NEW PROOFS OF TWO q -ANALOGUES OF KOSHY'S FORMULA

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ABSTRACT. In this paper we prove a q -analogue of Koshy's formula in terms of the Narayana polynomial due to Lassalle, and a q -analogue of Koshy's formula in terms of q -hypergeometric series due to Andrews, by applying the inclusion-exclusion principle on Dyck paths and on partitions. We generalize these two q -analogues of Koshy's formula for q -Catalan numbers to that for q -Ballot numbers. This work also answers an open question by Lassalle and two questions raised by Andrews in 2010. We conjecture that if n is odd, then for $m \geq n \geq 1$, the polynomial $(1 + q^n) \begin{bmatrix} m \\ n-1 \end{bmatrix}_q$ is unimodal. If n is even, for any even $j \neq 0$ and $m \geq n \geq 1$, the polynomial $(1 + q^n) [j]_q \begin{bmatrix} m \\ n-1 \end{bmatrix}_q$ is unimodal. This implies the answer to the second problem posed by Andrews.

1. INTRODUCTION AND BACKGROUND

Let $C_n = \frac{1}{n+1} \binom{2n}{n}$ be the n -th Catalan number. A recursive formula for Catalan numbers is given by Koshy as follows [8]:

$$(1.1) \quad \sum_{r=0}^n (-1)^r \binom{n-r+1}{r} C_{n-r} = 0.$$

Since this is a hypergeometric sum identity, it can be proved by Zeilberger's creative telescoping method [14]. Furthermore, Koshy's formula (1.1) follows immediately from the orthogonality of a special Fibonacci polynomial due to Cigler [5, 6]. However, these approaches do not provide structural insight; to this end a combinatorial proof based on the inclusion-exclusion principle is needed. Our proof of (1.1) in the context of labeled elevated Dyck paths is exactly of this kind. Here, a Dyck path of length $2n$ is a 2-dimensional lattice path from $(0, 0)$ to $(2n, 0)$ with allowed steps $(1, 1)$ (U -step) and $(1, -1)$ (D -step) that never goes below the x -axis. A Dyck path p is an elevated Dyck path if $p = U\bar{p}D$ where $\bar{p}$ is a Dyck path. We use D_n to denote the set of elevated Dyck paths of length $2n + 2$. Then $|D_n| = C_n$. We denote by P the peak UD . We call h an up-peak if $h = UUD$. Let $\mathcal{UP}(p)$ (resp. $\mathcal{U}(p)$) be the set of up-peaks (resp. U -steps) contained in a path p . We denote by $(\mathcal{M}^s, j, D_n)$ the set of elevated Dyck paths $p \in D_n$ having exactly j elements from the set $\mathcal{M}(p)$ labeled by s . We set $a_{n,j}(\mathcal{M}) = |(\mathcal{M}^s, j, D_n)|$. For the case $\mathcal{M} = \mathcal{UP}$, the number $a_{n,j}(\mathcal{UP})$ can be counted in two different ways.

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First, there is a bijection between the set $((\mathcal{UP})^s, j, D_n)$ and the set $(\mathcal{U}^s, j, D_{n-j})$. For a given path $p \in D_n$ that has j up-peaks labeled by s , we can get a unique path $p' \in D_{n-j}$ that has j U -steps labeled by s if we remove the peak P from each labeled up-peak. As a result, we have

$$(1.2) \quad a_{n,j}(\mathcal{UP}) = |((\mathcal{UP})^s, j, D_n)| = |(\mathcal{U}^s, j, D_{n-j})| = \binom{n-j+1}{j} C_{n-j}.$$

Second, we set $g_{n,m} = |\{p \in D_n : |\mathcal{UP}(p)| = m\}|$. For a given elevated Dyck path $p \in D_n$ such that $|\mathcal{UP}(p)| = m$, there are $\binom{m}{j}$ ways to label j up-peaks by s , which leads to $a_{n,j}(\mathcal{UP}) = \sum_{m \geq j} \binom{m}{j} g_{n,m}$. In combination with (1.2), we obtain

$$(1.3) \quad \sum_{m \geq j} \binom{m}{j} g_{n,m} = a_{n,j}(\mathcal{UP}) = \binom{n-j+1}{j} C_{n-j}.$$

