) e^{-i(\lambda_1 b_1 + \lambda_2 b_2 + \dots + \lambda_n b_n)}$$

and

$$(4.3) \quad \left\langle f(x_1, x_2, \dots, x_n), \psi \left(\frac{x_1 - b_1}{a_1}, \frac{x_2 - b_2}{a_2}, \dots, \frac{x_n - b_n}{a_n} \right) \right\rangle \\ = \left\langle Ff(x_1, x_2, \dots, x_n), F\psi \left(\frac{x_1 - b_1}{a_1}, \frac{x_2 - b_2}{a_2}, \dots, \frac{x_n - b_n}{a_n} \right) \right\rangle \\ \quad \quad \quad \text{[by Plancherel's lemma]} \\ = \left\langle \hat{f}(\lambda_1, \lambda_2, \dots, \lambda_n), \hat{\psi}(a_1 \lambda_1, a_2 \lambda_2, \dots, a_n \lambda_n) |a_1 a_2 \dots a_n| \right. \\ \quad \quad \quad \times e^{-i(\lambda_1 b_1 + \lambda_2 b_2 + \dots + \lambda_n b_n)} \left. \right\rangle \\ = |a_1 a_2 \dots a_n| \int_{\mathbb{R}^n} \hat{f}(\lambda_1, \lambda_2, \dots, \lambda_n) \overline{\hat{\psi}(a_1 \lambda_1, \dots, a_n \lambda_n)} \\ \quad \quad \quad \times e^{i(\lambda_1 b_1 + \lambda_2 b_2 + \dots + \lambda_n b_n)} d\lambda_1 d\lambda_2 \dots d\lambda_n,$$

where $\hat{\psi}(\lambda_1, \lambda_2, \dots, \lambda_n)$ is bounded uniformly on $\mathbb{R}^n$ and $\hat{f} \in L^2(\mathbb{R}^n)$, so $\widehat{f\bar{\psi}} \in L^2_{\lambda_1, \lambda_2, \dots, \lambda_n}$. Therefore the above integral in (4.2) belongs to $L^2_{b_1, b_2, \dots, b_n}$ by the property of the inverse Fourier transform. Now let

$$\begin{aligned} I &= \int_{\mathbb{R}^n} f(x_1, x_2, \dots, x_n) \bar{\psi} \left(\frac{x_1 - b_1}{a_1}, \frac{x_2 - b_2}{a_2}, \dots, \frac{x_n - b_n}{a_n} \right) dx_1 dx_2 \dots dx_n \\ &= \int_{-N_n}^{M_n} \int_{-N_{n-1}}^{M_{n-1}} \dots \int_{N_2}^{M_2} \int_{-N_1}^{M_1} f(x_1, x_2, \dots, x_n) \\ &\quad \times \bar{\psi} \left(\frac{x_1 - b_1}{a_1}, \frac{x_2 - b_2}{a_2}, \dots, \frac{x_n - b_n}{a_n} \right) dx_1 dx_2 \dots dx_n \\ &\quad + R(M, -N, b) \\ &= \int_{-\mathbf{N}}^{\mathbf{M}} f(\mathbf{x}) \bar{\psi} \left(\frac{\mathbf{x} - \mathbf{b}}{\mathbf{a}} \right) d\mathbf{x} + R(\mathbf{M}, -\mathbf{N}, \mathbf{b}) \\ &= I(\mathbf{M}, -\mathbf{N}, \mathbf{b}) + R(\mathbf{M}, -\mathbf{N}, \mathbf{b}), \end{aligned}$$

$$\begin{aligned} F_{\mathbf{b}} I &= F_{\mathbf{b}} I(\mathbf{M}, -\mathbf{N}, \mathbf{b}) + F_{\mathbf{b}} R(\mathbf{M}, -\mathbf{N}, \mathbf{b}) \\ &= \int_{-\mathbf{N}}^{\mathbf{M}} f(\mathbf{x}) F_{\mathbf{b}} \bar{\psi} \left(\frac{\mathbf{x} - \mathbf{b}}{\mathbf{a}} \right) d\mathbf{x} \\ &\quad + F_{\mathbf{b}} R(\mathbf{M}, -\mathbf{N}, \mathbf{b}) \quad [\text{by Fubini's theorem}]. \end{aligned} \tag{4.4}$$

