FINITENESS OF THE POLYHEDRAL $\mathbb{Q}$ -CODEGREE SPECTRUM

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ABSTRACT. In this note we prove Fujita’s spectrum conjecture for polarized varieties in the case of $\mathbb{Q}$ -Gorenstein projective toric varieties of index r . The theorem follows from a combinatorial result using the connection between lattice polytopes and polarized projective toric varieties. By this correspondence the spectral value of the polarized toric variety equals the $\mathbb{Q}$ -codegree of the polytope. Now the main theorem of the paper shows that the spectrum of the $\mathbb{Q}$ -codegree is finite above any positive threshold in the class of lattice polytopes with α -canonical normal fan for any fixed $\alpha > 0$.

1. INTRODUCTION

Let X be a normal projective algebraic variety. A line bundle L on X is *ample* if it has positive intersection with all irreducible curves on X , and L is *big* if the global sections of some multiple define a birational map to some projective space. Any ample line bundle is big. A *polarized variety* is a pair (X, L) of a normal projective algebraic variety X and an ample line bundle L on X . Let K_X be the canonical class on X . For a rational parameter $t > 0$ the *adjoint line bundles* on X are the line bundles $L + t \cdot K_X$. They define invariants that have been used for classifications of projective toric varieties [BS94, BS95, Som86]. In particular, the *unnormalized spectral value* $\mu(L)$ of the polarized variety is given by

$$\mu(L)^{-1} := \sup \{ t \in \mathbb{Q} \mid L + t \cdot K_X \text{ is big} \};$$

see [BS95, Ch. 7.1.1]. Note that $\mu(L)$ is finite as the line bundle L is big by assumption. The *spectral value* $\sigma(L) := d + 1 - \mu(L)$ has originally been considered by Sommese [Som86, §1]. Similar notions have been defined several times, e.g., $\kappa\epsilon(X, L) := -\mu(L)$ is the *Kodaira energy* of (X, L) [Fuj96]. Fujita has stated the following conjecture on the values of $\mu(L)$.

Conjecture 1.1 (Spectrum Conjecture, Fujita [Fuj92] and [Fuj96, (3.2)]). *For any $d \in \mathbb{Z}_{\geq 1}$, let S_d be the set of unnormalized spectral values of a smooth polarized d -fold. Then, for any $\varepsilon > 0$, the set $\{s \in S_d \mid s > \varepsilon\}$ is a finite set of rational numbers.*

This has been proved by Fujita for dimensions $d = 2, 3$ in 1996 [Fuj96]. Recently, Cerbo [Cer12] proved the related log spectrum conjecture [Fuj96, (3.7)].

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There is a fundamental connection between combinatorial and algebraic geometry which we introduce now. A *lattice polytope* $P \subseteq \mathbb{R}^d$ is a convex polytope whose vertices lie in a lattice N contained in $\mathbb{R}^d$. Choosing a basis of the lattice N fixes an isomorphism to the standard lattice $\mathbb{Z}^d$. Thus, in the following we restrict our attention to $N = \mathbb{Z}^d$. Any convex rational polytope P can be defined as

$$(1.1) \quad P = \{ \mathbf{x} \mid \langle \mathbf{a}_i, \mathbf{x} \rangle \leq b_i, \quad 1 \leq i \leq n \}$$

with *primitive* linear functionals $\mathbf{a}_i \in (\mathbb{Z}^d)^* \cong \mathbb{Z}^d$ for $1 \leq i \leq n$, where a functional is primitive if the greatest common divisor of its entries is 1. We may assume that this representation is *irredundant*, i.e., no inequality $\langle \mathbf{a}_i, \mathbf{x} \rangle \leq b_i$ can be omitted without changing P . If P is a lattice polytope, then also the b_i , $1 \leq i \leq n$, are integral.

Any lattice polytope P uniquely defines a pair (X_P, L_P) , where X_P is the projective toric variety corresponding to the normal fan of P polarized by an ample line bundle L_P , and vice versa (see, e.g., [Ful93] or [CLS11] for details). We use the same letter for the line bundle and its associated Weil divisor. If D_i denotes the divisor of the ray spanned by $\mathbf{a}_i$, then $L = \sum b_i D_i$. In this setting the canonical divisor is given by $K_{X_P} = -\sum D_i$.

This connection between toric varieties and polytopes allows us to set up a dictionary translating between combinatorial properties of the polytope (or fan) and algebraic properties of the variety. For example, the projective toric variety is smooth if and only if the primitive generators of each cone of the normal fan of P are part of a lattice basis of $\mathbb{Z}^d$ [CLS11, Thm. 2.4.3]. In this case we say that P is *smooth*.

