NEW OPERATOR INEQUALITIES IN FINITE-DIMENSIONAL VECTOR SPACES

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ABSTRACT. We establish some new operator inequalities in an n -dimensional vector space X equipped with a seminorm $\|\cdot\|$. Here is an example. If A is an invertible linear operator in X and ξ is a vector, then

$$\|\xi\|^r \leq \sum_{1 \leq |j| \leq \binom{n+r-1}{r}} \|A^j \xi\|^r.$$

Some special cases have been known and used in mathematical physics.

1. INTRODUCTION

Let X be an n -dimensional real or complex vector space equipped with a seminorm $\|\cdot\|$. One of the results established in the paper states: Suppose $A_i \in GL(X)$, $i = 1, \dots, r$ (that is, $A_1, \dots, A_r$ are invertible linear operators in X), and $\xi \in X$; then

$$\|\xi\|^r \leq \sum_{1 \leq |j| \leq n^r} \|A_1^j \xi\| \dots \|A_r^j \xi\|.$$

In the case of identical operators, this can be improved:

$$\|\xi\|^r \leq \sum_{1 \leq |j| \leq \binom{n+r-1}{r}} \|A^j \xi\|^r$$

for all $A \in GL(X)$ and all $\xi \in X$. The special cases $r = 1^1$ and $r = 2$ of this inequality (the latter case in a weaker form) have been known and used in the spectral theory of Schrödinger operators (see, e.g., [1, Theorem 10.4], [2], [3], [4], [5]).

Here is another example. For any two commuting operators $A, B \in GL(X)$ and any vector $\xi \in X$

$$\|\xi\| \leq \sum_{1 \leq j^2 + k^2 \leq 4n/\pi} \|A^j B^k \xi\|.$$

The remaining part of the paper is organized as follows. In Section 2 we formulate and prove the main lemmas, in Section 3 we derive some consequences, and in Section 4 we apply our results to the imperfect growth model.

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¹ See [4] for its history.

2. LEMMAS

Lemma 1. Suppose $g \mapsto T_i(g)$, $i = 1, \dots, r$, is a linear representation of a group G (with operation $(g_1, g_2) \mapsto g_1 g_2$ and the identity element e) on an n_i -dimensional vector space X_i . Suppose, furthermore, that K is a subset of G whose cardinality $|K|$ exceeds $n_1 \dots n_r$. Then for any r -linear form f on $X_1 \times \dots \times X_r$ and any vectors $\xi_1 \in X_1, \dots, \xi_r \in X_r$

$$(1) \quad |f(\xi_1, \dots, \xi_r)| \leq \sum_{g \in KK^{-1} \setminus \{e\}} |f(T_1(g)\xi_1, \dots, T_r(g)\xi_r)|.$$

Proof. We may assume that the set K is finite. Given $g \in G$ and $\eta_i \in X_i$ ($i = 1, \dots, r$), let $f_g(\eta_1, \dots, \eta_r) = f(T_1(g)\eta_1, \dots, T_r(g)\eta_r)$. The functions $f_g(\cdot)$ belong to the vector space M of all r -linear forms on $X_1 \times \dots \times X_r$; the dimension of M is $n_1 \dots n_r$. Since $|K| > \dim M$, the functions $f_g(\cdot)$, $g \in K$, are linearly dependent:

$$\sum_{g \in K} c_g f_g(\eta_1, \dots, \eta_r) = 0 \quad \text{for all } \eta_1 \in X_1, \dots, \eta_r \in X_r,$$

not all c_g 's being 0's. Choose $g_0 \in K$ such that $|c_{g_0}| = \max_{g \in K} |c_g|$. Then

$$f_{g_0}(\eta_1, \dots, \eta_r) = \sum_{g \in K \setminus \{g_0\}} (-c_g/c_{g_0}) f_g(\eta_1, \dots, \eta_r),$$

or

$$f(T_1(g_0)\eta_1, \dots, T_r(g_0)\eta_r) = \sum_{g \in K \setminus \{g_0\}} d_g f(T_1(g)\eta_1, \dots, T_r(g)\eta_r),$$

where $|d_g| \leq 1$ for all $g \in K \setminus \{g_0\}$.

