THE BEHAVIOR OF THE CHERN SCALAR CURVATURE UNDER THE CHERN-RICCI FLOW

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ABSTRACT. In this paper we study finite-time singularities in the Chern-Ricci flow. We show that finite-time singularities are characterized by the blow-up of the scalar curvature of the Chern connection.

1. INTRODUCTION

Let (M, g, J) be a complex manifold such that J is compatible with the Riemannian metric g . Associated to g , we have the Hermitian 2-form ω given by $\omega(X, Y) = g(J(X), Y)$ for vector fields X and Y . We call a manifold admitting such a structure a *Hermitian manifold*. Recently a number of geometric flows of Hermitian metrics have been introduced to study the underlying Hermitian structure of complex manifolds (e.g. [2], [8], [6], [7], and [3]). The Chern-Ricci flow was introduced by the first-named author in [2] and further developed by Tosatti-Weinkove in [8]. The Chern-Ricci flow is written

$$(1.1) \quad \begin{aligned} \frac{\partial}{\partial t} \omega &= -\text{Ric}^C(\omega), \\ \omega(0) &= \omega_0 \end{aligned}$$

where $\text{Ric}^C(\omega) = -\sqrt{-1}\partial\bar{\partial}\log\det(g)$. In other words, $\text{Ric}^C(\omega)$ is the curvature of the anti-canonical bundle. Notice that when ω is a Kähler metric, $\text{Ric}^C(\omega)$ coincides with the Ricci curvature of ω . Hence if ω_0 is Kähler, then (1.1) is the Kähler-Ricci flow.

The Chern-Ricci flow is a diffusion-reaction equation and therefore one expects the development of singularities in the flow. In fact, explicit examples of singularity development are described by Tosatti-Weinkove in [8], [10], and Tosatti-Weinkove-Yang in [9]. Moreover in [8] Tosatti-Weinkove give characterizations of finite-time singularities. To describe these characterizations, first notice that the Chern-Ricci flow (1.1) can be written

$$\begin{aligned} \frac{\partial}{\partial t} \omega &= -\text{Ric}^C(\omega_0) + \sqrt{-1}\partial\bar{\partial}\theta(t), \\ \omega(0) &= \omega_0 \end{aligned}$$

where $\theta(t) = \log \frac{\det(g(t))}{\det(g_0)}$. Hence the solution $\omega(t)$ can be expressed as

$$(1.2) \quad \omega(t) = \alpha_t + \sqrt{-1}\partial\bar{\partial}\phi(t),$$

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where $\alpha_t = \omega_0 - t \operatorname{Ric}^C(\omega_0)$ and $\phi(t)$ is a function satisfying $\frac{\partial}{\partial t} \phi(t) = \theta(t)$ and $\phi|_{t=0} = 0$. Let

$$T = \sup\{t \geq 0 \mid \exists \psi \in C^\infty(M) \text{ with } \alpha_t + \sqrt{-1} \partial \bar{\partial} \psi > 0\}.$$

It is clear that the flow cannot exist beyond time T . It was shown by Tosatti-Weinkove that the flow (1.1), in fact, exists up to time T .

Theorem (Tosatti-Weinkove [8]). *There exists a unique maximal solution to the Chern-Ricci flow on $[0, T)$.*

In addition, Tosatti-Weinkove give the following description of finite-time singularities on complex surfaces.

Theorem (Tosatti-Weinkove [8]). *Let M be a compact complex surface and ω_0 a $\partial\bar{\partial}$ -closed Hermitian metric. Then the Chern-Ricci flow starting at ω_0 exists until either the volume of M goes to zero, or the volume of a curve of negative self-intersection goes to zero.*

In this note we give an alternative characterization of finite-time singularities in terms of the blow-up of the scalar curvature. In particular we prove

Theorem 1.1. *Let M be a compact complex manifold of complex dimension n and ω_0 a Hermitian metric. Then the solution $\omega(t)$ of the Chern-Ricci flow (1.1) exists on the maximal interval $[0, T)$ and either $T = \infty$ or*

$$\limsup_{t \rightarrow T} \left(\sup_M R(g(t)) \right) = \infty$$

where R denotes the scalar curvature of the Chern connection.