Using this correspondence, we give a polyhedral interpretation of the spectral value in the case of projective *toric* varieties. Let P be given by an irredundant set of inequalities with primitive normals as in (1.1). We define the family of *adjoint polytopes* associated to P via

$$(1.2) \quad P^{(t)} = \{ \mathbf{x} \mid \langle \mathbf{a}_i, \mathbf{x} \rangle \leq b_i - t, \quad 1 \leq i \leq n \}$$

for a rational parameter $t \geq 0$. This notion has been introduced and studied by Dickenstein, Di Rocco, and Pienze [DDP09, Def. 1.1], and subsequently in [DHNP11, Def. 1.1]. If P is the polytope associated to a polarized projective toric variety (X_P, L_P) , then $(p \cdot P)^{(q)}$ for some integral $p, q > 0$ is the polytope that corresponds to the adjoint line bundle $p \cdot L + q \cdot K_X$ (where $p \cdot P$ is the polytope P scaled with p pointwise). Clearly, $P^{(t)}$ is empty for large t . Let $t = q/p > 0$ for $p, q \in \mathbb{Z}_{>0}$ be the maximal rational number such that $P^{(t)}$ (or, equivalently, $(p \cdot P)^{(q)}$) is non-empty. Its reciprocal, the $\mathbb{Q}$ -codegree $\text{codeg}_{\mathbb{Q}} P := t^{-1} = p/q$ of P equals precisely $\mu(L)$, see [DHNP11, Sec. 4.3]. We can now reformulate Conjecture 1.1 in the case of projective polarized toric varieties.

Conjecture 1.2 (Polyhedral Spectrum Conjecture). *For any $d \in \mathbb{Z}_{\geq 1}$, let S_d be the set of $\mathbb{Q}$ -codegrees of smooth d -dimensional lattice polytopes. Then, for any $\varepsilon > 0$, the set $\{s \in S_d \mid s > \varepsilon\}$ is a finite set of rational numbers.*

In this note we answer this question affirmatively. Theorem 3.1 shows that this conjecture is even true on the much larger class of lattice polytopes with α -canonical normal fan. This class includes the set of lattice polytopes with $\mathbb{Q}$ -Gorenstein normal fan of index r (for $\alpha = 1/r$), and in particular, for $\alpha = 1$, contains all smooth polytopes (see below for definitions).

A polytope P with $\mathbb{Q}$ -Gorenstein normal fan corresponds to a polarized $\mathbb{Q}$ -Gorenstein toric variety (X_P, L_P) of index r , i.e., the integer $r \in \mathbb{Z}_{\geq 1}$ is the minimal r such that rK_X is a Cartier divisor. Thus, with Theorem 3.1, we prove Conjecture 1.1 for the class of $\mathbb{Q}$ -Gorenstein varieties of index r (see Corollary 3.3), which contains all smooth polytopes (for $r = 1$).

2. BASIC DEFINITIONS

Let P be a rational d -dimensional polytope given by an irredundant set of normal vectors $\mathbf{a}_i$, $1 \leq i \leq n$, and corresponding right-hand sides b_i , $1 \leq i \leq n$, as in (1.1). Here, a set of normal vectors is *irredundant* if we cannot omit one of the inequalities $\langle \mathbf{a}_i, \mathbf{x} \rangle \leq b_i$ without changing P . Given some lattice $N \subseteq \mathbb{R}^d$, the polytope P is a *lattice polytope* if all vertices of P are contained in N . By choosing a basis of N , we can fix an isomorphism of the lattice to the standard lattice $\mathbb{Z}^d$, so in the following we will assume that all vertices of a lattice polytope are integral. See [CLS11, Ewa88] for background on these and the following notions.

For a lattice polytope we can assume that all normals in the inequality description (1.1) are *primitive*, i.e., all $\mathbf{a}_i$ for $1 \leq i \leq n$ are integral and there is no other integral point strictly between $\mathbf{a}_i$ and the origin.

Up to a lattice preserving transformation, we can always assume that P is full dimensional, so $P \subseteq \mathbb{R}^d$. Let $\Sigma_P \subseteq (\mathbb{R}^d)^*$ be the *normal fan* of P . This is a complete rational polyhedral fan. With $\Sigma_P(k)$ we denote the subset of Σ_P of cones of dimension at most k (the *k-skeleton*). The set of normals $\mathbf{a}_1, \dots, \mathbf{a}_n$ is irredundant, so they generate the rays in $\Sigma_P(1)$. For a rational parameter $t \geq 0$, the family of *adjoint polytopes* $P^{(t)}$ of P is given by (1.2).

Remark 2.1. Note that it is essential for the definition of the adjoint polytope that all inequalities $\langle \mathbf{a}_i, \mathbf{x} \rangle \leq b_i$ define a facet of P (i.e., the system is irredundant), and that all $\mathbf{a}_i$ are primitive vectors in the lattice $(\mathbb{Z}^d)^* \cong \mathbb{Z}^d$, for $1 \leq i \leq n$. For example, consider the triangle defined by

$$x \geq 0, \quad y \geq 0, \quad 3x + y \leq 3.$$

Then $P^{(t)} \neq \emptyset$ for $t \leq \frac{3}{5}$ and is empty otherwise. However, if we add the redundant inequality $x \leq 1$, then $P^{(t)} = \emptyset$ for $t > \frac{1}{2}$. If we replace the last inequality by $6x + 2y \leq 6$, then $P^{(t)}$ is non-empty for any $t \leq \frac{2}{3}$.