Given $\xi_1 \in X_1, \dots, \xi_r \in X_r$, let $\eta_i = T_i(g_0^{-1})\xi_i$, $i = 1, \dots, r$. Then the previous equation becomes

$$\begin{aligned} f(\xi_1, \dots, \xi_r) &= \sum_{g \in K \setminus \{g_0\}} d_g f(T_1(gg_0^{-1})\xi_1, \dots, T_r(gg_0^{-1})\xi_r) \\ &\equiv \sum_{h \in Kg_0^{-1} \setminus \{e\}} d_{hg_0} f(T_1(h)\xi_1, \dots, T_r(h)\xi_r), \end{aligned}$$

so that

$$\begin{aligned} |f(\xi_1, \dots, \xi_r)| &\leq \sum_{h \in Kg_0^{-1} \setminus \{e\}} |f(T_1(h)\xi_1, \dots, T_r(h)\xi_r)| \\ &\leq \sum_{h \in KK^{-1} \setminus \{e\}} |f(T_1(h)\xi_1, \dots, T_r(h)\xi_r)|. \end{aligned}$$

□

In the case where the vector spaces X_i are identical, as are representations $T_i(\cdot)$, and the form f is symmetric, the cardinality of the set K in Lemma 1 can be reduced.

Lemma 2. Suppose $g \mapsto T(g)$ is a linear representation of a group G on an n -dimensional vector space X and f is a symmetric r -linear form on X . If K is a subset of G such that $|K| > \binom{n+r-1}{r}$, then for any vectors $\xi_1, \dots, \xi_r \in X$

$$(2) \quad |f(\xi_1, \dots, \xi_r)| \leq \sum_{g \in KK^{-1} \setminus \{e\}} |f(T(g)\xi_1, \dots, T(g)\xi_r)|.$$

The proof is similar to that of Lemma 1; the only difference is that now the functions $f_g(\eta_1, \dots, \eta_r) = f(T(g)\eta_1, \dots, T(g)\eta_r)$ belong to the vector space of symmetric r -linear forms on X whose dimension is $\binom{n+r-1}{r}$. $\square$

Lemma 3. Suppose $g \mapsto T_i(g)$, $i = 1, \dots, r$, is a linear representation of a group G on an n_i -dimensional vector space X_i equipped with a seminorm $\|\cdot\|_i$. Let K be a subset of G such that $|K| > n_1 \dots n_r$. Then for any vectors $\xi_1 \in X_1, \dots, \xi_r \in X_r$

$$(3) \quad \|\xi_1\|_1 \dots \|\xi_r\|_r \leq \sum_{g \in KK^{-1} \setminus \{e\}} \|T_1(g)\xi_1\|_1 \dots \|T_r(g)\xi_r\|_r.$$

Proof. By the Hahn–Banach theorem, for each i there is a linear functional φ_i on X_i such that $|\varphi_i(\eta_i)| \leq \|\eta_i\|_i$ for all $\eta_i \in X_i$ and $\varphi_i(\xi_i) = \|\xi_i\|_i$. Given $\eta_1 \in X_1, \dots, \eta_r \in X_r$, let

$$f(\eta_1, \dots, \eta_r) = \varphi_1(\eta_1) \dots \varphi_r(\eta_r).$$

Then Lemma 1 applies, and (1) becomes

$$\|\xi_1\|_1 \dots \|\xi_r\|_r \leq \sum_{g \in KK^{-1} \setminus \{e\}} |\varphi_1(T_1(g)\xi_1)| \dots |\varphi_r(T_r(g)\xi_r)|,$$

which implies (3). $\square$

The following statement is the version of Lemma 3 in the case where the vector spaces $X_1, \dots, X_r$ are identical, the representations $g \mapsto T_i(g)$ are identical as well, and $\xi_1 = \dots = \xi_r$.

Lemma 4. Suppose $g \mapsto T(g)$ is a linear representation of a group G on an n -dimensional vector space X equipped with a seminorm $\|\cdot\|$. Let K be a subset of G such that $|K| > \binom{n+r-1}{r}$. Then for any vector $\xi \in X$

$$(4) \quad \|\xi\|^r \leq \sum_{g \in KK^{-1} \setminus \{e\}} \|T(g)\xi\|^r.$$

Proof. For $\eta_1, \dots, \eta_r \in X$, let $f(\eta_1, \dots, \eta_r) = \varphi(\eta_1) \dots \varphi(\eta_r)$, where φ is a linear functional on X such that $\varphi(\xi) = \|\xi\|$ and $|\varphi(\eta)| \leq \|\eta\|$ for all $\eta \in X$. Since f is a symmetric r -linear form on X , Lemma 2 is applicable, and (2) gives

$$\begin{aligned} \|\xi\|^r = f(\xi, \dots, \xi) &\leq \sum_{g \in KK^{-1} \setminus \{e\}} |\varphi(T(g)\xi)| \dots |\varphi(T(g)\xi)| \\ &\leq \sum_{g \in KK^{-1} \setminus \{e\}} \|T(g)\xi\|^r. \end{aligned} \quad \square$$

3. THEOREMS

In this section, $\|\cdot\|$ is an arbitrary seminorm in an n -dimensional vector space X , and $\|\cdot\|_i$ is any seminorm in an n_i -dimensional vector space X_i ($i = 1, \dots, r$).

