A NOTE ON THE TWO NESTED REGULAR POLYGONAL CENTRAL CONFIGURATIONS

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ABSTRACT. This paper is concerned with nested polygonal central configurations for the Newtonian $2n$ -body problem. We show that when two nested regular n -polygons ($n \geq 3$) with masses located at the vertices form a central configuration where the twisted angle θ is zero, then the value of masses in each separate polygon must be equal.

1. INTRODUCTION AND THE MAIN RESULT

Newtonian N -body problems concern the motion of N point masses $m_k > 0$ and their positions $x_k \in \mathbb{R}^d$, $k = 1, \dots, N$ and $d \in \mathbb{N}^+$. For $N \geq 2$, this motion is governed by Newton's second Law and the Universal Gravitational Law:

$$(1.1) \quad m_k \ddot{x}_k = \frac{\partial U(\vec{x})}{\partial x_k}, \quad k = 1, 2, \dots, N,$$

where $\vec{x} = (x_1, x_2, \dots, x_N)^T$ with Newtonian potential

$$U(\vec{x}) = \sum_{1 \leq k < j \leq N} \frac{m_j m_k}{|x_j - x_k|}.$$

In order to avoid collisions, we consider the configuration space

$$X \setminus \Delta = \{(x_1, x_2, \dots, x_N)^T \in \mathbb{C}^N : x_k \neq x_j \text{ when } k \neq j\}.$$

Definition 1.1. We say that $\vec{x} = (x_1, x_2, \dots, x_N) \in X \setminus \Delta$ is a central configuration at some moment if there exists a $\lambda \in \mathbb{R}$ such that

$$(1.2) \quad \sum_{\substack{j \neq k \\ j=1}}^N \frac{x_j - x_k}{|x_j - x_k|^3} m_j m_k = -\lambda m_k (x_k - x_0), \quad k = 1, 2, \dots, N,$$

where the center of masses is defined as $x_0 = \frac{\sum_{k=1}^N m_k x_k}{\sum_{k=1}^N m_k}$.

If at every moment, $\vec{x} = (x_1, x_2, \dots, x_N)^T$ satisfies (1.1) and (1.2), then, by direct calculation, we see that $\ddot{x}_k = -\lambda(x_k - x_0)$ which means the acceleration vector of every mass points toward the center of masses and its magnitude is proportional to its distance from the center of masses. In [5], L. M. Perko and E. L. Walter studied configurations of masses on the vertices of a regular N -polygon ($N \geq 4$)

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and proved they form a central configuration if and only if their masses are equal; later, the authors in [6] and [2] provide simplifications of their arguments. In the paper of F. Diacu [1], we find a study of these configurations in spaces with constant curvature. In [4], R. Moeckel and C. Simó studied those central configurations consisting of two regular n -polygons ($n \geq 3$) lying on horizontal planes, centered on a common vertical axis, and aligned so that the corresponding vertices lie in the same vertical half-plane. They showed that under certain conditions, spatial configurations bifurcate from planar ones as the masses are varied. In [8], S. Q. Zhang and Q. Zhou studied the necessary conditions for nested regular polygonal central configurations.

In studying the nested regular polygonal central configurations, we first suppose that all of the particles which lie on the vertices of one n -polygon ($n \geq 3$) have mass μ , and the other particles on the other n -polygon have mass 1. A natural question would be: Is the assumption of the equality of masses on one n -polygon a necessary condition to form a central configuration? Do there exist any other assignments of mass values which form central configurations? Let $q_k = e^{\frac{2k\pi}{n}i}$ for $k = 0, \dots, n-1$ be the n -th roots of unity, and suppose there are $2n$ masses $(M_1, M_2) = (m_1, \dots, m_n, m_{n+1}, \dots, m_{2n})^T \in \mathbb{R}^{2n}$, with their position $x_j = (q_j, 0), x_{n+j} = (rq_j, h)$ where $r \in \mathbb{R}$. In the following, we will prove

Theorem 1.2. *For every $r, h \in \mathbb{R}^+$ and $(r, h) \neq (1, 0)$, if the masses (M_1, M_2) with their position $\vec{x}$ form a central configuration, then we must have $m_1 = m_2 = \dots = m_n$ and $m_{n+1} = m_{n+2} = \dots = m_{2n}$.*