For sufficiently large t the adjoint polytopes $P^{(t)}$ are empty. The $\mathbb{Q}$ -codegree $\text{codeg}_{\mathbb{Q}}(P)$ of P is the inverse of the largest $t > 0$ such that $P^{(t)}$ is non-empty, i.e.,

$$\text{codeg}_{\mathbb{Q}}(P)^{-1} := \max \left(t \mid P^{(t)} \neq \emptyset \right) = \sup \left(t \mid \dim P^{(t)} = d \right).$$

See [DHN11, Def. 1.5 and Prop. 1.6] for a proof that these two definitions of the $\mathbb{Q}$ -codegree coincide and for more background. For convenience we define

$$c := \text{codeg}_{\mathbb{Q}}(P)^{-1}.$$

The *core* of P is

$$\text{core } P := P^{(c)}.$$

FIGURE 2.1. The lattice polytope of Example 2.2. Its core is the segment drawn with a thick line.

The core of P is a (rational, not full-dimensional) polytope defined by the (usually redundant set of) inequalities $\langle \mathbf{a}_i, \mathbf{x} \rangle \leq b_i - c$. A vector $\mathbf{a}_i$ is a *core normal* if

$$\langle \mathbf{a}_i, \mathbf{y} \rangle = b_i - c \quad \text{for all } \mathbf{y} \in \text{core } P.$$

The set of all *core normals* of P will be denoted by $\mathcal{N}_{\text{core}}(P)$. It is a subset of the primitive generators of the rays in $\Sigma_P(1)$ (but they do not necessarily span a subfan of Σ_P). Up to relabeling, we can assume that $\mathcal{N}_{\text{core}}(P) = \{\mathbf{a}_1, \dots, \mathbf{a}_m\}$ for some $m \leq n$. The *affine hull* $C_{\text{aff}}(P)$ of core P is the smallest affine space that contains core P . This is the set of solutions of all equations given by the core normals, i.e.,

$$C_{\text{aff}}(P) := \left\{ \mathbf{x} \mid \langle \mathbf{a}_i, \mathbf{x} \rangle = b_i - c, \ 1 \leq i \leq m \right\}.$$

Example 2.2. Consider the polytope shown in Figure 2.1 given by the five inequalities

$$\begin{array}{lll} -y \leq 0, & -x \leq 0, & x - y \leq 4, \\ y \leq 3, & x \leq 5. & \end{array}$$

The core of P is the segment with vertices $(3/2, 3/2)$ and $(7/2, 3/2)$ given by the inequalities

$$\begin{array}{lll} -y \leq -3/2 = 0 - 3/2, & -x \leq -3/2 = 0 - 3/2, & x - y \leq 5/2 = 4 - 3/2, \\ y \leq 3/2 = 3 - 3/2, & x \leq 7/2 = 5 - 3/2. & \end{array}$$

It is drawn with a thick line in Figure 2.1. The inequalities in the left column actually span the affine hull of core P , so $\mathcal{N}_{\text{core}}(P) = \{(0, \pm 1)\}$. The inequalities in the center define the two facets (vertices in this case), and the inequality in the right column is redundant. We have $C_{\text{aff}} = \{(x, y) \mid y = 3/2\}$.

Let $\sigma \subset (\mathbb{R}^d)^*$ be a rational polyhedral cone with primitive generators $\mathbf{a}_1, \dots, \mathbf{a}_k$. The cone σ is called *$\mathbb{Q}$ -Gorenstein* of index r if there is a primitive vector $\mathbf{u}$ such that $\langle \mathbf{a}_i, \mathbf{u} \rangle = r$ for $1 \leq i \leq k$. The cone is *Gorenstein* if $r = 1$. A complete rational polyhedral fan Σ is *$\mathbb{Q}$ -Gorenstein* of index r if all cones are $\mathbb{Q}$ -Gorenstein and r is the least common multiple of the indices of all cones. The fan Σ is *Gorenstein* if $r = 1$.