Theorem 1. Let $\xi_i \in X_i$ and $A_i \in GL(X_i)$ ($i = 1, \dots, r$). Then

$$(5) \quad \|\xi_1\|_1 \dots \|\xi_r\|_r \leq \sum_{1 \leq |j| \leq n_1 \dots n_r} \|A_1^j \xi_1\|_1 \dots \|A_r^j \xi_r\|_r.$$

Proof. Apply Lemma 3 with $G = \mathbf{Z}$, $K = \{j \in \mathbf{Z} \mid 0 \leq j \leq n_1 \dots n_r\}$ and $T_i(j) = A_i^j$ ($1 \leq i \leq r$; $j \in \mathbf{Z}$). $\square$

Theorem 2. If $\xi \in X$ and $A \in GL(X)$, then

$$\|\xi\|^r \leq \sum_{1 \leq |j| \leq \binom{n+r-1}{r}} \|A^j \xi\|^r.$$

Proof. Apply Lemma 4 with $G = \mathbf{Z}$, $K = \{j \in \mathbf{Z} \mid 0 \leq j \leq \binom{n+r-1}{r}\}$ and $T(j) = A^j$ ($j \in \mathbf{Z}$). $\square$

Example 1. Suppose for each $i = 1, \dots, r$ the function $y_i(\cdot)$ is a solution of the differential equation

$$(6) \quad y_i^{(p_i)}(t) + \sum_{m=0}^{p_i-1} a_{i,m}(t) y_i^{(m)}(t) = 0, \quad -\infty < t < \infty,$$

whose coefficients $a_{i,m}(\cdot)$ are C^∞ -smooth and T -periodic. Then for any integers $k_i \geq 0$ and any $a \in \mathbf{R}$

$$(7) \quad |y_1^{(k_1)}(a)| \dots |y_r^{(k_r)}(a)| \leq \sum_{1 \leq |j| \leq p_1 \dots p_r} |y_1^{(k_1)}(a + jT)| \dots |y_r^{(k_r)}(a + jT)|.$$

Proof. The vector space X_i of all solutions of (6) is p_i -dimensional. The operator $A_i: z(\cdot) \rightarrow z(\cdot + T)$ maps X_i onto itself. It remains to apply Theorem 1 to these operators, the seminorms $\|z\|_i = |z^{(k_i)}(a)|$ ($z \in X_i$) and the vectors $y_i \in X_i$, $i = 1, \dots, r$. $\square$

Let, in particular, $y(\cdot)$ be a solution of the differential equation

$$y''(t) + b(t)y'(t) + c(t)y(t) = 0, \quad t \in \mathbf{R},$$

with T -periodic $\mathbf{C}^\infty$ -smooth complex-valued coefficients $b(\cdot)$ and $c(\cdot)$. Then for any $a \in \mathbf{R}$ and any $r \in \mathbf{N}$, $k \in \mathbf{Z}_+$

$$|y^{(k)}(a)|^r \leq \sum_{1 \leq |j| \leq r+1} |y^{(k)}(a + jT)|^r$$

(here we use Theorem 2 rather than Theorem 1).

Example 2. Let $P(\cdot)$ be a polynomial of degree m . Then for any $z, h \in \mathbf{C} \setminus \{0\}$

$$|P(z)| \leq \sum_{1 \leq |j| \leq m+1} |P(h^j z)|.$$

The following statement generalizes this example.

Corollary 1. Let $P(\zeta) \equiv P(\zeta_1, \dots, \zeta_v)$ be a polynomial in v variables of degree d . Let B be an invertible affine transformation of $\mathbf{C}^v$. Then for any $\zeta \in \mathbf{C}^v$

$$|P(\zeta)| \leq \sum_{1 \leq |j| \leq \binom{d+v}{v}} |P(B^j \zeta)|.$$

Proof. We note that the vector space of all such polynomials has dimension $\binom{d+v}{v}$; then we apply Theorem 2 with $r = 1$. $\square$