Remark 1.3. In Theorem 2 of [8], Zhang-Zhou studied the case $h = 0$, and arrived at a more general conclusion (but without detailed proof): when $h = 0$, $1 \neq r \in \mathbb{C}$, if (M_1, M_2) and $\vec{x}$ are solutions of (1.2), then $m_1 = m_2 = \dots = m_n$ and $m_{n+1} = m_{n+2} = \dots = m_{2n}$. Here $r = |r|e^{i\theta} \in \mathbb{C}$ means that the two polygons have a twisted angle θ . In [7], Yu-Zhang showed that when the $2n$ masses $m_1 = m_2 = \dots = m_n$ and $m_{n+1} = m_{n+2} = \dots = m_{2n}$ form a central configuration, the twisted angle θ must be 0 or $\frac{\pi}{n}$.

We end this section with the concept of circulant matrices; especially, the general formulas for their eigenvalues λ_k and eigenvectors v_k ([3] p. 66).

Definition 1.4. An $n \times n$ matrix $C = (c_{kj})$ is circulant if $c_{kj} = c_{k-1, j-1}$, where $c_{0, j}$ and $c_{k, 0}$ are identified with $c_{n, j}$ and $c_{k, n}$, respectively.

Using powers of the circulant matrix

$$P = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix},$$

every circulant matrix C can be represented as

$$C = \sum_{j=1}^n c_{1j} P^{j-1}.$$

The matrix C has eigenvalues $\lambda_k = \sum_{j=1}^n c_{1j} q_{k-1}^{j-1}$ with corresponding eigenvectors $v_k = (q_{k-1}, q_{k-1}^2, \dots, q_{k-1}^n)^T$. Let $A_1 = \sum_{j=1}^{n-1} \frac{1-q_j}{|1-q_j|^3} P^j$, and for $r \neq 1$, we denote

the circulant matrix $A_r = \sum_{j=1}^n \frac{1-rq_j}{|1-rq_j|^3} P^j$, i.e.,

$$A_r = \begin{pmatrix} \frac{1-rq_n}{|1-rq_n|^3} & \frac{1-rq_1}{|1-rq_1|^3} & \frac{1-rq_2}{|1-rq_2|^3} & \cdots & \frac{1-rq_{n-1}}{|1-rq_{n-1}|^3} \\ \frac{1-rq_{n-1}}{|1-rq_{n-1}|^3} & \frac{1-rq_n}{|1-rq_n|^3} & \frac{1-rq_1}{|1-rq_1|^3} & \cdots & \frac{1-rq_{n-2}}{|1-rq_{n-2}|^3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1-rq_1}{|1-rq_1|^3} & \frac{1-rq_2}{|1-rq_2|^3} & \frac{1-rq_3}{|1-rq_3|^3} & \cdots & \frac{1-rq_n}{|1-rq_n|^3} \end{pmatrix}.$$

Then the k -th eigenvalue of A_r is

$$\lambda_{r,k} = \begin{cases} \sum_{j=1}^n \frac{1-rq_j}{|1-rq_j|^3} q_{k-1}^j, & r \neq 1, \\ \sum_{j=1}^{n-1} \frac{1-q_j}{|1-q_j|^3} q_{k-1}^j, & r = 1. \end{cases}$$

2. SOME LEMMAS

Lemma 2.1. *For all $r \in (0, +\infty)$ and $k = 1, 2, \dots, n$, $\lambda_{r,k}$ are real numbers and $\lambda_{r^{-1},k} = -r^2 \lambda_{r,n+1-k}$.*