We define the height of a point $\mathbf{y} \in \sigma$ as

$$\text{height}(\mathbf{y}) := \max \left(\sum_{i=1}^k \lambda_i \mid \mathbf{y} = \sum_{i=1}^k \lambda_i \mathbf{a}_i, \text{ and } \lambda_i \geq 0 \text{ for } 1 \leq i \leq k \right).$$

The cone σ is *α -canonical* for some $\alpha > 0$ if $\text{height}(\mathbf{y}) \geq \alpha$ for any integral point $\mathbf{y} \in \sigma \cap (\mathbb{Z}^d)^*$. A complete rational polyhedral fan Σ is *α -canonical* if every cone

in Σ is. A cone or fan is *canonical* if it is α -canonical for $\alpha = 1$. Note that one-dimensional cones are always canonical, so $\alpha \leq 1$ for all fans. Any $\mathbb{Q}$ -Gorenstein cone or fan of index r is $\frac{1}{r}$ -canonical. See [DHN11, Sec. 2.3] for more properties of α -canonical fans. We define

$$\mathcal{P}(d) := \{ P \mid P \text{ is a } d\text{-dimensional lattice polytope} \}$$

and $\mathcal{P}_\alpha^{\text{can}}(d) := \{ P \mid P \in \mathcal{P}(d) \text{ with } \alpha\text{-canonical normal fan} \}$.

A lattice polytope is *smooth* if the primitive generators of any cone in its normal fan are a subset of a lattice basis. So, in particular, the normal fan of a smooth polytope is Gorenstein (with index 1) and canonical.

We can define analogous notions for projective toric varieties. A polarized toric variety (X, L) is non-singular if and only if the associated lattice polytope is smooth (see e.g., [CLS11, Thm. 2.4.3]). The polarized toric variety (X, L) is $\mathbb{Q}$ -Gorenstein if rK_X is a Cartier divisor for some integer r . The minimal r such that this holds is the *index* of the polarized toric variety. Again, using the toric dictionary, (X, L) is $\mathbb{Q}$ -Gorenstein of index r if the normal fan of the associated polytope is $\mathbb{Q}$ -Gorenstein of index r .

3. THE $\mathbb{Q}$ -CODEGREE SPECTRUM

The purpose of this note is to study the set of lattice polytopes with α -canonical normal fan whose $\mathbb{Q}$ -codegree is bounded from below. For $\alpha, \varepsilon > 0$ and $d \in \mathbb{Z}_{\geq 1}$, we define

$$\mathcal{S}(d, \varepsilon) := \{ P \mid P \in \mathcal{P}(d) \text{ and } \text{codeg}_{\mathbb{Q}}(P) \geq \varepsilon \}$$

$$\begin{aligned} \text{and} \quad \mathcal{S}_\alpha^{\text{can}}(d, \varepsilon) &:= \{ P \mid P \in \mathcal{P}_\alpha^{\text{can}}(d) \text{ and } \text{codeg}_{\mathbb{Q}}(P) \geq \varepsilon \} \\ &= \mathcal{S}(d, \varepsilon) \cap \mathcal{P}_\alpha^{\text{can}}(d). \end{aligned}$$

For $\alpha = 1$, this set contains all smooth lattice polytopes and, for $\alpha = 1/r$, all lattice polytopes with $\mathbb{Q}$ -Gorenstein normal fan of index r and $\mathbb{Q}$ -codegree bounded from below by ε . Our main theorem is the following.

Theorem 3.1. *Let $d \in \mathbb{N}$, and let $\alpha, \varepsilon \geq 0$ be given. Then*

$$\{ \text{codeg}_{\mathbb{Q}}(P) \mid P \in \mathcal{S}_\alpha^{\text{can}}(d, \varepsilon) \}$$

is finite.

In other words, in any fixed dimension and for any fixed α , the set of values the $\mathbb{Q}$ -codegree assumes above any positive threshold is finite in the class of polytopes with α -canonical normal fan. We obtain the following corollary.

Corollary 3.2. *Conjecture 1.2 holds for d -dimensional lattice polytopes with α -canonical normal fan, for any given $\alpha > 0$.*

Using the correspondence between combinatorial and toric geometry, we can translate this into a statement about polarized toric varieties.

Corollary 3.3. *Conjecture 1.1 holds for d -dimensional polarized $\mathbb{Q}$ -Gorenstein toric varieties of index r , for any integer $r > 0$.*

This proves a generalization of Conjecture 1.1 in the toric case. For smooth two- and three-dimensional polarized varieties this has previously been proved by Fujita [Fuj96].

FIGURE 3.1. The mountain of the polytope in Figure 2.1. The top (thick) face is the core of P , up to a projection.

Remark 3.4. The following family of examples shows that we cannot expect the conjecture to be true without any further assumptions. For positive integers $a \in \mathbb{Z}_{>0}$, consider the polytopes

$$\Delta_d(a) := \text{conv}(\mathbf{0}, a\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_d).$$

They have $\mathbb{Q}$ -codegree $\text{codeg}_{\mathbb{Q}}(\Delta_d(a)) = d - 1 + \frac{2}{a}$ arbitrarily close to $d - 1$, and the normal fan of $\Delta_d(a)$ is $\mathbb{Q}$ -Gorenstein with index a . Hence, the theorem cannot hold without restrictions to the normal fan unless $\varepsilon > d - 1$.

We prove Theorem 3.1 by a series of lemmas. The proof has two main steps.