This is a generalization of Zhang's analogous result for Kähler-Ricci flow [11] and the proof employs similar arguments. Tosatti-Weinkove-Yang have also investigated the Chern scalar curvature along the normalized Chern-Ricci flow in the case of elliptic surfaces (with genus strictly larger than one). In particular, they show that the Chern scalar curvature is bounded above by $Ce^{t/2}$ and below by $-C$ and apply this estimate to show that the flow converges exponentially fast to the base in the Gromov-Hausdorff sense [10].

Now we specialize to the case when (M, g_0) is a compact Gauduchon surface, i.e. the associated $(1, 1)$ form ω_0 of g_0 has the property that

$$(1.3) \quad \partial \bar{\partial} \omega_0 = 0.$$

A theorem of Gauduchon shows that every Hermitian metric on a complex surface is conformal to a unique Gauduchon metric. When the Chern-Ricci flow terminates at a finite time T , there are two cases to consider: finite-time collapsing and non-collapsing. If $\operatorname{Vol}(M, g(t)) \rightarrow 0$ as $t \rightarrow T$, we say that the flow is *collapsing*, otherwise we say it is *non-collapsing*. Tosatti-Weinkove consider the case of finite-time non-collapsing on compact Gauduchon surfaces and show that there are finitely many (-1) curves on M away from which the flow converges smoothly on compact sets [10].

Using the proof of Theorem 1.1 and its result, we are able to show that the set on which singularities develop on M is equal to the region on which Chern scalar curvature blows up. Collins-Tosatti have an analogous result in the Kähler case [1]. Let $g(t)$ be the solution to the Chern-Ricci flow that is non-collapsing in finite

time. Let E denote the union of the (-1) curves on M that are collapsing as $t \rightarrow T$. Define the singularity formation set Σ to be the complement of all points x in M such that there exists an open neighborhood U of x on which $|\text{Rm}(t)| \leq C$ for all $0 \leq t < T$. Also define the set Σ' to be the complement of all points x in M such that there exists an open neighborhood U of x on which $|R(t)| \leq C$ for all $0 \leq t < T$. Then we have the following result.

Theorem 1.2. *Let (M, g_0) be a compact Gauduchon surface that is non-collapsing in finite time along the Chern-Ricci flow. Then*

$$E = \Sigma = \Sigma'.$$

If instead the flow is collapsing in finite time, we have

$$M = \Sigma = \Sigma'.$$

In the final section we include an explicit example of the Chern scalar curvature blowing up like $C(T - t)^{-1}$ along the Chern-Ricci flow in the case of the Hopf manifold.

2. PRELIMINARIES

Let (M, g, J) be a closed Hermitian manifold of complex dimension n . The *Chern connection* is the unique connection ∇ satisfying: $\nabla\omega = 0$, $\nabla J = 0$, and $T^{1,1} = 0$. In local holomorphic coordinates, $z_1, \dots, z_n$, the only non-vanishing components of the Chern connection are

$$(2.1) \quad \Gamma_{i\bar{j}}^k = g^{k\bar{l}} \partial_i g_{j\bar{l}}$$

and its conjugate. The curvature of the Chern connection is given locally by

$$(2.2) \quad R_{i\bar{j}k}^l = -\partial_{\bar{j}} \Gamma_{ik}^l.$$

Taking a trace of the Chern curvature over the last two components yields the Chern-Ricci curvature, that is,

$$(2.3) \quad R_{i\bar{j}}^C = g^{k\bar{l}} R_{i\bar{j}k\bar{l}} = -\partial_i \partial_{\bar{j}} \log \det(g).$$

Interestingly, the parabolic flow of Hermitian metrics defined by Streets-Tian in [7] is the trace of the Chern curvature over the first two components plus a term that is quadratic in the torsion of ∇ . Finally, the *Chern scalar curvature*, denoted R , is defined as the trace of the Chern-Ricci curvature

$$R = g^{i\bar{j}} R_{i\bar{j}}^C.$$

One of the reasons for the recent success of the Chern-Ricci flow is that, like Kähler-Ricci flow, the Chern-Ricci flow is equivalent to a scalar flow. To see this, first recall from (1.2) that the solution of the Chern-Ricci flow can be written

$$(2.4) \quad \omega(t) = \alpha_t + \sqrt{-1} \partial \bar{\partial} \phi(t).$$