Let $G_n(q)$ be the ordinary generating function of $g_{n,m}$, i.e., $\sum_{m \geq 1} g_{n,m} q^m = G_n(q)$. Then from (1.3), we can derive

$$(1.4) \quad G_n(q) = \sum_{j \geq 0} \frac{G_n^{(j)}(1)}{j!} (q-1)^j = \sum_{j \geq 0} \binom{n-j+1}{j} C_{n-j} (q-1)^j,$$

where $G_n^{(j)}(1) = \frac{\partial^j G_n(q)}{\partial q^j} \big|_{q=1}$. By setting $q = 0$ on both sides of (1.4), Koshy's formula (1.1) follows. Our proof uses the generating function approach to illustrate the inclusion-exclusion principle; see Chapter 2.3 of [12] and Chapter 4.2 of [13]. Here we will use this approach to prove (1.6) and (1.7). We can also prove the sieving identity by a direct application of the inclusion-exclusion method. To make this point clear, we will prove (2.3) in both ways in Section 2. Next we give two q -analogues of Koshy's formula due to Lassalle and Andrews [3, 10]. The Narayana number $N_{n,k}$ and the Narayana polynomial $N_n(q)$ are defined by

$$(1.5) \quad N_{n,k} = \frac{1}{n} \binom{n}{k-1} \binom{n}{k}, \quad N_n(q) = \sum_{k=1}^n N_{n,k} q^{k-1},$$

where $N_{n,k}$ counts the number of Dyck paths of length $2n$ that have k peaks P . By utilizing a λ -identity of complete functions, Lassalle [10] proved for $n \geq 1$,

$$(1.6) \quad N_n(q) = (1-q)^{n-1} + q \sum_{k=1}^{n-1} N_{n-k}(q) \sum_{m=0}^{k-1} (-1)^m \binom{k-1}{m} \binom{n-m}{k} (1-q)^{k-m-1}.$$

Furthermore, Andrews gave another q -analogue of Koshy's formula in terms of q -hypergeometric series. Here we adopt the standard notations of q -series, i.e.,

$$\begin{aligned} [n]_q &= 1 + q + \cdots + q^{n-1}, \quad [n]_q! = [1]_q [2]_q \cdots [n]_q, \\ (x; q)_n &= (1-x)(1-qx) \cdots (1-q^{n-1}x), \\ \left[\begin{matrix} m \\ n \end{matrix} \right]_q &= \frac{[m]_q!}{[n]_q! [m-n]_q!} = \frac{(q; q)_m}{(q; q)_n (q; q)_{m-n}}, \\ C_n(q) &= \frac{1}{[n+1]_q} \left[\begin{matrix} 2n \\ n \end{matrix} \right]_q. \end{aligned}$$

Then the before-mentioned q -analogue of Koshy's formula is given by

$$(1.7) \quad C_n(q) = \sum_{r=1}^n (-1)^{r-1} T_r(n, q),$$

$$(1.8) \quad T_r(n, q) = q^{r^2-r} \frac{(-q^{n-r+1}; q)_r}{(-q; q)_r} \cdot \begin{bmatrix} n-r+1 \\ r \end{bmatrix}_q \cdot C_{n-r}(q).$$

Andrews [3] raised open questions about $T_r(n, q)$, which are:

- (1) Is $T_r(n, q)$ a polynomial in q ?
- (2) Does $T_r(n, q)$ only have nonnegative coefficients of q if $n \geq 2r$?
- (3) Does $T_{r-1}(2r-1, -q)$ only have nonnegative coefficients?
- (4) What is the partition-theoretic combinatorial interpretation of $T_r(n, q)$ for $n \geq 2r$, what is that of $T_{r-1}(2r-1, -q)$, and what is the sieving process on the partitions to eliminate all the non-Catalan partitions?