- (1) We first show that there are, up to lattice equivalence and for fixed $\alpha > 0$ and $d \in \mathbb{Z}_{>0}$, only finitely many sets of core normals for polytopes in the class $\mathcal{P}_{\alpha}^{\text{can}}(d)$ (Corollary 3.9) by reducing this to a finiteness result of Lagarias and Ziegler (Theorem 3.7).
- (2) In a second step we show that each such configuration gives rise to a finite number of different $\mathbb{Q}$ -codegrees above any positive threshold (Lemma 3.10).

We start by studying the set of normal vectors that define the core of a lattice polytope. Let P be a d -dimensional lattice polytope with α -canonical normal fan given by

$$P = \left\{ \mathbf{x} \mid \langle \mathbf{a}_i, \mathbf{x} \rangle \leq b_i \quad 1 \leq i \leq n \right\}$$

with an irredundant set of primitive normals $\mathbf{a}_i$. Let $C = \text{core } P$ be the core of P with affine hull $C_{\text{aff}} = \text{aff } C$, and let c^{-1} be the $\mathbb{Q}$ -codegree of P . Up to relabeling, we can assume that $\mathbf{a}_1, \dots, \mathbf{a}_m$ for some $m \leq n$ is the (not irredundant) set of facet normals defining C_{aff} , i.e.,

$$C_{\text{aff}} = \left\{ \mathbf{x} \mid \langle \mathbf{a}_i, \mathbf{x} \rangle = b_i - c, \quad 1 \leq i \leq m \right\},$$

and no other functional $\mathbf{a}_i$ is constant on C_{aff} . We define a new polytope, the *mountain* of P , via

$$\text{mountain}(P) = \left\{ (\mathbf{x}, \lambda) \mid \langle \mathbf{a}_i, \mathbf{x} \rangle + \lambda \leq b_i \quad \text{for } 1 \leq i \leq n \quad \text{and } \lambda \geq 0 \right\}.$$

See Figure 3.1 for the mountain of the polytope in Example 2.2. This notion has been introduced in [DHN11, Prop. 1.14]. Up to a projection we can recover all adjoint polytopes by intersecting the mountain with the hyperplane $\lambda = t$ for some $t > 0$, i.e.,

$$P^{(t)} \times \{t\} = \text{mountain } P \cap \{(\mathbf{x}, \lambda) \mid \lambda = t\}.$$

Let F be the face of mountain P defined by the inequality $\lambda \leq c$ for some (appropriately chosen) $c \in \mathbb{R}$. Then $\text{core } P$ is the projection of F onto the first d

FIGURE 3.2. The polytope A_{core} and the linear space $L_{\mathcal{A}}$ for the example in Figure 2.1.

coordinates, and the $\mathbb{Q}$ -codegree of P is the inverse $1/c$ of the height of this face over the base $P \times \{0\}$.

Let $A_{\text{core}} := \text{conv}(\mathbf{a}_1, \dots, \mathbf{a}_m)$ be the convex hull of the core normals. The following two lemmas show that all $\mathbf{a}_i$ are vertices and $\mathbf{0}$ is a relatively interior point of A_{core} (i.e., the vectors $\mathbf{a}_1, \dots, \mathbf{a}_m$ positively span the linear space they generate). See Figure 3.2(a). The first lemma has been proved in [DHNP11].

Lemma 3.5 (Di Rocco et al. [DHNP11, Lemma 2.2]). *The origin is in the relative interior of A_{core} .*

Proof. We consider the mountain of P . The vector $(\mathbf{0}, 1)$ defining the face F projecting onto the core of P as above is in the relative interior of the normal cone of F , and the generators of this normal cone are the normals defining C_{aff} .

By construction, this normal cone is spanned by the vectors $(\mathbf{a}_i, 1)$ for $1 \leq i \leq m$, so there are positive coefficients $\eta_1, \dots, \eta_m$ such that

$$(\mathbf{0}, 1) = \sum_{i=1}^m \eta_i (\mathbf{a}_i, 1).$$

This reduces to a positive linear combination of $\mathbf{0}$ in the $\mathbf{a}_i$, $1 \leq i \leq m$, which proves that $\mathbf{0} \in \text{relint } A_{\text{core}}$. $\square$

Lemma 3.6. *The vertices of A_{core} are $\mathbf{a}_1, \dots, \mathbf{a}_m$.*

Proof. Assume on the contrary that this is not the case. Then one of the $\mathbf{a}_j$, $1 \leq j \leq m$, is a convex combination of the others. We can assume that this is $\mathbf{a}_m$. So there are $\eta_1, \dots, \eta_{m-1} \geq 0$ such that $\mathbf{a}_m = \sum \eta_i \mathbf{a}_i$ and $\sum \eta_i = 1$. Let $\mathbf{x}_{\text{core}}$ be a relative interior point of $\text{core}(P)$. Then $\langle \mathbf{a}_j, \mathbf{x}_{\text{core}} \rangle = b_j - c$ for $1 \leq j \leq m$, so we compute

$$\begin{aligned} -c + b_m &= \langle \mathbf{a}_m, \mathbf{x}_{\text{core}} \rangle = \sum \eta_i \langle \mathbf{a}_i, \mathbf{x}_{\text{core}} \rangle \\ &= \sum \eta_i b_i - c \sum \eta_i = -c + \sum \eta_i b_i. \end{aligned}$$