Differentiating (2.4) in time, we have

$$(2.5) \quad -\text{Ric}^C(t) = \sqrt{-1} \partial \bar{\partial} \log \det(g_0) + \sqrt{-1} \partial \bar{\partial} \left(\frac{\partial}{\partial t} \phi(t) \right).$$

Rearranging terms and using (2.4) yields

$$\begin{aligned}\sqrt{-1}\partial\bar{\partial}\left(\frac{\partial}{\partial t}\phi(t)\right) &= \sqrt{-1}\partial\bar{\partial}(\log \det(g(t)) - \log \det(g_0)) \\ &= \sqrt{-1}\partial\bar{\partial} \log \frac{(\alpha_t + \sqrt{-1}\partial\bar{\partial}\phi(t))^n}{\omega_0^n}.\end{aligned}$$

Therefore the Chern-Ricci flow is equivalent to the scalar complex Monge-Ampère equation

$$(2.6) \quad \frac{\partial}{\partial t}\phi = \log \frac{(\alpha_t + \sqrt{-1}\partial\bar{\partial}\phi(t))^n}{\omega_0^n}, \text{ with } \phi|_{t=0} = 0.$$

The proof of Theorem 1.1 will repeatedly exploit the fact that the Chern-Ricci flow (1.1) is equivalent to the scalar flow (2.6).

3. PROOF OF THEOREM 1.1

In this section we show that finite-time singularities of the Chern-Ricci flow occur if and only if the scalar curvature is unbounded as t approaches the singular time.

Proof. We will proceed in the same fashion as Zhang in [11] (cf. Section 7.2 of [5]). Let $\omega(t)$ be a solution of the Chern-Ricci flow (1.1) on the maximal interval $[0, T)$ with $T < \infty$. Suppose, by way of contradiction, that there exists a constant C such that

$$(3.1) \quad \limsup_{t \rightarrow T} \left(\sup_M R(g(t)) \right) \leq C.$$

Throughout this section C will denote a constant which is independent of time, however this constant may change from line to line. We will show that if (3.1) holds, then for any $k \geq 0$ we can produce uniform C^k bounds on the metric $\omega(t)$ and hence T is not maximal. We begin by obtaining uniform C^0 estimates on $\omega(t)$ for $t \in [0, T)$.

First, by studying the evolution of the scalar curvature $R(t)$ under (1.1), we will show that the C^0 norm of $R(t)$ is uniformly bounded on $[0, T)$.

Lemma 3.1. *Under the Chern-Ricci flow (1.1) the scalar curvature of the Chern connection evolves by*

$$\frac{\partial}{\partial t}R = \Delta R + |\operatorname{Ric}^C(\omega)|^2.$$

Proof. In local holomorphic coordinates, at a point $p \in M$, we compute

$$\begin{aligned}\frac{\partial}{\partial t}R &= \frac{\partial}{\partial t} \left(g^{i\bar{j}} R_{i\bar{j}}^C \right) \\ &= g^{k\bar{j}} g^{i\bar{p}} R_{k\bar{p}}^C R_{i\bar{j}}^C - g^{i\bar{j}} \partial_i \partial_{\bar{j}} \left(g^{k\bar{l}} \frac{\partial}{\partial t} g_{k\bar{l}} \right) \\ &= \Delta R + |\operatorname{Ric}^C(\omega)|^2.\end{aligned}$$

□

Hence by Lemma 3.1 and the maximum principle, it follows that $R(t)$ is uniformly bounded below on $[0, T)$. Combining this lower bound with (3.1), we have that $\|R(t)\|_{C^0}$ is uniformly bounded on $[0, T)$.