In Section 4 we completely answer (1) and (3), and show (2) for even n . Furthermore, in Sections 2 and 5 we prove (1.6) and (1.7) by the inclusion-exclusion method. In Sections 3 and 6 we generalize (1.6) and (1.7) for $N_n(q)$ and $C_n(q)$ to that for Ballot numbers $B_{n,r}$ and q -Ballot numbers $B_r(n, q)$. Finally we conjecture that if n is odd, then for $m \geq n \geq 1$, the polynomial $(1+q^n) \begin{bmatrix} m \\ n-1 \end{bmatrix}_q$ is unimodal. If n is even, for any even $j \neq 0$ and $m \geq n \geq 1$, the polynomial $(1+q^n) [j]_q \begin{bmatrix} m \\ n-1 \end{bmatrix}_q$ is unimodal. This would answer question (2) from above for odd n .

2. PROOF OF (1.6)

Before we prove (1.6), we give some main ingredients needed in the proof.

First we introduce the notions of a *tower* and a *colored tower* for an elevated Dyck path. We will choose to label the colored towers by s in order to apply the inclusion-exclusion method. Recall that D_n is the set of elevated Dyck paths of length $2n+2$. For any $p \in D_n$, we will use $U^i D^j \subseteq p$ to express the fact that p contains i consecutive U -steps that are followed by j consecutive D -steps. We call t a *tower* of height i contained in an elevated Dyck path p , if $t = U^i D^i \subseteq p$ and $UtD \not\subseteq p$. For any elevated Dyck path $p \in D_n$, let $\mathcal{T}^*(p)$ be the set of towers contained in p . We will color the towers in $\mathcal{T}^*(p)$ in the following way for any $p \in D_n$:

Step 1. Color all the towers $t \subseteq p$ that immediately follow a U -step.

Step 2. Color all the towers $t \subseteq p$ that immediately follow an uncolored tower $t' \subseteq p$ after Step 1.

We use $\mathcal{T}(p) \subseteq \mathcal{T}^*(p)$ to denote the set of towers contained in a path p which are colored according to Steps 1 and 2. We call every t in the set $\mathcal{T}(p)$ a *colored tower*. We set $\mathcal{T}^c(p) = \mathcal{T}^*(p) - \mathcal{T}(p)$ and let $\mathcal{T}_2(p) \subseteq \mathcal{T}(p)$ (resp. $\mathcal{T}_1(p) \subseteq \mathcal{T}(p)$) be the set of colored towers of height ≥ 2 (resp. height = 1). As an example, consider the path $p = U^3 D^2 U D U D^2 \in D_4$. The colored towers in the set $\mathcal{T}(p)$ are depicted using double lines in the rightmost elevated Dyck path of Figure 1. After Step 1, the first tower in the left-to-right order is colored. The second tower remains uncolored after Step 1, and therefore after Step 2, the third tower is colored.

Second we will use some one-to-one correspondences between two sets of labeled elevated Dyck paths. If there is a bijection between two sets A and B , we write

FIGURE 1

$A \simeq B$. Recall that $(\mathcal{M}^s, j, D_n)$ is the set of elevated Dyck paths $p \in D_n$ having exactly j elements from the set $\mathcal{M}(p)$ labeled by s . We consider the set of elevated Dyck paths $p \in D_n$ that have m colored towers labeled by s , which is the particular case $\mathcal{M} = \mathcal{T}$ and $j = m$. Since each colored tower has height 1 or at least 2, we have $(\mathcal{T}^s, m, D_n) = ((\mathcal{T}_1 \cup \mathcal{T}_2)^s, m, D_n)$. Let $\mathcal{U}^2(p)$ be the set of UU -steps of an elevated Dyck path $p \in D_n$. We will show

Lemma 2.1. $(\mathcal{T}_1^s, 1, D_n) \simeq ((\mathcal{U}^2 \cup \mathcal{T}^c)^s, 1, D_{n-1})$, $(\mathcal{T}_2^s, 1, D_n) \simeq (\mathcal{T}^s, 1, D_{n-1})$ and $(\mathcal{T}^s, m, D_n) \simeq (\mathcal{U}^s, m, D_{n-m})$ for any m , $1 \leq m \leq n$.