Hence, $\sum \eta_i b_i = b_m$. On the other hand, by irredundancy of the $\mathbf{a}_i$ as facet normals of P there is $\mathbf{y}$ such that $\langle \mathbf{a}_m, \mathbf{y} \rangle = b_m$, but $\langle \mathbf{a}_i, \mathbf{y} \rangle < b_i$ for $1 \leq i \leq m-1$ (in fact, also for $m+1 \leq i \leq n$). Indeed, we can choose any point in the relative

interior of the facet defined by $\mathbf{a}_m$. Hence, we can continue with

$$\sum \eta_i b_i = b_m = \langle \mathbf{a}_m, \mathbf{y} \rangle = \sum \eta_i \langle \mathbf{a}_i, \mathbf{y} \rangle < \sum \eta_i b_i.$$

This is clearly a contradiction, so $\mathbf{a}_m$ is not a convex combination of the other $\mathbf{a}_j$. $\square$

From now on we fix some parameter $\alpha > 0$ and restrict to lattice polytopes in $\mathcal{P}_\alpha^{\text{can}}(d)$. We subdivide this set according to the configuration of primitive normal vectors spanning the core face of a polytope in $\mathcal{P}_\alpha^{\text{can}}(d)$, i.e., according to the polytope A_{core} it generates. To this end, let $\mathcal{A} := \{\mathbf{a}_1, \dots, \mathbf{a}_m\} \subset (\mathbb{Z}^d)^* \cong \mathbb{Z}^d$ be a set of primitive vectors that positively span the linear space they generate (i.e., there are $\lambda_1, \dots, \lambda_m \geq 0$ and not all 0 such that $\sum \lambda_i \mathbf{a}_i = \mathbf{0}$). For such a set $\mathcal{A}$ and parameters $d \in \mathbb{N}$ and $\alpha > 0$, we consider the set

$$\mathcal{N}(d, \alpha, \mathcal{A}) := \{P \mid P \in \mathcal{P}_\alpha^{\text{can}}(d) \text{ and } \mathcal{N}_{\text{core}}(P) = \mathcal{A}\}.$$

Note that we do not require that the vectors in $\mathcal{A}$ are the generators of rays of a subfan of Σ_P . We just assume that they generate rays in $\Sigma_P(1)$ and are normal to $C_{\text{aff}}(P)$. We will show that, up to lattice equivalence and for fixed $d \in \mathbb{Z}_{>0}$ and $\alpha > 0$, only finitely many of the sets $\mathcal{N}(d, \alpha, \mathcal{A})$ are non-empty. This is obtained by reducing the problem to the following result of Lagarias and Ziegler.

Theorem 3.7 (Lagarias and Ziegler [LZ91]). *Let integers $d, k, r \geq 1$ be given. There are, up to lattice equivalence, only finitely many different lattice polytopes of dimension d with exactly k interior points in the lattice $r\mathbb{Z}^d$.* $\square$

This theorem is based on work of Hensley [Hen83], who proved this for $r = 1$. The bound has later been improved by Pikhurko [Pik01]. The proof of Lagarias and Ziegler has two steps. They first show that the volume of a d -dimensional lattice polytope with exactly k interior points in $r\mathbb{Z}^d$ is bounded. In a second step, they show that any polytope with volume V can be transformed into a polytope inside a cube with volume $d!V$.

The next lemma gives the reduction to this theorem by showing that for $P \in \mathcal{P}_\alpha^{\text{can}}(d)$ the origin is the only relatively interior point of A_{core} in the lattice $\lceil \frac{1}{\alpha} \rceil \mathbb{Z}^d$.

Lemma 3.8. *For A_{core} and α as above we have $\text{relint } \alpha A_{\text{core}} \cap (\mathbb{Z}^d)^* = \{\mathbf{0}\}$.*