Next we will use the bounds on the scalar curvature $R(t)$ to obtain bounds on the determinant of $\omega(t)$, which are independent of t . Recall from (2.6) that the Chern-Ricci flow is equivalent to the complex Monge-Ampère equation $\frac{\partial}{\partial t}\phi = \log\left(\frac{\omega^n}{\omega_0^n}\right)$. Now, since $\frac{\partial}{\partial t}\log(\omega^n) = -R$, we have

$$(3.2) \quad \frac{\partial}{\partial t}\dot{\phi} = -R.$$

Consequently,

$$(3.3) \quad \left|\frac{\partial}{\partial t}\dot{\phi}\right| = |R| \leq C.$$

Integrating (3.3) in time we have that there exists a constant C such that

$$(3.4) \quad \left|\log\left(\frac{\omega^n}{\omega_0^n}\right)\right| = \left|\frac{\partial}{\partial t}\phi\right| \leq C.$$

Notice that (3.4) gives uniform bounds on the determinant of $\omega(t)$ on $[0, T)$. Integrating in time again gives uniform bounds for $|\varphi|$. Given uniform estimates on the determinant of the metric, to prove that $\omega(t)$ is uniformly bounded in C^0 , it suffices to show that the trace of the metric $\text{tr}_{g_0} g$ is bounded above.

For a bound on the trace of the metric, we repeat the estimates derived by Tosatti-Weinkove in Section 4 of [8] with minor modifications. First let

$$(3.5) \quad Q_1 \doteq t\dot{\phi} - \phi - nt$$

and notice that Q_1 is uniformly bounded. Combining the trace of (2.5) with (3.2), it follows that

$$\left(\frac{\partial}{\partial t} - \Delta\right)\dot{\phi} = \text{tr}_{\omega}(-\text{Ric}^C(\omega_0)).$$

Taking the trace of (2.4), we have $\Delta\phi = \text{tr}_{\omega}(t\text{Ric}^C(\omega_0) - \omega_0) + n$ and so,

$$(3.6) \quad \left(\frac{\partial}{\partial t} - \Delta\right)Q_1 = -\text{tr}_{\omega}\omega_0.$$

Let $\tilde{C}$ denote a constant which is sufficiently large so that $\phi + \tilde{C} \geq 1$ and let B be a large constant to be chosen below. Next, define

$$(3.7) \quad Q_2 \doteq \log \text{tr}_{g_0} g - \phi + \frac{1}{\phi + \tilde{C}} + BQ_1.$$

Notice that each term in Q_2 , with the possible exception of $\log \text{tr}_{g_0} g$, is uniformly bounded. Therefore it is enough to show that Q_2 is bounded above at a point where Q_2 achieves its maximum.

By Proposition 3.1 in [8], we have the following estimate on the heat operator applied to $\log \text{tr}_{g_0} g$.

Proposition 3.2 (Tosatti-Weinkove [8]). *Given a solution $g(t)$ of the Chern-Ricci flow (1.1),*

$$(3.8) \quad \left(\frac{\partial}{\partial t} - \Delta\right)\log \text{tr}_{g_0} g \leq \frac{2}{(\text{tr}_{g_0} g)^2} \text{Re}\left(g^{\bar{l}k}(T_0)_{kp}^p \partial_{\bar{l}} \text{tr}_{g_0} g\right) + C \text{tr}_g g_0.$$

Now we may assume that at a maximum of Q_2 that $\text{tr}_{g_0} g \geq 1$ since otherwise we have the desired upper bound. For a bound on the first term in (3.8), notice that at a maximum of Q_2 , $\partial_i Q_2 = 0$. Equivalently,

$$(3.9) \quad \frac{1}{\text{tr}_{g_0} g} \partial_i \text{tr}_{g_0} g - \partial_i \phi - \frac{1}{(\phi + \tilde{C})^2} \partial_i \phi = 0.$$

Hence at a maximum of Q_2 ,

$$(3.10) \quad \begin{aligned} & \left| \frac{2}{(\text{tr}_{g_0} g)^2} \text{Re} \left(g^{\bar{l}k} (T_0)_{kp}^p \partial_{\bar{l}} \text{tr}_{g_0} g \right) \right| \\ & \leq \left| \frac{2}{\text{tr}_{g_0} g} \text{Re} \left(\left(1 + \frac{1}{(\phi + \tilde{C})^2} \right) g^{\bar{l}k} (T_0)_{kp}^p (\partial_{\bar{l}} \phi) \right) \right| \\ & \leq \frac{|\partial \phi|_g^2}{(\phi + \tilde{C})^3} + C(\phi + \tilde{C})^3 \frac{\text{tr}_g g_0}{(\text{tr}_{g_0} g)^2}. \end{aligned}$$