Proof. We use U^s , $U^s U^s$, and t^s to represent a U -step, a UU -step, and a tower labeled by s . For a path $p \in D_n$ that has only one colored tower t labeled by s , if the labeled and colored tower t has height 1, i.e., $t \in \mathcal{T}_1(p)$, then t is either located between two U -steps, i.e., $U t^s U \subseteq p$, or t follows an uncolored tower t_1 from $\mathcal{T}^c(p)$, i.e., $t_1 t^s \subseteq p$ where $t_1 \in \mathcal{T}^c(p)$. Let $g_1 : (\mathcal{T}_1^s, 1, D_n) \rightarrow ((\mathcal{U}^2 \cup \mathcal{T}^c)^s, 1, D_{n-1})$ be the map defined as follows. If $p = \cdots U t^s U \cdots \in (\mathcal{T}_1^s, 1, D_n)$, then $g_1(p) = \cdots U^s U^s \cdots \in ((\mathcal{U}^2)^s, 1, D_{n-1})$. If $p = \cdots t_1 t^s \cdots \in (\mathcal{T}_1^s, 1, D_n)$, then $g_1(p) = \cdots t_1^s \cdots \in ((\mathcal{T}^c)^s, 1, D_{n-1})$. Notice that the tower t_1 is also an uncolored tower of the labeled path $g_1(p)$. This follows from the way we define the colored towers. It is obvious that the map g_1 is a bijection.

For a path $p \in D_n$ that has only one colored tower t labeled by s , if the labeled and colored tower t has height at least 2, i.e., $t \in \mathcal{T}_2(p)$, then $t = U t_2 D$ where t_2 is a tower. Let $g_2 : (\mathcal{T}_2^s, 1, D_n) \rightarrow (\mathcal{T}^s, 1, D_{n-1})$ be the map defined as follows. If $p = \cdots t^s \cdots = \cdots (U t_2 D)^s \cdots \in (\mathcal{T}_2^s, 1, D_n)$, then $g_2(p) = \cdots t_2^s \cdots \in (\mathcal{T}^s, 1, D_{n-1})$. Notice that the tower t_2 is also a colored tower of the labeled path $g_2(p)$. This follows from the way we define the colored towers. It is clear that the map g_2 is a bijection.

We will next prove $(\mathcal{T}^s, m, D_n) \simeq ((\mathcal{U}^2 \cup \mathcal{T} \cup \mathcal{T}^c)^s, m, D_{n-m})$ for any m . The bijection for the case $m = 1$ can be obtained by combining the bijections g_1 and g_2 , namely, let $f_1 : (\mathcal{T}^s, 1, D_n) \rightarrow ((\mathcal{U}^2 \cup \mathcal{T}^c \cup \mathcal{T})^s, 1, D_{n-1})$ be the bijection defined as follows: $f_1(p) = g_1(p)$ if $p \in (\mathcal{T}_1^s, 1, D_n)$ and $f_1(p) = g_2(p)$ if $p \in (\mathcal{T}_2^s, 1, D_n)$. We can extend the bijection f_1 to the general bijection f_m from the set $(\mathcal{T}^s, m, D_n)$ to the set $((\mathcal{U}^2 \cup \mathcal{T} \cup \mathcal{T}^c)^s, m, D_{n-m})$ as follows. For any $p \in (\mathcal{T}^s, m, D_n)$, suppose p has $t_1, t_2, \dots, t_m$ colored towers labeled by s from left to right. For every labeled tower t_i that has height 1 and $U t_i^s U \subseteq p$, we label the U -steps next to t_i by s and remove t_i from p , i.e., $U^s U^s \subseteq f_m(p)$. For every labeled tower t_i that has height 1 and $t t_i^s \subseteq p$, we label the tower t by s and remove t_i from p , i.e., $t^s \subseteq f_m(p)$. For every labeled tower t_i that has height at least 2, i.e., $t_i^s = (U t D)^s \subseteq p$, we label t by s and remove a U -step and a D -step from the tower t_i , i.e., $t^s \subseteq f_m(p)$. In fact, the map f_m on the path p is equivalent to applying the bijection f_1 on