Theorem 3. Let $\xi_i \in X_i$ and $A_{i,k} \in GL(X_i)$ ($i = 1, \dots, r$; $k = 1, \dots, s$). Suppose the operators $A_{i,k}$ with any fixed i commute. If K is a subset of $\mathbf{Z}^s$ such that $|K| > n_1 \dots n_r$, then

$$\|\xi_1\|_1 \dots \|\xi_r\|_r \leq \sum_{j \in (K - K) \setminus \{0\}} \|A_{1,1}^{j_1} \dots A_{1,s}^{j_s} \xi_1\|_1 \dots \|A_{r,1}^{j_1} \dots A_{r,s}^{j_s} \xi_r\|_r.$$

Proof. Apply Lemma 3 with $G = \mathbf{Z}^s$ and, for any $\mathbf{j} = (j_1, \dots, j_s) \in \mathbf{Z}^s$, $T_i(\mathbf{j}) = A_{i,1}^{j_1} \dots A_{i,s}^{j_s}$. $\square$

Theorem 4. Suppose operators $A_1, \dots, A_s \in GL(X)$ commute. If K is a subset of $\mathbf{Z}^s$ such that $|K| > \binom{n+r-1}{r}$, then for any vector $\xi \in X$

$$\|\xi\|^r \leq \sum_{\mathbf{j} \in (K-K) \setminus \{0\}} \|A_1^{j_1} \dots A_s^{j_s} \xi\|^r.$$

Proof. Apply Lemma 4 with $G = \mathbf{Z}^s$ and $T(\mathbf{j}) = A_1^{j_1} \dots A_s^{j_s}$ for all $\mathbf{j} = (j_1, \dots, j_s) \in \mathbf{Z}^s$. $\square$

Theorem 5. Let H be a closed convex centrally symmetric set in $\mathbf{R}^s$. If operators $A_1, \dots, A_s$ commute and $\text{Vol}(H) \geq 2^s \binom{n+r-1}{r}$, then for any vector $\xi \in X$

$$(8) \quad \|\xi\|^r \leq \sum_{\mathbf{j} \in (H \cap \mathbf{Z}^s) \setminus \{0\}} \|A_1^{j_1} \dots A_s^{j_s} \xi\|^r.$$

Proof. We may assume that H is bounded and hence $\text{Vol}(H) < \infty$. Let $G = \frac{1}{2}H$. There is $x_0 \in \mathbf{R}^s$ such that $|(G - x_0) \cap \mathbf{Z}^s| > \text{Vol}(G)$. Indeed, by shifting G we may assume that G has a boundary point, say a , in the interior of the cube $Q = \{x \in \mathbf{R}^s: 0 \leq x_i < 1, i = 1, \dots, s\}$. Let $G_{\mathbf{j}} = (G - \mathbf{j}) \cap Q$ ($\mathbf{j} \in \mathbf{Z}^s$) and $f(x) = \sum_{\mathbf{j} \in \mathbf{Z}^s} \mathbf{1}_{G_{\mathbf{j}}}(x)$ ($x \in Q$). Then

$$(9) \quad \int_Q f(x) dx = \sum_{\mathbf{j} \in \mathbf{Z}^s} \int_Q \mathbf{1}_{G_{\mathbf{j}}}(x) dx = \sum_{\mathbf{j} \in \mathbf{Z}^s} \text{Vol}(G_{\mathbf{j}}) = \text{Vol}(G).$$

There is $x_0 \in Q$ such that $f(x_0) > \text{Vol}(G)$. (Assume the converse; then, in particular, $f(a) \leq \text{Vol}(G)$. Consequently, we have $f(x) = f(a) - 1 \leq \text{Vol}(G) - 1$ for all points $x \in Q \setminus G$ that are close enough to a , hence on a nonempty open subset of Q ; together with our assumption, this implies that $\int_Q f(x) dx < \text{Vol}(G)$, which contradicts (9).) It remains to note that $f(x_0) = |\{\mathbf{j} \in \mathbf{Z}^s: x_0 \in G - \mathbf{j}\}| = |\{\mathbf{j} \in \mathbf{Z}^s: \mathbf{j} \in G - x_0\}| = |(G - x_0) \cap \mathbf{Z}^s|$.