$R(\mathbf{M}, -\mathbf{N}, \mathbf{b})$ as a function of $\mathbf{b}$ goes to zero in $L^2(\mathbb{R}^n)$ as $\mathbf{M}, \mathbf{N} \rightarrow \infty$ independently of each other; therefore $F_{\mathbf{b}} R(\mathbf{M}, -\mathbf{N}, \mathbf{b})$ also goes to zero in $L^2(\mathbb{R}^n)$ as a function of $\mathbf{b}$. Hence, letting $\mathbf{M}, \mathbf{N} \rightarrow \infty$ in (4.4) we obtain the desired result. The desired result could have been obtained by switching the order of integration in $F_{\mathbf{b}} I$, but we did not do so as neither of the iterated integrals $\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(\mathbf{x}) F_{\mathbf{b}} \bar{\psi} \left(\frac{\mathbf{x} - \mathbf{b}}{\mathbf{a}} \right) d\mathbf{x} d\mathbf{b}$ and $\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(\mathbf{x}) F_{\mathbf{b}} \bar{\psi} \left(\frac{\mathbf{x} - \mathbf{b}}{\mathbf{a}} \right) d\mathbf{b} d\mathbf{x}$ is absolutely convergent, nor is there any other way of proving that the double integral exists. So we applied Fubini's theorem as shown above in (4.3). $\square$

Theorem 4.2. *Let ψ be a basic or regular wavelet function in $L^2(\mathbb{R}^n)$ satisfying the admissibility condition (3.1) and assume that $f \in L^2(\mathbb{R}^n)$. Then the following inversion formula holds:*

$$\begin{aligned} f(x) &= \frac{1}{C_{\psi}} \int_{\mathbb{R}^n} |a_1 a_2 \dots a_n|^{-1/2} \psi \left(\frac{x_1 - b_1}{a_1}, \frac{x_2 - b_2}{a_2}, \dots, \frac{x_n - b_n}{a_n} \right) \\ &\quad \times W_f(a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n) \frac{db_1 db_2 \dots db_n \dots da_1 da_2 \dots da_n}{|a_1 a_2 \dots a_n|^2}, \end{aligned} \tag{4.5}$$

where $W_f(a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n)$ is defined to be the integral wavelet transform of $f \in L^2(\mathbb{R}^n)$ by

$$\begin{aligned} &\int_{\mathbb{R}^n} |a_1 a_2 \dots a_n|^{-1/2} f(x_1, x_2, \dots, x_n) \\ &\quad \times \bar{\psi} \left(\frac{x_1 - b_1}{a_1}, \frac{x_2 - b_2}{a_2}, \dots, \frac{x_n - b_n}{a_n} \right) dx_1 dx_2 \dots dx_n. \end{aligned}$$

The a_i 's and b_i 's are real numbers, and the a_i 's are non-zero. When a_i is zero the above wavelet transform of f is defined to be zero. In duality notation we write it

in the form

$$(4.6) \quad W_f(a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n) = \left\langle f(x_1, x_2, \dots, x_n), \frac{\psi\left(\frac{x_1-b_1}{a_1}, \frac{x_2-b_2}{a_2}, \dots, \frac{x_n-b_n}{a_n}\right)}{\sqrt{|a_1 a_2 \dots a_n|}} \right\rangle.$$

The inversion formula (4.5) is proved interpreting convergence in $L^2(\mathbb{R}^n)$; at the point of continuity of f the convergence in (4.5) takes place in the pointwise sense, as indicated in section 3 immediately preceding 3.3.

Proof. Our proof will be sketchy, leaving the verification of the lengthy computations to the readers.

