BOUNDARIES AND POLYHEDRAL BANACH SPACES

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ABSTRACT. We show that if X and Y are Banach spaces, where Y is separable and polyhedral, and if $T : X \rightarrow Y$ is a bounded linear operator such that $T^*(Y^*)$ contains a boundary B of X , then X is separable and isomorphic to a polyhedral space. Some corollaries of this result are presented.

1. THE MAIN RESULT

Let X be a Banach space. A subset $B \subset S_{X^*}$ of the unit sphere S_{X^*} of X is called a *boundary* of X if, for any $x \in X$, there is $f \in B$ satisfying $f(x) = \|x\|$. From the Krein-Milman Theorem, it follows that the set $\text{ext } B_{X^*}$ of all extreme points of the unit ball B_{X^*} of X^* is a boundary. Easy examples show that a boundary may be a proper subset of $\text{ext } B_{X^*}$. A separation theorem shows that, for any boundary B , we have $w^*\text{-cl}(\text{co } B) = B_{X^*}$. It is clear that if X is infinite-dimensional, then any boundary of X must be infinite.

Recall that a Banach space X is *polyhedral* if, given any finite-dimensional subspace $E \subseteq X$, the unit ball B_E of E is a polytope. If X has a countably infinite boundary, then it is separable and isomorphic to a polyhedral space (see [F]).

The following theorem is the main result of this paper.

Theorem 1.1. *Let X and Y be Banach spaces, where Y is separable and polyhedral. Assume that $T : X \rightarrow Y$ is a bounded linear operator such that $T^*(Y^*)$ contains a boundary B of X . Then X is separable and isomorphic to a polyhedral space.*

The proof of Theorem 1.1 uses the following result.

Proposition 1.2. *Let $T : X \rightarrow Y$ be a linear bounded operator such that $T^*(Y^*)$ contains a boundary B of X . Then $T^*(Y^*)$ is norm dense in X^* .*

Proof. We make use of the so-called (I)-property (see [FL]). Let $K \subset X^*$ be w^* -compact and convex, and suppose that $B \subset K$. We say that B has the (I)-property if, whenever $B = \bigcup_{i=1}^{\infty} B_i$, where $B_i \subset B_{i+1}$, then the set $\bigcup_{i=1}^{\infty} w^*\text{-cl}(\text{co } B_i)$ is norm-dense in K . It is proved in [FL] that, for K as above, any boundary B of K has the (I)-property ($B \subset K$ is a boundary of K if, for any $x \in X$, there is $f \in B$ such that $f(x) = \max x(K)$).

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Put $K = B_{X^*}$, and $B_i = T^*(iB_{Y^*}) \cap B$ for $i = 1, 2, \dots$. Clearly, $\bigcup_{i=1}^{\infty} w^*\text{-cl}(\text{co } B_i) \subset T^*(Y^*)$, and the result follows. $\square$

Proof of Theorem 1.1. Without loss of generality, we assume that $\|T\| = 1$. Since Y is a separable polyhedral space, it follows that Y^* is separable (see [F]), and hence by Proposition 1.2, X^* and X are separable too. Next, it is easily seen that T is injective. Also, without loss of generality, we can assume that T is a dense embedding, i.e. $\text{cl } T(X) = Y$ (if necessary, pass to the subspace $Y_1 = \text{cl } T(X)$ and the operator $T_1 : X \rightarrow Y_1$). In particular, we can assume that T^* is injective. Now set

$$W_n = T^{*-1}(nT^*(B_{Y^*}) \cap B_{X^*}), \quad n = 1, 2, \dots$$

Clearly,

$$B_{Y^*} \subset W_n \subset nB_{Y^*}, \quad n = 1, 2, \dots$$

(recall that $\|T\| = 1$).

The W_n are convex centrally symmetric w^* -compact bodies in Y^* (with Y separable and polyhedral). Choose a sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ of positive numbers tending to 0. By [DFH, Theorem 1.1], each such body W_n can be approximated by a convex centrally symmetric w^* -compact body A_n having a countable boundary, say $\{\pm h_i^n\}_{i=1}^{\infty}$, such that

- (a) $(1 - \varepsilon_n)A_n \subset W_n \subset A_n$, and
- (b) no w^* -limit point of $\{\pm h_i^n\}_{i=1}^{\infty}$ is a support point of A_n , for any $y \in Y$.

Define

$$U^* = w^*\text{-cl} \left(\text{co} \bigcup_{n=1}^{\infty} (1 + \varepsilon_n)T^*(A_n) \right), \quad t_i^n = T^*h_i^n, \quad n = 1, 2, \dots,$$

$$\tilde{B} = \{\pm(1 + \varepsilon_n)t_i^n\}_{i,n=1}^{\infty}.$$

It is easily seen that U^* is a convex centrally symmetric w^* -compact set in X^* , and $U^* = w^*\text{-cl}(\text{co } \tilde{B})$. Moreover, $B \subset U^*$ and hence $B_{X^*} \subset U^*$.