Let $K = (G - x_0) \cap \mathbf{Z}^s$. Then $K - K \subset (G - G) \cap \mathbf{Z}^s = H \cap \mathbf{Z}^s$. Since $|K| > \text{Vol}(G) = 2^{-s} \text{Vol}(H) \geq \binom{n+r-1}{r}$, the inequality (8) follows from Theorem 4. $\square$

Example 3. Let $A, B \in GL(X)$ be two commuting operators. Let $r \in \mathbf{N}$ and $H = \{(x, y) \in \mathbf{R}^2: x^2 + y^2 \leq 4 \binom{n+r-1}{r} / \pi\}$. The area of the disc H equals $4 \binom{n+r-1}{r}$ and, according to Theorem 5 with $s = 2$, for any vector $\xi \in X$

$$(10) \quad \|\xi\|^r \leq \sum_{1 \leq j^2 + k^2 \leq 4 \binom{n+r-1}{r} / \pi} \|A^j B^k \xi\|^r.$$

Consider, as a special case, the equation

$$(11) \quad (\Delta + v)y = \lambda y$$

on the “discrete torus” $\mathbf{Z}_L^2 = (\mathbf{Z}/L\mathbf{Z})^2$. Here $(\Delta y)(x_1, x_2) = y(x_1 + 1, x_2) + y(x_1 - 1, x_2) + y(x_1, x_2 + 1) + y(x_1, x_2 - 1)$ and $(vy)(x) = v(x)y(x)$, where $v(\cdot)$ is a (τ, τ) -periodic function on $\mathbf{Z}_L^2$, τ being a divisor of L . If $y(\cdot)$ is a solution of (11), then for any $x = (x_1, x_2) \in \mathbf{Z}_L^2$

$$(12) \quad |y(x_1, x_2)| \leq \sum_{1 \leq j^2 + k^2 \leq 8L/\pi} |y(x_1 + j\tau, x_2 + k\tau)|.$$

Indeed, consider the space X of all solutions of (11); its dimension is $\leq 2L$. Also consider the operators A and B acting in X as follows:

$$(Ay)(x_1, x_2) = y(x_1 + \tau, x_2) \quad \text{and} \quad (By)(x_1, x_2) = y(x_1, x_2 + \tau).$$

They commute; it remains to apply (10) with $r = 1$. Note that the inequality (12) is nontrivial only if $\tau^2 < \pi L/8$ (otherwise, the right-hand side contains terms identical to the one on the left).

4. AN APPLICATION

In this section we show how inequalities of the type considered in the present work can be used in spectral theory. Theorem 6 below is a stronger version of the main result (Theorem 1) of [4].

We consider a discrete model of the d -dimensional medium whose Hamiltonian is the Schrödinger operator h_v acting in $l^2(\mathbf{Z}^d)$: $h_v = \Delta + v$, where Δ is the lattice Laplacian and v is the operator of multiplication by the lattice potential $v(\cdot)$. In this model, the potential $v(\cdot)$ is the result of the following recursive process. Consider a nested sequence of discrete cubes Q_n ($n = 0, 1, 2, \dots$) of the form

$$(13) \quad Q_n = \{x \in \mathbf{Z}^d: -\frac{T_n}{2} < x_i \leq \frac{T_n}{2}, \quad i = 1, \dots, d\}$$

whose sides $T_n \in \mathbf{N}$ are such that

$$(14) \quad \frac{T_n}{T_{n-1}} \in \mathbf{N} \setminus \{1\}, \quad n = 1, 2, 3, \dots$$

Initially, the potential is defined on the cube Q_0 . At the n th step ($n = 1, 2, \dots$) the potential, already defined on the cube Q_{n-1} , is extended Q_{n-1} -periodically to the cube Q_n ; then “impurities” are being introduced by arbitrarily changing the values of the potential at m_n points of Q_n selected at random so that all such selections are equiprobable. After infinitely many steps, we obtain a potential $v(\cdot)$ on $\mathbf{Z}^d$, which is almost surely (a.s.) well-defined if

$$(15) \quad \sum_{n=1}^{\infty} m_n / T_n^d < \infty$$

(due to the Borel-Cantelli Lemma).

We are going to show that if the sequence T_n grows fast enough while the sequence m_n grows not too fast, then a.s. the operator h_v does not have eigenvalues—in other words, there are no localized states.² Before formulating the result precisely, we will describe our model in an equivalent but more convenient way.

We consider a Schrödinger operator $h_v = \Delta + v$ acting in $l^2(\mathbf{Z}^d)$ as follows:

$$(16) \quad (h_v y)(x) = \sum_{z \in \mathbf{Z}^d: \|z\|_1=1} y(x+z) + v(x)y(x), \quad x \in \mathbf{Z}^d,$$

where $\|z\|_1 = |z_1| + \dots + |z_d|$ and

$$(17) \quad v(x) = \sum_{n=0}^{\infty} u_n(x).$$

² Note that if $m_n = 0$ for $n \geq n_0$, then the potential v is periodic, in which case the spectrum of h_v is well known to be not only continuous (no eigenvalues) but absolutely continuous.