Notice that we may also assume that at a maximum of Q_2 , $(\text{tr}_{g_0} g)^2 \geq (\phi + \tilde{C})^3$ since if this is not the case then we again have the desired upper bound. And so, by (3.6), (3.8), and (3.10), at a maximum

$$(3.11) \quad \begin{aligned} 0 & \leq \left(\frac{\partial}{\partial t} - \Delta \right) Q_2 \\ & \leq \frac{|\partial \phi|_g^2}{(\phi + \tilde{C})^3} + C \text{tr}_g g_0 - \left(1 + \frac{1}{(\phi + \tilde{C})^2} \right) \dot{\phi} + \left(1 + \frac{1}{(\phi + \tilde{C})^2} \right) \text{tr}_g (g - \alpha_t) \\ & \quad - \frac{2}{(\phi + \tilde{C})^3} |\partial \phi|_g^2 - B \text{tr}_g g_0 \\ & \leq (C - B) \text{tr}_g g_0 + C' \end{aligned}$$

where C' comes from the bounds on ϕ and $\dot{\phi}$. Also notice that we used the bound

$$\begin{aligned} -\text{tr}_g \alpha_t &= -\text{tr}_g g_0 + t \text{tr}_g (\text{Ric}^C(g_0)) \\ &\leq -\text{tr}_g g_0 + \hat{C} \text{tr}_g g_0 \end{aligned}$$

where $\hat{C}$ is a constant that depends on the geometry of g_0 and harmlessly on t . Now we choose $B = C + 1$. Hence

$$(3.12) \quad \text{tr}_g g_0 \leq C'.$$

Finally, using the bounds on the determinant of the metric (3.4) and (3.12),

$$(3.13) \quad \text{tr}_{g_0} g \leq \frac{1}{(n-1)!} (\text{tr}_g g_0)^{n-1} \frac{\det g}{\det g_0} \leq C''.$$

Thus we have shown that Q_2 is uniformly bounded above and so $\text{tr}_{g_0} g$ is bounded above. This proves the uniform C^0 bounds on $\omega(t)$; hence there exists a constant C so that

$$\frac{1}{C} \omega_0 \leq \omega(t) \leq C \omega_0$$

for $t \in [0, T)$.

The higher order estimates on the metric follow from the first-named author in [2]. This proves the theorem. $\square$

4. PROOF OF THEOREM 1.2

The proof follows rather quickly from the previous proof and the results of Tosatti-Weinkove and follows a method similar to that in [1]. In the case of finite-time non-collapsing, Tosatti-Weinkove have shown that there exist finitely many disjoint (-1) curves $E_1, \dots, E_l$ on M with $\text{Vol}(E_i, g(t)) \rightarrow 0$ as $t \rightarrow T$ [8]. This immediately implies that $E \subset \Sigma$. Tosatti-Weinkove also proved that the flow converges smoothly on compact sets away from E , giving the reverse containment since no singularities develop outside of E and hence $E = \Sigma$ [10].

Trivially, we have $\Sigma' \subset \Sigma = E$. It remains to show that for any x in M with an open neighborhood U on which $|R(t)| \leq C$ for all $0 \leq t < T$, we have smooth convergence on a smaller open neighborhood U' of x . A key observation is that most of the estimates from the proof of Theorem 1.1 are in fact local and therefore will be applied again here.

First let K be a compact subset of U and let U' be an open neighborhood of x contained in K . We will derive uniform C^0 estimates on $\omega(t)$ in K and thus on U' . Since $|R(t)|$ is bounded on U , it follows from (3.3) and (3.4) that the determinant of $\omega(t)$ is bounded on $K \times [0, T)$. To show that $\omega(t)$ is bounded in C^0 on $K \times [0, T)$, it is enough to show that $\text{tr}_{g_0} g$ is bounded above on this set.