Proof. The proof of the case $r = 1$ can be found as Lemma 9 in [5] or a simpler proof in [2]. When $r \neq 1$, we have

$$\bar{\lambda}_{r,k} = \sum_{j=1}^n \frac{1-rq_{n-j}}{|1-rq_{n-j}|^3} q_{k-1}^{n-j} = \lambda_{r,k},$$

which implies $\lambda_{r,k} \in \mathbb{R}$. Also,

$$\begin{aligned} \lambda_{r^{-1},k} &= \sum_{j=1}^n \frac{1-r^{-1}q_j}{|1-r^{-1}q_j|^3} q_{k-1}^j = r^2 \sum_{j=1}^n \frac{r-q_j}{|r-q_j|^3} q_{-j} q_{jk} \\ &= -r^2 \sum_{j=1}^n \frac{1-rq_{-j}}{|1-rq_{-j}|^3} q_k^j = -r^2 \lambda_{r,n+1-k}. \end{aligned}$$

□

Lemma 2.2 (Lemma 2 in [4]). *Let $\phi_s(x) = \sum_{j=1}^n \frac{1}{|1-xq_j|^s}$, where $s > 0$. Then, for $0 < x < 1$, $\phi_s(x)$ and all of its derivatives are positive; moreover, the same is true for $\psi_s(x) = \sum_{j=1}^n \frac{q_{jt}}{|1-xq_j|^s}$, where $t \in \mathbb{Z}$.*

Proof. An explicit power series expansion for $\psi_s(x)$ will be obtained, from which it will follow that all of the coefficients are nonnegative. Recall the expansion

$$\frac{1}{(1-z)^{s/2}} = \sum_{k=0}^{\infty} \alpha_k z^k,$$

where the coefficients are positive numbers depending on s . Following Euler, one can write

$$\frac{1}{|1-xq_j|^s} = \frac{1}{(1-xq_j)^{s/2}} \frac{1}{(1-xq_{n-j})^{s/2}}.$$

Using the above power series, we have

$$\frac{q_{jt}}{|1-xq_j|^s} = \sum_{k=0}^{\infty} c_k^j x^k,$$

where

$$c_k^j = \sum_{h+l=k} \alpha_h \alpha_l q_{(h-l+t)j}.$$

This must be summed for $j = 1, 2, \dots, n$. But the sum over the roots of unity is n if $h + t \equiv l \pmod n$, and 0 otherwise. Thus,

$$\psi_s(x) = n \sum_{k=0}^{\infty} \left(\sum_{\substack{h+l=k \\ h+t \equiv l \pmod n}} \alpha_h \alpha_l \right) x^k.$$

Here, all of the coefficients are nonnegative and infinitely many are positive. The claim about $\psi_s(x)$ and its derivatives follows. $\square$

Corollary 2.3. *Let $\varphi_s(x) = \sum_{j=1}^n \frac{q_{jt}}{(|1-xq_j|^2+h)^s}$, where $s, h > 0$. Then for $0 < x \leq 1$, $\varphi_s(x) > 0$.*

Proof. Calculating,

$$\begin{aligned} \varphi_s(x) &= \sum_{j=1}^n \frac{q_{jt}}{(1+x^2+h-2x \cos \frac{2j\pi}{n})^s} \\ &= (1+x^2+h)^{-s} \sum_{j=1}^n \frac{q_{jt}}{(1-2\frac{x}{1+x^2+h} \cos \frac{2j\pi}{n})^s} \\ &= x^{-s} y_1^s \sum_{j=1}^n \frac{q_{jt}}{(1-2y_1 \cos \frac{2j\pi}{n})^s}, \end{aligned}$$

where $y_1 = \frac{x}{1+x^2+h} \in (0, \frac{1}{2})$. Let $y_2 = \frac{1-\sqrt{1-4y_1^2}}{2y_1} = \frac{2y_1}{1+\sqrt{1-4y_1^2}} \in (0, 1)$, then

$$(1-2y_1 \cos \frac{2j\pi}{n})^s = y_1^s y_2^{-s} (1+y_2^2-2y_2 \cos \frac{2j\pi}{n})^s = y_1^s y_2^{-s} |1-y_2 q_j|^{2s}.$$

That is,

$$\varphi_s(x) = x^{-s} y_2^s \psi_{2s}(y_2) > 0.$$

$\square$

Lemma 2.4. *When $0 < r < 1$, we have $\lambda_{r,k+1} > r\lambda_{r,k}$ for $k = 1, 2, \dots, n-1$.*