Proof. We prove this by contradiction. Assume that there is some vector $\mathbf{a} \in (\mathbb{Z}^d)^* \setminus \{\mathbf{0}\}$ contained in the relative interior of αA_{core} . As $\mathbf{0} \in \text{relint } \alpha A_{\text{core}}$, the point $\mathbf{a}$ is contained in the cone spanned by the vertices of some facet F of A_{core} . Hence, we can find $\lambda_1, \dots, \lambda_m \geq 0$ with $\lambda_i = 0$ if $\mathbf{a}_i \notin F$ and

$$\Lambda := \sum \lambda_i < \alpha$$

such that $\mathbf{a} = \sum_{i=1}^m \lambda_i \mathbf{a}_i$. Note that Λ is strictly less than α as $\mathbf{a}$ is in the relative interior of αA_{core} . The normal fan Σ_P of P is complete. Hence, there is some (possibly not unique) maximal cone $\sigma \in \Sigma_P(d)$ that contains $\mathbf{a}$. Then $\sigma = \text{cone}(\mathbf{a}_i \mid i \in I)$ for some set $I \subseteq [n]$ (where $\mathbf{a}_1, \dots, \mathbf{a}_n$ is the list of all normal vectors of facets of P). This gives a second representation of $\mathbf{a}$ in the form $\mathbf{a} = \sum_{i \in I} \mu_i \mathbf{a}_i$, where $\mu_i \geq 0$. By assumption, the normal fan Σ_P is α -canonical, i.e., the height of $\mathbf{a}$ in σ is at least α . Hence, we can choose the μ_i in such a way that

$$M := \sum_{i \in I} \mu_i \geq \alpha.$$

Note that this implies

$$(3.1) \quad \Lambda < \alpha \leq M.$$

The cone $\sigma = \text{cone}(\mathbf{a}_i \mid i \in I)$ is a maximal cone in Σ_P . Hence, it is the normal cone of a vertex $\mathbf{v}$ of P . By its definition, this implies that $\langle \mathbf{a}_i, \mathbf{v} \rangle = b_i$ if $i \in I$ and $\langle \mathbf{a}_i, \mathbf{v} \rangle < b_i$ otherwise. We compute

$$(3.2) \quad \sum_{i=1}^m \lambda_i b_i \geq \sum_{i=1}^m \lambda_i \langle \mathbf{a}_i, \mathbf{v} \rangle = \langle \mathbf{a}, \mathbf{v} \rangle = \sum_{i \in I} \mu_i \langle \mathbf{a}_i, \mathbf{v} \rangle = \sum_{i \in I} \mu_i b_i.$$

Now let $\mathbf{x}_{\text{core}}$ be a relative interior point of $C = \text{core } P$. Then

$$\begin{aligned} \langle \mathbf{a}_i, \mathbf{x}_{\text{core}} \rangle &= b_i - c \quad \text{for } 1 \leq i \leq m \\ \text{and } \langle \mathbf{a}_i, \mathbf{x}_{\text{core}} \rangle &< b_i - c \quad \text{otherwise.} \end{aligned}$$

Using this we compute

$$\begin{aligned} \sum_{i \in I} \mu_i b_i - Mc &= \sum_{i \in I} \mu_i (b_i - c) \\ &\geq \sum_{i \in I} \mu_i \langle \mathbf{a}_i, \mathbf{x}_{\text{core}} \rangle = \langle \mathbf{a}, \mathbf{x}_{\text{core}} \rangle = \sum_{i=1}^m \lambda_i \langle \mathbf{a}_i, \mathbf{x}_{\text{core}} \rangle \\ &= \sum_{i=1}^m \lambda_i (b_i - c) = \sum_{i=1}^m \lambda_i b_i - \Lambda c. \end{aligned}$$

We can shorten this to

$$\sum_{i \in I} \mu_i b_i \geq \sum_{i=1}^m \lambda_i b_i + (M - \Lambda)c > \sum_{i=1}^m \lambda_i b_i,$$

where the last strict inequality follows from (3.1). This contradicts (3.2). $\square$

We can now combine this result with Theorem 3.7 to obtain the desired finiteness of configurations with non-empty $\mathcal{N}(d, \alpha, \mathcal{A})$.

Corollary 3.9. *Let $d \in \mathbb{N}$ and $\alpha > 0$ be given. Then there are, up to unimodular transformation, only finitely many sets $\mathcal{A} \subseteq (\mathbb{Z}^d)^*$ such that $\mathcal{N}(d, \alpha, \mathcal{A})$ is non-empty.*

In other words, Corollary 3.9 shows that there are, up to lattice equivalence, only finitely many configurations that occur as the set $\mathcal{N}_{\text{core}}(P)$ of core normals of a lattice polytope $P \in \mathcal{P}_{\alpha}^{\text{can}}(d)$.