Let Q_1 be defined as above in (3.5) and let

$$Q_2 = \log \text{tr}_{g_0} g - \phi + \frac{1}{\phi + \tilde{C}} + BQ_1$$

as in (3.7). Since $|R(t)|$ is uniformly bounded on K , we again have that each term in Q_2 , except possibly $\log \text{tr}_{g_0} g$, is uniformly bounded. It remains to show that Q_2 is bounded above at a point where Q_2 achieves its maximum in K .

For the term $\log \text{tr}_{g_0} g$ in Q_2 , we have that

$$\left(\frac{\partial}{\partial t} - \Delta \right) \log \text{tr}_{g_0} g \leq \frac{2}{(\text{tr}_{g_0} g)^2} \text{Re} \left(g^{\bar{l}k}(T_0)_{kp}^p \partial_{\bar{l}} \text{tr}_{g_0} g \right) + C \text{tr}_g g_0$$

by Proposition 3.1 in [8]. Furthermore at a maximum of Q_2 in K , from (3.10) we have

$$\left| \frac{2}{(\text{tr}_{g_0} g)^2} \text{Re} \left(g^{\bar{l}k}(T_0)_{kp}^p \partial_{\bar{l}} \text{tr}_{g_0} g \right) \right| \leq \frac{|\partial \phi|_g^2}{(\phi + \tilde{C})^3} + C(\phi + \tilde{C})^3 \frac{\text{tr}_g g_0}{(\text{tr}_{g_0} g)^2}.$$

Combining the two estimates above with (3.6) yields

$$0 \leq \left(\frac{\partial}{\partial t} - \Delta \right) Q_2 \leq (C - B) \text{tr}_g g_0 + C';$$

see (3.11). So by choosing B sufficiently large, in particular $B = C + 1$, it follows that

$$\text{tr}_g g_0 \leq C'$$

at a maximum in $K \times [0, T)$. And so, as in (3.13), there exists a constant C'' so that

$$\text{tr}_{g_0} g \leq C''.$$

This proves that Q_2 , and thus $\text{tr}_{g_0} g$, is bounded above on $K \times [0, T)$. Finally, since $U' \subset K$, we have local C^0 estimates on $\omega(t)$ in $U' \times [0, T)$. The local higher order estimates follow by work of Sherman-Weinkove [4]. $\square$

5. THE HOPF MANIFOLD

Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n \setminus \{0\}$ with $|\alpha_1| = \dots = |\alpha_n| \neq 1$. Then consider the Hopf manifold $M_\alpha = (\mathbb{C} \setminus \{0\})/\sim$ where $(z_1, \dots, z_n) \sim (\alpha_1 z_1, \dots, \alpha_n z_n)$. Let

$$(5.1) \quad \omega_H = \frac{\delta_{i\bar{j}}}{r^2} \sqrt{-1} dz^i \wedge dz^{\bar{j}},$$

$$(5.2) \quad \text{Ric}^C(\omega_H) = \frac{n}{r^2} \left(\delta_{i\bar{j}} - \frac{z_i z_{\bar{j}}}{r^2} \right) \sqrt{-1} dz^i \wedge dz^{\bar{j}}.$$

Tosatti-Weinkove show that the metric

$$(5.3) \quad \omega(t) = \omega_H - t \text{Ric}^C(\omega_H) = \left(\frac{\delta_{i\bar{j}}}{r^2} - \frac{nt}{r^2} \left(\delta_{i\bar{j}} - \frac{z_i z_{\bar{j}}}{r^2} \right) \right) \sqrt{-1} dz^i \wedge dz^{\bar{j}}$$

is a solution to the Chern-Ricci flow on the Hopf manifold [8]. In particular, they show that $\text{Ric}^C(\omega(t)) = \text{Ric}^C(\omega_H)$.

Computing directly, we see that the Chern scalar curvature blows up at $T = \frac{1}{n}$ and additionally

$$(5.4) \quad R(\omega(t)) = \text{tr}_{\omega(t)} \text{Ric}^C(\omega_H) \leq \frac{C}{\frac{1}{n} - t}.$$

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