Here $u_n(\cdot)$ is a $(T_n, \dots, T_n)$ -periodic function on $\mathbf{Z}^d$, where the numbers $T_n \in \mathbf{N}$ ($n = 0, 1, 2, \dots$) satisfy the condition (14). We assume that the function $u_n(\cdot)$ ($n \geq 1$) on its elementary cell Q_n defined by (13) vanishes everywhere except for a uniformly distributed random subset $X_n \subset Q_n$ of cardinality m_n . The values of $u_n(\cdot)$ on X_n are arbitrary. Under the condition (15) the potential v is a.s. well-defined, and h_v is a linear operator in $l^2(\mathbf{Z}^d)$ (in general, unbounded) with the domain $D(h_v) = \{y \in l^2(\mathbf{Z}^d) : \sum_{x \in \mathbf{Z}^d} |v(x)y(x)|^2 < \infty\}$. In our model, the functions u_n are real-valued so the operator h_v is self-adjoint. However, in what follows the functions u_n may be complex-valued.

Theorem 6. *If*

$$(18) \quad \sum_{n=0}^{\infty} \frac{m_{n+1} T_n^{2d-1}}{T_{n+1}} < \infty,$$

then with probability 1 the operator (16), (17) has no eigenvalues.

The proof of Theorem 6 is based on a theorem regarding solutions of periodic lattice equations, which is an improvement of the corresponding statement (Theorem 2) in [4]. For a function $y: \mathbf{Z}^d \rightarrow \mathbf{C}$, let

$$\sigma(k) = \sum_{x \in \mathbf{Z}^d : x_d = k} |y(x)|^2, \quad k \in \mathbf{Z} \quad (0 \leq \sigma(k) \leq \infty).$$

Theorem 7. *Let $y: \mathbf{Z}^d \rightarrow \mathbf{C}$ be a solution of $(\Delta + q)y = \lambda y$, where $q(\cdot)$ is a $(\tau_1, \dots, \tau_d)$ -periodic complex-valued function on $\mathbf{Z}^d$. Suppose $\sigma(k) < \infty$ for all $k \in \mathbf{Z}$. Then*

$$\sigma(0) \leq \sum_{1 \leq |m| \leq 2N^2 + N} \sigma(m\tau_d),$$

where $N = \tau_1 \dots \tau_{d-1}$.

In order to simplify the notation, we will present the proofs of Theorems 6 and 7 only for $d = 2$. The proofs in the general case ($d \geq 2$) follow the same lines.

Proof of Theorem 7 ($d = 2$). The periods τ_1 and τ_2 of the function q will be alternatively denoted as J and K , and the coordinates of a generic point $x \in \mathbf{Z}^2$ as j and k . We want to show that if $y(\cdot)$ satisfies the equation $h_q y = \lambda y$ and

$$(19) \quad \sigma(k) \equiv \sum_{j \in \mathbf{Z}} |y(j, k)|^2 < \infty \quad \text{for all } k \in \mathbf{Z},$$

then

$$(20) \quad \sigma(0) \leq \sum_{1 \leq |m| \leq 2J^2 + J} \sigma(mK).$$

(a) Let $y: \mathbf{Z}^2 \rightarrow \mathbf{C}$ be a solution of the equation $h_q y = \lambda y$, i.e., of the equation (21)

$$y(j, k+1) + y(j, k-1) + y(j+1, k) + y(j-1, k) + (q(j, k) - \lambda)y(j, k) = 0 \quad (j, k \in \mathbf{Z}),$$

satisfying (19). Consider the “discrete cylinder” $C_J = \mathbf{Z}_J \times \mathbf{Z}$, where $\mathbf{Z}_J = \mathbf{Z}/J\mathbf{Z}$, and the function $Z(\nu, k; \theta)$ on $C_J \times \mathbf{T}$ (where $\mathbf{T} = \mathbf{R}/\mathbf{Z}$), setting

$$Z(\nu, k; \theta) = \sum_{j \in \nu} y(j, k) e^{2\pi i j \theta}.$$

In view of (19), for any fixed $\nu \in \mathbf{Z}_J$ and $k \in K$, this series converges in $L^2(\mathbf{T})$ and, by Parseval's identity,

$$(22) \quad \int_{\mathbf{T}} |Z(\nu, k; \theta)|^2 d\theta = \sum_{j \in \nu} |y(j, k)|^2.$$