It follows that U^* , as the unit ball, defines an equivalent dual norm on X^* . We denote the corresponding norm on X by $\|\cdot\|$, and show that $(X, \|\cdot\|)$ is polyhedral. To prove this statement, it is enough to check that $\tilde{B}$ has (*) (see [FLP]), i.e. no w^* -limit point of $\tilde{B}$ is a support point of U^* , for any $x \in X$.

If f is a w^* -limit point of $\tilde{B}$, then, by the separability of X , we may find a sequence of distinct points in $\tilde{B}$ that converges to f in the w^* -topology. There are two possibilities. The first possibility is that, for some fixed n , we can write $f = w^*\text{-lim}_k (1 + \varepsilon_n)t_{i_k}^n \in (1 + \varepsilon_n)T^*(A_n)$, where $i_1 < i_2 < \dots$. Then $T^{*-1}f$ is a w^* -limit point of the set $\{\pm(1 + \varepsilon_n)h_i^n\}_{i=1}^{\infty}$ and, by (b) above, it cannot be a support point of $(1 + \varepsilon_n)A_n$ for any $y \in Y$. Therefore, f is not a support point of $(1 + \varepsilon_n)T^*(A_n)$, for any $x \in X$. Since $f \in (1 + \varepsilon_n)T^*(A_n) \subset U^*$ it follows that f is not a support point of U^* , for any $x \in X$.

If the first possibility above does not hold, then we can write

$$f = w^*\text{-lim}_k (1 + \varepsilon_{n_k})t_{i_k}^{n_k},$$

where $n_k \rightarrow \infty$ as $k \rightarrow \infty$. Assume to the contrary that f is a support point of U^* with $f(x_0) = \max x_0(U^*) \neq 0$, $x_0 \in X$. By (a), it is easily seen that $f \in B_{X^*}$. Since $B_{X^*} \subset U^*$, it follows that $f(x_0) = \max x_0(B_{X^*}) = \|x_0\| \neq 0$. As B is a boundary of X , there is $g \in B$ satisfying $g(x_0) = \|x_0\|$. Next, because $B \subset T^*(Y^*)$, it easily follows that $B \subset \bigcup_{n=1}^{\infty} T^*(A_n)$, and hence $g \in T^*(A_m)$ for

some m . However, $(1+\varepsilon_m)g \in U^*$ implies $\max x_0(U^*) \geq (1+\varepsilon_m)\|x_0\|$, contradicting $\|x_0\| = f(x_0) = \max x_0(U^*) \neq 0$. The proof is complete. $\square$

2. CONSEQUENCES

Corollary 2.1. *Let X be a Banach space and Y_i , $i = 1, 2, \dots$, be a sequence of separable isomorphically polyhedral Banach spaces. Let B be a boundary of X and $T_i : X \rightarrow Y_i$ a sequence of bounded linear operators such that*

$$B \subset \bigcup_{i=1}^{\infty} T_i^*(Y_i^*).$$

Then X is separable and isomorphically polyhedral.

Proof. Without loss of generality we can assume that Y_i is polyhedral and $\|T_i\| = 1$ for all i . Let $Y = (\bigoplus_{i=1}^{\infty} Y_i)_{c_0}$. Clearly Y is a separable isomorphically polyhedral space. Define $T : X \rightarrow Y$ by $(Tx)_i = i^{-1}T_i x \in Y_i$. Evidently, $Y^* = (\bigoplus_{i=1}^{\infty} Y_i^*)_{\ell_1}$, and if $f \in Y^*$ and $x \in X$, we have

$$(T^*f)(x) = f(Tx) = \sum_{i=1}^{\infty} i^{-1}f_i(T_i x) = \sum_{i=1}^{\infty} (i^{-1}T_i^*f)(x),$$

where $f = (f_i)_{i=1}^{\infty}$ and $\|f\| = \sum_{i=1}^{\infty} \|f_i\|$. In particular, if $g \in Y_i^* \subset Y^*$ we have $T^*g = i^{-1}T_i^*g$, whence

$$B \subset \bigcup_{i=1}^{\infty} T_i^*(Y_i^*) \subset T^*(Y^*).$$

Apply Theorem 1.1 to finish the proof. $\square$

Corollary 2.2. *Assume that X has a boundary B which is contained in a set of the form*

$$\bigcup_{i=1}^{\infty} w^*\text{-cl}(\text{co } K_i),$$

where the K_i are countable w^ -compact subsets of X^* . Then X is isomorphically polyhedral.*

Proof. For $i \in \mathbb{N}$, let Y_i be the isomorphically polyhedral space $C(K_i)$, and define $T_i : X \rightarrow Y_i$ by $(T_i x)(t) = t(x)$, $x \in X$, $t \in K_i$. Then $w^*\text{-cl}(\text{co } K_i) \subset T_i^*(B_{C(K_i)^*})$, and we can apply Corollary 2.1. $\square$

We will see later in Remark 2.6 that we cannot drop the requirement that the K_i above are w^* -compact. Our final result includes an application of the material above, stated in terms of M-bases.