Step A. In the one-dimensional case the proof of this step is given in the book of Bogess and Narcovich [3]. This result is

$$F_b\left(\bar{\psi}\left(\frac{x-b}{a}\right)\right) = |a|e^{-i\lambda x} \left[\widehat{\psi}(a\lambda)\right].$$

Therefore,

$$(4.7) \quad \overline{F_b\bar{\psi}\left(\frac{x-b}{a}\right)}(\lambda) = |a|e^{i\lambda x} \hat{\psi}(a\lambda).$$

In the n -dimensional case using substitutions $\frac{x_i-b_i}{a_i} = v_i$, $i = 1, 2, 3, \dots, n$, and the same trick of calculation as used by Bogess and Narcovich to prove (4.5), we get

$$(4.8) \quad \begin{aligned} & F_{b_1, b_2, \dots, b_n} \bar{\psi}\left(\frac{x_1-b_1}{a_1}, \frac{x_2-b_2}{a_2}, \dots, \frac{x_n-b_n}{a_n}\right) \\ &= |a_1 a_2 \dots a_n| e^{-i(\lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_n x_n)} \bar{\psi}(a_1 \lambda_1, a_2 \lambda_2, \dots, a_n \lambda_n). \end{aligned}$$

Step B. So, using a similar technique we get

$$F_{b_1, b_2, \dots, b_n} \left\{ W_f(a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n) \right\} = \prod_{i=1}^n \sqrt{2\pi|a_i|} \bar{\psi} \hat{f}(\lambda_1, \lambda_2, \dots, \lambda_n).$$

Here the wavelet transform of f is defined as $W_f : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{C}$:

$$\begin{aligned} & W_f(a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n) \\ &= \int_{\mathbb{R}^n} \frac{f(x_1, x_2, \dots, x_n)}{\sqrt{|a_1 a_2 \dots a_n|}} \bar{\psi}\left(\frac{x_1-b_1}{a_1}, \frac{x_2-b_2}{a_2}, \dots, \frac{x_n-b_n}{a_n}\right) dx_1 dx_2 \dots dx_n \end{aligned}$$

and we also use the fact proved in the previous theorem that

$$\begin{aligned} & F_{b_1, b_2, \dots, b_n} W_f(a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n) \\ &= \int_{\mathbb{R}^n} \frac{f(x_1, x_2, \dots, x_n)}{\sqrt{|a_1 a_2 \dots a_n|}} F_{b_1, b_2, \dots, b_n} \bar{\psi}\left(\frac{x_1-b_1}{a_1}, \frac{x_2-b_2}{a_2}, \dots, \frac{x_n-b_n}{a_n}\right) dx_1 dx_2 \dots dx_n \\ &= \int_{\mathbb{R}^n} \frac{f(x_1, x_2, \dots, x_n)}{\sqrt{|a_1 a_2 \dots a_n|}} |a_1 a_2 \dots a_n| e^{-i(\lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_n x_n)} \\ & \quad \times \bar{\psi}(a_1 \lambda_1, a_2 \lambda_2, \dots, a_n \lambda_n) dx_1 dx_2 \dots dx_n \\ &= \prod_{i=1}^n |2\pi a_i|^{1/2} \hat{f}(\lambda_1, \lambda_2, \dots, \lambda_n) \bar{\psi}(a_1 \lambda_1, a_2 \lambda_2, \dots, a_n \lambda_n). \end{aligned}$$

So

$$(4.9) \quad \begin{aligned} & F_{b_1, b_2, \dots, b_n} W_f(a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n)(\lambda_1, \lambda_2, \dots, \lambda_n) \\ &= \prod_{i=1}^n \sqrt{2\pi|a_i|} \, \overline{\hat{\psi}}(a_1\lambda_1, a_2\lambda_2, \dots, a_n\lambda_n) \hat{f}(\lambda_1, \lambda_2, \dots, \lambda_n). \end{aligned}$$

Step C. Final derivation of the inversion formula. Now take

$$(4.10) \quad \begin{aligned} F(x) &= \frac{1}{C_\psi} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |a_1 a_2 \dots a_n|^{-1/2} \psi\left(\frac{x_1 - b_1}{a_1}, \frac{x_2 - b_2}{a_2}, \dots, \frac{x_n - b_n}{a_n}\right) \\ &\quad \times W_f(a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n) db_1 db_2 \dots db_n \frac{da_1 da_2 \dots da_n}{|a_1 a_2 \dots a_n|^2}. \end{aligned}$$