The (J, K) -periodic function $q(j, k)$ carries over naturally to C_J , generating a function $Q(\nu, k)$ ($\nu \in \mathbf{Z}_J$, $k \in \mathbf{Z}$), which is K -periodic in k . It is easy to derive from (21) that for any $\nu \in \mathbf{Z}_J$ and $k \in K$ one has the following equality in $L^2(\mathbf{T})$:

$$(23) \quad \begin{aligned} Z(\nu, k+1; \theta) + Z(\nu, k-1; \theta) + e^{-2\pi i \theta} Z(\nu+1, k; \theta) \\ + e^{2\pi i \theta} Z(\nu-1, k; \theta) + (Q(\nu, k) - \lambda) Z(\nu, k; \theta) = 0. \end{aligned}$$

Each $Z(\nu, k; \cdot)$, as an element of $L^2(\mathbf{T})$, is a class of equivalent square integrable Lebesgue measurable functions on $\mathbf{T}$. Let us select a Borel measurable function in this class. From now on, $Z(\nu, k; \cdot)$ will denote that function on $\mathbf{T}$, and each equation (23) will be understood as a pointwise equation that holds for Lebesgue a.e. $\theta \in \mathbf{T}$. Let B be a full measure Borel subset of $\mathbf{T}$ such that (23) holds for any $\theta \in B$ and all $\nu \in \mathbf{Z}_J$, $k \in \mathbf{Z}$.

(b) Denote by $\mathcal{M}$ the J -dimensional coordinate vector space over $\mathbf{C}$ with coordinates labeled by elements of $\mathbf{Z}_J$. Pick an arbitrary $\theta \in B$. Setting $\zeta_k(\theta) = (Z(\nu, k; \theta))_{\nu \in \mathbf{Z}_J} \in \mathcal{M}$, we transform (23) into a recurrence relation:

$$(24) \quad \zeta_{k+1}(\theta) = L_k(\theta) \zeta_k(\theta) - \zeta_{k-1}(\theta),$$

where $L_k(\theta)$ is the linear operator in $\mathcal{M}$ acting on vectors $(\zeta(\nu))_{\nu \in \mathbf{Z}_J} \in \mathcal{M}$ as follows:

$$(L_k(\theta) \zeta)(\nu) = -e^{2\pi i \theta} \zeta(\nu-1) - e^{-2\pi i \theta} \zeta(\nu+1) - V(\nu, k) \zeta(\nu).$$

Since $V(\nu, k)$ is K -periodic in k , we have $L_{k+K}(\theta) = L_k(\theta)$ for any $k \in \mathbf{Z}$.

Let $\mathcal{L} = \mathcal{M} \oplus \mathcal{M}$ and $\xi_k(\theta) = (\zeta_k(\theta), \zeta_{k-1}(\theta)) \in \mathcal{L}$. The equation (24) can be further transformed into

$$(25) \quad \xi_{k+1}(\theta) = T_k(\theta) \xi_k(\theta),$$

where the operator $T_k(\theta) \in GL(\mathcal{L})$ acts as follows: given $\zeta, \eta \in \mathcal{M}$,

$$T_k(\theta)((\zeta, \eta)) = (-\eta + L_k(\theta) \zeta, \zeta).$$

Again, $T_k(\theta)$ is K -periodic in k :

$$(26) \quad T_{k+K}(\theta) = T_k(\theta).$$

Let, furthermore,

$$S(\theta) = T_{K-1}(\theta) T_{K-2}(\theta) \dots T_1(\theta) T_0(\theta).$$

In view of (25) and (26),

$$(27) \quad \xi_{rK}(\theta) = (S(\theta))^r \xi_0(\theta)$$

for all $r \in \mathbf{Z}$.

(c) By (27) and Theorem 2 with $r = 2$, for any seminorm $\|\cdot\|$ in $\mathcal{L}$

$$(28) \quad \|\xi_0(\theta)\|^2 \leq \sum_{1 \leq |m| \leq \binom{2J+1}{2}} \|\xi_{mK}(\theta)\|^2.$$

Define a seminorm $\|\cdot\|$ in $\mathcal{L}$ as follows. Let $\xi \in \mathcal{L}$, that is, $\xi = (\zeta, \eta)$, where $\zeta, \eta \in \mathcal{M}$. Then we set $\|\xi\| = (\sum_{\nu \in \mathbf{Z}_J} |\zeta_\nu|^2)^{1/2}$. Now the inequality (28) implies that

$$\sum_{\nu \in \mathbf{Z}_J} |Z(\nu, 0; \theta)|^2 \leq \sum_{1 \leq |m| \leq 2J^2+J} \left(\sum_{\nu \in \mathbf{Z}_J} |Z(\nu, mK; \theta)|^2 \right),$$

which is true for all $\theta \in B$. Integrating both sides of this inequality over B or, equivalently, over $\mathbf{T}$ and using (22), we derive that