Proof. By Lemma 3.8 $\text{relint}(\alpha A_{\text{core}})$ only contains the origin. Hence, the set $\text{relint}(A_{\text{core}}) \cap \lceil \frac{1}{\alpha} \rceil \mathbb{Z}^d$ contains exactly one point. By Theorem 3.7 there are only finitely many possible such configurations. $\square$

Let P be a polytope in $\mathcal{N}(d, \alpha, \mathcal{A})$ with $\mathbb{Q}$ -codegree $c^{-1} = \text{codeg}_{\mathbb{Q}}(P)$ given by

$$P := \{ \mathbf{x} \mid \langle \mathbf{a}_i, \mathbf{x} \rangle \leq b_i \quad 1 \leq i \leq n \}$$

for some b_i , $n \geq m$, and $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_m\}$. Choose a relative interior point $\mathbf{x}_{\text{core}}$ in core P . Then $\langle \mathbf{a}_i, \mathbf{x}_{\text{core}} \rangle + c = b_i$ for $1 \leq i \leq m$. The b_i are the right-hand sides of our inequality description of a lattice polytope, so in particular, $\langle \mathbf{a}_i, \mathbf{x}_{\text{core}} \rangle + c \in \mathbb{Z}$ for $1 \leq i \leq m$. Let A be the $(m \times d)$ -matrix whose rows are the $\mathbf{a}_j$, $1 \leq j \leq m$. Let $\mathbf{1} \in \mathbb{R}^m$ be the all-ones vector. Then, in matrix form, this reads

$$A\mathbf{x}_{\text{core}} + c\mathbf{1} \in \mathbb{Z}^m.$$

In other words, whenever $\mathcal{A}$ is the set of core normals of a lattice polytope with $\mathbb{Q}$ -codegree c , then there is a rational point $\mathbf{y}$ such that

$$(3.3) \quad A\mathbf{y} + c\mathbf{1} \in \mathbb{Z}^m.$$

Hence, we have found a necessary condition that any pair $\mathcal{A}$ and c must satisfy if they come from a lattice polytope $P \in \mathcal{N}(d, \alpha, \mathcal{A})$. We use this to study possible values for c .

Lemma 3.10. *Let $\alpha, \varepsilon > 0$, $d \in \mathbb{N}$, and $\mathcal{A}$ be as above. Then*

$$\{\text{codeg}_{\mathbb{Q}}(P) \mid P \in \mathcal{N}(d, \alpha, \mathcal{A}) \text{ and } \text{codeg}_{\mathbb{Q}}(P) \geq \varepsilon\}$$

is finite.

Proof. Let A be the matrix with rows given by the vectors in $\mathcal{A}$ as above, and let $\bar{\mathbf{a}}_k$, $1 \leq k \leq d$ be the columns of A . Let $L_{\mathcal{A}} \subseteq \mathbb{R}^m$ be the linear span of $\bar{\mathbf{a}}_1, \dots, \bar{\mathbf{a}}_d$. See Figure 3.2.(b).

We will now show that (3.3) has only finitely many possible solutions for c in the range $0 < c \leq \varepsilon^{-1}$ (but, of course, for any given c such that (3.3) has at least one solution there are infinitely many solutions $\mathbf{y}$ for this fixed c). By assumption, the vectors $\mathbf{a}_1, \dots, \mathbf{a}_m \in \mathcal{A}$ positively span their linear span. Hence, $\text{rank}(\mathbf{a}_1, \dots, \mathbf{a}_m) \leq m - 1$, and $L_{\mathcal{A}}$ is a proper subspace of $\mathbb{R}^d$. Further, as there are $\lambda_1, \dots, \lambda_m \geq 0$ and not all zero such that $\mathbf{0} = \sum \lambda_i \mathbf{a}_i$, there is no $\mathbf{w} \in \mathbb{R}^d$ such that $\langle \mathbf{a}_j, \mathbf{w} \rangle > 0$ for all $1 \leq j \leq m$. In other words, $L_{\mathcal{A}}$ meets the positive orthant of $\mathbb{R}^m$ only in the origin. In particular, $\mathbf{1} \notin L_{\mathcal{A}}$.

Thus, there are only finitely many translates of the form $L_{\mathcal{A}} + c\mathbf{1}$ for $0 < c \leq \varepsilon^{-1}$ that contain a lattice point (namely those that are at distance $\frac{p}{\det \bar{M}}$ in direction $\mathbf{1}$, where M is the row vector matrix of any lattice basis of $L_{\mathcal{A}}$ and $\bar{M} = (M, \mathbf{1})$ the matrix M with an additional column of $\mathbf{1}$). Put differently, the set

$$\{c \mid A\mathbf{x} + c\mathbf{1} \in \mathbb{Z}^m \text{ for some } \mathbf{x} \in \mathbb{R}^d \text{ and } 0 < c \leq \varepsilon^{-1}\}$$

is finite. This proves the claim. $\square$

Combining this with Corollary 3.9 finally proves Theorem 3.1.

Remark 3.11. The proof of Lemma 3.1 also gives a way to explicitly compute a (finite superset of) the possible values of $\text{codeg}_{\mathbb{Q}}(P)$. Indeed, all possible values are contained in the set $\left\{ p (\det \bar{M})^{-1} \mid p \in \mathbb{Z}_{>0} \right\}$, where M is any matrix whose rows are a lattice basis of the lattice in the linear subspace $L_{\mathcal{A}}$ of the previous lemma and $\bar{M}$ is this matrix with an additional column of ones.

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