Using Plancherel's formula we get

$$(4.11) \quad \begin{aligned} F(x) &= \frac{1}{C_\psi} \int_{\mathbb{R}^n} \frac{da_1 da_2 \dots da_n}{\sqrt{|a_1 a_2 \dots a_n|} (a_1 a_2 \dots a_n)^2} \\ &\quad \times \overline{\int_{\mathbb{R}^n} F_{b_1, b_2, \dots, b_n} \psi\left(\frac{x_1 - b_1}{a_1}, \frac{x_2 - b_2}{a_2}, \dots, \frac{x_n - b_n}{a_n}\right)(\lambda_1, \lambda_2, \dots, \lambda_n)} \\ &\quad \times F_{b_1, b_2, \dots, b_n} W_f(a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n)(\lambda_1, \lambda_2, \dots, \lambda_n) d\lambda_1 d\lambda_2 \dots d\lambda_n \\ &= \frac{1}{C_\psi} \int_{\mathbb{R}^n} \frac{da_1 da_2 \dots da_n}{\sqrt{|a_1 a_2 \dots a_n|} (a_1 a_2 \dots a_n)^2} |a_1 a_2 \dots a_n| \\ &\quad \times \int_{\mathbb{R}^n} \left[e^{i(\lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_n x_n)} \hat{\psi}(a_1 \lambda_1, a_2 \lambda_2, \dots, a_n \lambda_n) \right. \\ &\quad \left. \times \hat{f}(\lambda_1, \lambda_2, \dots, \lambda_n) \prod_{i=1}^n \sqrt{2\pi|a_i|} \, \overline{\hat{\psi}}(a_1 \lambda_1, a_2 \lambda_2, \dots, a_n \lambda_n) \right] d\lambda_1 d\lambda_2 \dots d\lambda_n \\ &= \frac{1}{C_\psi} (2\pi)^{n/2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{da_1 da_2 \dots da_n}{|a_1 a_2 \dots a_n|} e^{i\lambda_1 x_1 + i\lambda_2 x_2 + \dots + i\lambda_n x_n} \\ &\quad \times |\hat{\psi}(a_1 \lambda_1, a_2 \lambda_2, \dots, a_n \lambda_n)|^2 \hat{f}(\lambda_1, \lambda_2, \dots, \lambda_n) d\lambda_1 d\lambda_2 \dots d\lambda_n \\ &= \frac{1}{C_\psi} (2\pi)^n \int_{\mathbb{R}^n} \frac{|\hat{\psi}(a_1 \lambda_1, a_2 \lambda_2, \dots, a_n \lambda_n)|^2}{|a_1 a_2 \dots a_n|} da_1 da_2 \dots da_n \\ &\quad \times \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\lambda_1 x_1 + i\lambda_2 x_2 + \dots + i\lambda_n x_n} \hat{f}(\lambda_1, \lambda_2, \dots, \lambda_n) d\lambda_1 d\lambda_2 \dots d\lambda_n \\ &= \frac{C_\psi}{C_\psi} f(x_1, x_2, \dots, x_n) \end{aligned}$$

Using the transformation $a_i \lambda_i = u_i$, $i = 1, 2, \dots, n$ one can show that

$$\begin{aligned} & \int_{\mathbb{R}^n} \frac{|\hat{\psi}(a_1 \lambda_1, a_2 \lambda_2, \dots, a_n \lambda_n)|^2}{|a_1 a_2 \dots a_n|} da_1 da_2 \dots da_n \\ &= \int_{\mathbb{R}^n} \frac{|\hat{\psi}(u_1, u_2, \dots, u_n)|^2}{|u_1 u_2 \dots u_n|} du_1 du_2 \dots du_n. \end{aligned}$$

The convergence here is in $L^2(\mathbb{R}^n)$. But at points of continuity of f the convergence is in the pointwise sense in view of the Fourier inversion theorem we used in (4.10). $\square$

ACKNOWLEDGEMENT

The authors express their gratefulness to Professor Mike Moore of the School of Mathematics and Statistics, Carleton University, Ottawa, who carefully went through the manuscript and gave us some valuable suggestions. The authors also express their gratefulness to a referee for his constructive criticisms.

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SCHOOL OF MATHEMATICS AND STATISTICS, CARLETON UNIVERSITY, OTTAWA, CANADA

E-mail address: Jimpandey1936@gmail.com

DEPARTMENT OF MATHEMATICAL SCIENCES, INDIAN INSTITUTE OF TECHNOLOGY, DST-CIMS
BANARAS HINDU UNIVERSITY, INDIA

E-mail address: sk-upadhyay2001@yahoo.com