each labeled tower t_i of p from left to right. It follows that f_m is a bijection and therefore $(\mathcal{T}^s, m, D_n) \simeq ((\mathcal{U}^2 \cup \mathcal{T} \cup \mathcal{T}^c)^s, m, D_{n-m})$ holds for any m . It remains to prove $\mathcal{U}^2(p) \cup \mathcal{T}(p) \cup \mathcal{T}^c(p) \simeq \mathcal{U}(p)$ for any $p \in D_n$.

In view of $\mathcal{T}^c(p) \cup \mathcal{T}(p) = \mathcal{T}^*(p)$ for any $p \in D_n$, we next show for any $p \in D_n$, $\mathcal{U}^2(p) \cup \mathcal{T}^*(p) \simeq \mathcal{U}(p)$. Each U -step is either part of a UU -step or part of a peak P by considering the step that follows this U -step. Each peak P is contained in only one tower from $\mathcal{T}^*(p)$ and each tower from $\mathcal{T}^*(p)$ has only one peak. Therefore $\mathcal{U}(p) \simeq \mathcal{U}^2(p) \cup \mathcal{T}^*(p)$ for any $p \in D_n$ and the proof is complete. $\square$

For a path p , we name the first U -step of a tower $t \in \mathcal{T}_2(p)$ *bottom*. Let $\mathcal{B}(p)$ be the set of bottoms contained in a path p , and let $((\mathcal{T}^s, \mathcal{B}^w), (m, r), D_n)$ be the set of elevated Dyck paths $p \in D_n$ having exactly m colored towers labeled by s and among these m colored towers there are exactly r bottoms labeled by w . Let $((\mathcal{T}^s, \mathcal{T}^w), (m, r), D_n)$ be the set of elevated Dyck paths $p \in D_n$ having exactly m colored towers labeled by s and among these m colored towers there are exactly r colored towers labeled by w . We will prove

Lemma 2.2. $((\mathcal{T}^s, \mathcal{B}^w), (m, r), D_n) \simeq ((\mathcal{T}^s, \mathcal{T}^w), (m, r), D_{n-r})$.

Proof. For a path $p \in ((\mathcal{T}^s, \mathcal{B}^w), (m, r), D_n)$ and for every colored tower $t \in \mathcal{T}_2(p)$ labeled by s whose bottom is labeled by w , we remove the bottom and a D -step from t , and label the remaining colored tower by s and w . For a path $\bar{p} \in ((\mathcal{T}^s, \mathcal{T}^w), (m, r), D_{n-r})$ and for every colored tower $\bar{t} \in \mathcal{T}(\bar{p})$ labeled by s, w , we replace the tower $\bar{t}^{s,w}$ by the tower $U^w \bar{t}^s D$. This yields a bijection. $\square$

Now we are in position to prove (1.6).