Proof. When $0 < r < 1$,

$$\lambda_{r,k+1} - r\lambda_{r,k} = \sum_{j=1}^n \frac{1-rq_j}{|1-rq_j|^3} (1-rq_{-j}) q_{jk} = \sum_{j=1}^n \frac{q_{jk}}{|1-rq_j|}.$$

By Lemma 2.2, $\lambda_{r,k+1} - r\lambda_{r,k} > 0$. $\square$

Lemma 2.5. *For $k = 1, 2, \dots, n$, we have $\lambda_{r,k} > 0$ if $0 < r < 1$; and $\lambda_{r,k} < 0$ if $r > 1$.*

Proof. By Lemma 2.1, we know that $\lambda_{r,k}$ are real numbers, so for $0 < r < 1$ we have

$$\lambda_{r,1} = \sum_{j=1}^n \frac{1-rq_j}{|1-rq_j|^3} = \sum_{j=1}^n \frac{1-r \cos \frac{2j\pi}{n}}{|1-rq_j|^3} > 0.$$

By Lemma 2.4, $\lambda_{r,k} > 0$ holds for all $k = 1, 2, \dots, n$, when $0 < r < 1$. The case $r > 1$ follows from Lemma 2.1. $\square$

3. PROOF OF THEOREM 1.2

In this section we will give a proof of Theorem 1.2. From the symmetry of the configuration, we need only consider the situation $r \in (0, 1]$. By (1.2) and our assumption in Theorem 1.2, we have, for $k = 1, 2, \dots, n$,

$$\begin{cases} \sum_{j=1}^n \frac{(q_j - q_k, 0)}{(|q_j - q_k|^2 + 0)^{\frac{3}{2}}} m_j m_k + \sum_{j=1}^n \frac{(rq_j - q_k, h)}{(|rq_j - q_k|^2 + h^2)^{\frac{3}{2}}} m_{n+j} m_k \\ \quad = -\lambda m_k (q_k - c_0, -h_0), \\ \sum_{j=1}^n \frac{(q_j - rq_k, -h)}{(|q_j - rq_k|^2 + h^2)^{\frac{3}{2}}} m_j m_{n+k} + \sum_{j=1}^n \frac{(rq_j - rq_k, 0)}{(|rq_j - rq_k|^2 + 0)^{\frac{3}{2}}} m_{n+j} m_{n+k} \\ \quad = -\lambda m_{n+k} (rq_k - c_0, h - h_0), \end{cases}$$

where the center of masses $x_0 = (c_0, h_0)$, and it implies

$$(3.1) \quad \begin{cases} \sum_{j=1}^n \frac{(1 - q_{j-k}, 0)}{|1 - q_{j-k}|^{\frac{3}{2}}} m_j + \sum_{j=1}^n \frac{(1 - rq_{j-k}, h)}{(|1 - rq_{j-k}|^2 + h^2)^{\frac{3}{2}}} m_{n+j} = \lambda(1 - c_0 q_{-k}, h_0), \\ \frac{1}{|r|^3} \sum_{j=1}^n \frac{(1 - (r)^{-1} q_{j-k}, h)}{(|1 - (r)^{-1} q_{j-k}|^2 + (\frac{h}{|r|})^2)^{\frac{3}{2}}} m_j + \frac{1}{|r|^3} \sum_{j=1}^n \frac{(1 - q_{j-k}, 0)}{|1 - q_{j-k}|^{\frac{3}{2}}} m_{n+j} \\ \quad = \lambda(1 - \frac{c_0}{r} q_{-k}, h - h_0). \end{cases}$$

The system of equations can be rewritten more simply in terms of circulant matrices as

$$(3.2) \quad \begin{pmatrix} A_1 & A_{r,h} \\ |r|^{-3} A_{r^{-1}, \frac{h}{|r|}} & |r|^{-3} A_1 \end{pmatrix} \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} = \lambda \begin{pmatrix} v_1 - c_0 v_n \\ v_1 - \frac{c_0}{r} v_n \end{pmatrix},$$

$$(3.3) \quad \begin{cases} B_{r,h} M_2 = \lambda \frac{h_0}{h} v_1, \\ \frac{1}{|r|^3} B_{r^{-1}, \frac{h}{|r|}} M_1 = \lambda(1 - \frac{h_0}{h}) v_1, \end{cases} \quad h \neq 0,$$