Definition 2.3. Let $\{x_i\} \subset X$ be an M-basis of a Banach space X , having biorthogonal sequence $\{x_i^*\} \subset X^*$. We call a subset $A \subset X^*$ *summable* if $\sum_{i=1}^{\infty} |f(x_i)| < \infty$ for all $f \in A$.

Lemma 2.4. *Let $B = \{\pm f_i\} \subset S_{X^*}$ be a countable boundary of X , take a sequence of numbers $\{\varepsilon_i\}$, $0 < \varepsilon_i < \frac{1}{2}$, $\lim_i \varepsilon_i = 0$, and a sequence of vectors $\{t_i\} \subset X^*$ satisfying $\|f_i - t_i\| < \varepsilon_i$. Then the sequence $\{\pm h_i\}$, where $h_i = (1 + 2\varepsilon_i)t_i$, $i = 1, 2, \dots$, is a boundary having $(*)$ with respect to the norm it generates, given by*

$$\|x\| = \sup_i |h_i(x)|, \quad x \in X.$$

Proof. First, note that for any $x \in X$, $x \neq 0$, we have

$$(1) \quad |||x||| > \|x\|.$$

Indeed, if $f_i(x) = \|x\|$, then

$$\begin{aligned} |||x||| &\geq (1 + 2\varepsilon_i)t_i(x) \\ &\geq (1 + 2\varepsilon_i)f_i(x) - (1 + 2\varepsilon_i)\|t_i - f_i\| \|x\| \\ &\geq (1 + 2\varepsilon_i)(1 - \varepsilon_i)\|x\| > \|x\|. \end{aligned}$$

Put $V^* = \{f \in X^* : |||f||| \leq 1\}$, $S_{V^*} = \partial V^*$. By using the Hahn-Banach Theorem and (1), we easily obtain $B_{X^*} \subset V^*$ and, again by using (1), we see that no functional $f \in S_{V^*} \cap B_{X^*}$ (if any) attains its norm with respect to $|||\cdot|||$. However, any w^* -limit point g of B_1 satisfying $|||g||| = 1$ (if any) lies in B_{X^*} (recall that $\lim_i \varepsilon_i = 0$). The proof is complete. $\square$

Theorem 2.5. *For a separable Banach space X , the following assertions are equivalent:*

- (a) X admits a boundary B and a bounded linear operator $T : X \rightarrow c_0$ such that $T^*(c_0^*) \supset B$.
- (b) X admits a boundary B and a bounded linear operator $T : X \rightarrow Y$ into a polyhedral space Y such that $T^*(Y^*) \supset B$.
- (c) X is isomorphically polyhedral.
- (d) X admits an equivalent norm having a boundary B , which is summable with respect to a normalized M-basis $\{x_i\}$ with bounded biorthogonal sequence $\{x_i^*\}$.

Proof. (a) $\Rightarrow$ (b) is trivial, while (b) $\Rightarrow$ (c) is Theorem 1.1. To prove (c) $\Rightarrow$ (d), we can assume without loss of generality that X polyhedral. By [F1], X admits a countable boundary $\{f_i\}$, and X^* is separable. It is well-known (see for instance [FHHMVZ, Theorem 4.59]) that X admits a (shrinking) normalized M-basis $\{x_i\}$ with bounded biorthogonal sequence $\{x_i^*\}$. By using Lemma 2.4, we easily obtain a sequence $B = \{h_i\} \subset \text{span}\{x_i^*\}$ which is a boundary of X with respect to an equivalent norm. Clearly, B is summable with respect to the M-basis $\{x_i\}$. Finally, we prove (d) $\Rightarrow$ (a). Let B be a boundary of X which is summable with respect to a normalized M-basis $\{x_i\}$ with bounded biorthogonal sequence $\{x_i^*\}$. Define $T : X \rightarrow c_0$ by

$$Tx = (x_i^*(x))_{i=1}^\infty, \quad x \in X.$$

Evidently, $\|T\| = \sup_i \|x_i^*\| < \infty$. If $\{e_i\}$ is the natural basis of $\ell_1 = c_0^*$, then it is easily seen that $T^*e_i = x_i^*$, $i = 1, 2, \dots$. Since B is summable, it follows that $T^*(c_0^*) \supset B$. $\square$

Remark 2.6. We cannot replace the x_i^* in Theorem 2.5 (d), even with a normalized basis of X^* , if its biorthogonal sequence does not belong to X . Indeed, if X^* is isomorphic to ℓ_1 , then it admits a normalized basis, with respect to which every element of X^* is an absolutely summable combination. However, there exist non-isomorphically polyhedral Banach spaces having duals isomorphic to ℓ_1 . The spaces in e.g. [BD] and [AH] have duals isomorphic to ℓ_1 but do not contain any isomorphic copies of c_0 , and in order for a Banach space to be isomorphically polyhedral, it

is necessarily c_0 -saturated [F]. In fact, any space X having a separable dual space (many of which are not isomorphically polyhedral) embeds isomorphically into a space having a dual space isomorphic to ℓ_1 [FOS]. The same examples show that it is necessary for the K_i in Corollary 2.2 to be w^* -compact.

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