Proof. Let $F_n(q) = qN_n(q)$. Then (1.6) is equivalent to

(2.1)

$$\frac{F_n(q)}{(1-q)^n} = \frac{q}{1-q} + \sum_{k=1}^{n-1} \frac{F_{n-k}(q)}{(1-q)^{n-k}} \sum_{m=0}^{k-1} (-1)^m \binom{k-1}{m} \binom{n-m}{k} \frac{q}{(1-q)^{m+1}},$$

where $\frac{F_n(q)}{(1-q)^n} = \sum_{k=1}^n \frac{N_{n,k} q^k}{(1-q)^n}$ counts the number of elevated Dyck paths of length $2n+2$ where each U -step, except the first U -step, has weight $\frac{1}{1-q}$ and each peak UD has weight q . Before we proceed to the combinatorial proof, we first transform $\frac{q}{(1-q)^{m+1}}$ according to our needs, namely,

$$\begin{aligned} & \sum_{m=0}^{k-1} (-1)^m \binom{k-1}{m} \binom{n-m}{k} \frac{q}{(1-q)^{m+1}} \\ &= \sum_{i=0}^{k-1} \sum_{m \geq i} (-1)^m \binom{k-1}{m} \binom{n-m}{k} \binom{m}{i} \left(\frac{q}{1-q}\right)^{i+1} \\ &= \sum_{m=1}^k (-1)^{m-1} \binom{k-1}{m-1} \binom{n-k+1}{m} \left(\frac{q}{1-q}\right)^m. \end{aligned}$$

Consequently, we can express (2.1) as

$$(2.2) \quad \frac{F_n(q)}{(1-q)^n} = \frac{q}{1-q} + \sum_{m=1}^{n-1} (-1)^{m-1} \sum_{k=m}^{n-m+1} \frac{F_{n-k}(q)}{(1-q)^{n-k}} \binom{k-1}{m-1} \binom{n-k+1}{m} \left(\frac{q}{1-q}\right)^m.$$

We observe that adding a peak P right between double U -steps contributes weight $\frac{q}{1-q}$, while adding a peak P right between UD -steps contributes weight $\frac{1}{1-q}$. In contrast to the proof of (1.1), the reduction from $\frac{F_n(q)}{(1-q)^n}$ to $\frac{F_{n-1}(q)}{(1-q)^{n-1}}$ by removing a peak highly depends on the type of the U -step we choose.

Next we will derive an identity for the generating function of weighted elevated Dyck paths. This can also be obtained by directly applying the inclusion-exclusion principle. We will show this identity in both ways. We set $g_{n,k}(\mathcal{T}) = |\{p \in D_n : |\mathcal{T}(p)| = k\}|$ and define $g_{n,k}(\mathcal{T}, q)$ to be the generating function of $g_{n,k}(\mathcal{T})$ that weights each U -step, except the first U -step, of an elevated Dyck path $p \in D_n$ by $\frac{1}{1-q}$ and each peak UD of an elevated Dyck path $p \in D_n$ by q . Recall that we denote by $(\mathcal{M}^s, m, D_n)$ the set of elevated Dyck paths $p \in D_n$ having exactly m elements from the set $\mathcal{M}(p)$ labeled by s , and that $a_{n,m}(\mathcal{M}) = |(\mathcal{M}^s, m, D_n)|$. Furthermore, we denote by $(\bigcup_{i=1}^k \mathcal{M}_i)^s, m, D_n)$ the set of elevated Dyck paths $p \in D_n$ having exactly m elements from the set $\bigcup_{i=1}^k \mathcal{M}_i(p)$ labeled by s , and define $a_{n,m}(\bigcup_{i=1}^k \mathcal{M}_i) = |((\bigcup_{i=1}^k \mathcal{M}_i)^s, m, D_n)|$. Then, analogous to the proof of (1.1), we consider $\mathcal{M} = \mathcal{T}$. Let $a_{n,m}(\mathcal{T}, q)$ be the generating function of $a_{n,m}(\mathcal{T})$ where each U -step, except the first U -step, of an elevated Dyck path $p \in D_n$ is weighted by $\frac{1}{1-q}$ and each peak P of an elevated Dyck path $p \in D_n$ is weighted by q . Then

$$a_{n,m}(\mathcal{T}) = \sum_{k \geq m} \binom{k}{m} g_{n,k}(\mathcal{T}), \quad a_{n,m}(\mathcal{T}, q) = \sum_{k \geq m} \binom{k}{m} g_{n,k}(\mathcal{T}, q).$$