where $A_{r,0}$ is the circulant matrix A_r defined at the end of Section 1, for $h \neq 0$, $A_{r,h} = \sum_{j=1}^n \frac{1 - rq_j}{(|1 - rq_j|^2 + h^2)^{\frac{3}{2}}} P^j$ and $B_{r,h} = \sum_{j=1}^n \frac{1}{(|1 - rq_j|^2 + h^2)^{\frac{3}{2}}} P^j$, with $\beta_k(r, h) = \sum_{j=1}^n \frac{1 - rq_j}{(|1 - rq_j|^2 + h^2)^{\frac{3}{2}}} q_j(k-1)$ and $\gamma_k(r, h) = \sum_{j=1}^n \frac{q_j(k-1)}{(|1 - rq_j|^2 + h^2)^{\frac{3}{2}}}$ their respective eigenvalues.

3.1. The planar case $h = 0$. When $h = 0$, the system (3.2) implies

$$(3.4) \quad \begin{pmatrix} A_1 & A_r \\ |r|^{-3} A_{r^{-1}} & |r|^{-3} A_1 \end{pmatrix} \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} = \lambda \begin{pmatrix} v_1 - c_0 v_n \\ v_1 - \frac{c_0}{r} v_n \end{pmatrix}.$$

Suppose $M_1 = \sum_{k=1}^n \alpha_k v_k$ and $M_2 = \sum_{k=1}^n \beta_k v_k$. It is straightforward to see that (3.4) holds if and only if

$$(3.5) \quad \begin{cases} \lambda_{1,k} \alpha_k + \lambda_{r,k} \beta_k = 0, \\ \lambda_{r^{-1},k} \alpha_k + \lambda_{1,k} \beta_k = 0, \end{cases} \quad k = 2, 3, \dots, n-1,$$

and

$$(3.6) \quad \begin{cases} \lambda_{1,1} \alpha_1 + \lambda_{r,1} \beta_1 = \lambda, \\ |r|^{-3} \lambda_{r^{-1},1} \alpha_1 + |r|^{-3} \lambda_{1,1} \beta_1 = \lambda, \end{cases} \quad \begin{cases} \lambda_{1,n} \alpha_n + \lambda_{r,n} \beta_n = -\lambda c_0, \\ |r|^{-3} \lambda_{r^{-1},1} \alpha_1 + |r|^{-3} \lambda_{1,1} \beta_1 = -\lambda c_0 r^{-1}. \end{cases}$$

We see that if (3.5) has nonzero solutions for (α_k, β_k) , we must have

$$\det \begin{pmatrix} \lambda_{1,k} & \lambda_{r,k} \\ \lambda_{r^{-1},k} & \lambda_{1,k} \end{pmatrix} = 0,$$

which means

$$\lambda_{1,k}^2 = \lambda_{r,k} \lambda_{r-1,k};$$

however, this contradicts Lemma 2.5. So $\alpha_k = \beta_k = 0$ for $k = 2, 3, \dots, n-1$. That is to say $M_1 = \alpha_1 v_1 + \alpha_n v_n$ and $M_2 = \beta_1 v_1 + \beta_n v_n$. We notice that M_1 , M_2 and v_1 are real vectors in $\mathbb{R}^n$ but $v_n \notin \mathbb{R}^n$ when $n \geq 3$, so $\alpha_n = \beta_n = 0$ and $M_1 = \alpha_1 v_1$, $M_2 = \beta_1 v_1$, i.e., $m_1 = m_2 = \dots = m_n$ and $m_{n+1} = m_{n+2} = \dots = m_{2n}$.

3.2. The spatial case $h \neq 0$. From (3.3) we see that if we have a solution with unequal masses, there must exist some r, h and $k \neq 1$ such that $\gamma_k(r, h) = 0$. But by Corollary 2.3, we have $\gamma_k(r, h) > 0$ for all r, k, h . So we also must have $m_1 = m_2 = \dots = m_n$ and $m_{n+1} = m_{n+2} = \dots = m_{2n}$. This completes the proof of Theorem 1.2.

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