$$\sum_{j \in \mathbf{Z}} |y(j, 0)|^2 \leq \sum_{1 \leq |m| \leq 2J^2+J} \left(\sum_{j \in \mathbf{Z}} |y(j, mK)|^2 \right),$$

which is equivalent to (20). □

Proof of Theorem 6 ($d = 2$). For all $n \geq 0$ and $x \in \mathbf{Z}^2$, let $v_n(x) = u_0(x) + \dots + u_n(x)$. Consider the elementary cell Q_{n+1} of $v_{n+1}(\cdot)$:

$$Q_{n+1} = \{x = (j, k) \in \mathbf{Z}^2: -T_{n+1}/2 < j, k \leq T_{n+1}/2\}$$

and the strip $S_n = \{x = (j, k) \in \mathbf{Z}^2: -W_n \leq k \leq W_n\}$, where $W_n = 2T_n^3 + T_n^2$. Let

$$p_{n+1} = \Pr\{X_{n+1} \cap S_n \neq \emptyset\} \equiv \Pr\{X_{n+1} \cap R_{n+1} \neq \emptyset\},$$

where $R_{n+1} = Q_{n+1} \cap S_n$. It is easy to see that

$$p_{n+1} \leq m_{n+1} \frac{2W_n + 1}{T_{n+1}}.$$

By the Borel-Cantelli Lemma, the inequality $\sum_{n \geq 0} p_n < \infty$, which follows from (18), implies that a.s. there is a random number $N(\omega)$ such that $X_{n+1} \cap S_n = \emptyset$ for all $n \geq N(\omega)$. Consequently, for all $n \geq N(\omega)$ we have $(X_{n+1} \cup X_{n+2} \cup \dots) \cap S_n = \emptyset$ so that on the strip S_n $u_{n+1}(x) = u_{n+2}(x) = \dots = 0$ and, therefore,

$$(29) \quad v(x) = v_n(x) \quad \text{on } S_n \text{ for } n \geq N(\omega).$$

Suppose $y(\cdot)$ is a solution of $(\Delta + v)y = \lambda y$ that belongs to $l^2(\mathbf{Z}^2)$, so that $\sum_{k \in \mathbf{Z}} \sigma(k) < \infty$. For any $n \geq 0$, denote by $y_n(\cdot)$ the unique function on $\mathbf{Z}^2$ satisfying the equation $(\Delta + v_n)y_n = \lambda y_n$ and such that $y_n(j, k) = y(j, k)$ for $k = -1, 0$ and all $j \in \mathbf{Z}$. Let $\sigma_n(k) = \sum_{j \in \mathbf{Z}} |y_n(j, k)|^2$, $k \in \mathbf{Z}$.

It follows from Theorem 7 (with $\tau_1 = \tau_2 = T_n$) that

$$\sigma_n(0) \leq \sum_{1 \leq |m| \leq 2T_n^2+T_n} \sigma_n(mT_n) \leq \sum_{T_n \leq |k| \leq W_n} \sigma_n(k).$$

Due to (29), we have $y_n(x) = y(x)$ on the strip S_n if $n \geq N(\omega)$, and hence

$$\sigma(0) \leq \sum_{T_n \leq |k| \leq W_n} \sigma(k)$$

for $n \geq N(\omega)$. Therefore, a.s. $\sigma(0) \leq \sum_{|k| \geq T_n} \sigma(k)$ for all large enough n ; since $\sum_{k \in \mathbf{Z}} \sigma(k) < \infty$ and $T_n \rightarrow \infty$ as $n \rightarrow \infty$, it follows that $\sigma(0) = 0$, that is, $y(j, 0) = 0$ for all $j \in \mathbf{Z}$.

Similarly, the function $y^*(j, k) = y(j, k+1)$ satisfies the equation $(\Delta + v^*)y^* = \lambda y^*$, where $v^*(j, k) = v(j, k+1)$ for all j, k . The potential $v^*(\cdot)$ has the same structure as $v(\cdot)$ (since in the construction of $v(\cdot)$ each square Q_n could be replaced

by any shift thereof), so the inclusion $y^* \in l^2(\mathbf{Z}^2)$ a.s. implies that $y^*(j, 0) = 0$ for all j .

Therefore, if $(\Delta + v)y = \lambda y$ and $y \in l^2(\mathbf{Z}^2)$, then a.s. $y(j, 0) = y(j, 1) = 0$ for all $j \in \mathbf{Z}$ and hence $y \equiv 0$ on $\mathbf{Z}^2$. $\square$

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