In particular for $m = 0$, we have $a_{n,0}(\mathcal{T}) = C_n$ and $a_{n,0}(\mathcal{T}, q) = \frac{F_n(q)}{(1-q)^n}$. By introducing the generating function $G_n(x, q)$ of $g_{n,k}(\mathcal{T}, q)$, we obtain

$$G_n(x, q) = \sum_{k \geq 1} g_{n,k}(\mathcal{T}, q) x^k = \sum_{m=0}^n \frac{G_n^{(m)}(1, q)}{m!} (x-1)^m = \sum_{m=0}^n a_{n,m}(\mathcal{T}, q) (x-1)^m,$$

where $G_n^{(m)}(1, q) = \frac{\partial^m G_n(x, q)}{\partial x^m} \big|_{x=1}$. By setting $x = 0$, we get

$$(2.3) \quad a_{n,0}(\mathcal{T}, q) = \frac{F_n(q)}{(1-q)^n} = \sum_{m=1}^n (-1)^{m-1} a_{n,m}(\mathcal{T}, q).$$

In fact (2.3) can also be derived by directly applying the inclusion-exclusion method. To be precise, we adopt the notations from [12]. Let A_i be the set of elevated Dyck paths in D_n whose i -th tower (in the left-to-right order) is a colored tower. Then $A_1, \dots, A_n$ are all the subsets of the set D_n and $A_1 \cup \dots \cup A_n = D_n$. For each subset T of $[n]$, let $A_T = \bigcap_{i \in T} A_i$ with $A_\emptyset = D_n$. Here A_T is the set of elevated Dyck paths in D_n whose i -th tower (in the left-to-right order) is a colored tower for every $i \in T$. For $0 \leq m \leq n$, we set $S_m = \sum_{|T|=m} |A_T|$ where the sum runs over all the m -subsets T of $[n]$. Therefore S_m equals the number of elevated Dyck paths in D_n that have m colored towers labeled by s , which is $a_{n,m}(\mathcal{T})$. So according to the

principle of inclusion-exclusion, the number $|\bar{A}_1 \cap \cdots \cap \bar{A}_n|$ of elevated Dyck paths in D_n that have no colored towers equals

$$|\bar{A}_1 \cap \cdots \cap \bar{A}_n| = \sum_{m=0}^n (-1)^m S_m = \sum_{m=0}^n (-1)^m a_{n,m}(\mathcal{T}).$$

Since every elevated Dyck path has at least one colored tower, we have $|\bar{A}_1 \cap \cdots \cap \bar{A}_n| = g_{n,0}(\mathcal{T}) = 0$. Consequently, $a_{n,0}(\mathcal{T}) = \sum_{m=1}^n (-1)^{m-1} a_{n,m}(\mathcal{T})$. In terms of the weighted elevated Dyck paths, (2.3) follows.

In order to use (2.3) to prove (2.2), we will need to give an explicit expression for $a_{n,m}(\mathcal{T}, q)$. For any path $p \in D_n$ and any two colored towers $t_1, t_2 \in \mathcal{T}(p)$, t_1 and t_2 are disjoint. We start by counting $a_{n,1}(\mathcal{T}, q)$, which is the generating function for the elevated Dyck paths $p \in D_n$ having one colored tower labeled by s . Equivalently, $|(\mathcal{T}^s, 1, D_n)| = a_{n,1}(\mathcal{T})$. In view of Lemma 2.1, we get $a_{n,1}(\mathcal{T}) = a_{n-1,1}(\mathcal{U})$ and

$$\begin{aligned} a_{n,1}(\mathcal{T}, q) &= \frac{q}{1-q} a_{n-1,1}(\mathcal{U}^2 \cup \mathcal{T}^c, q) + \frac{1}{1-q} a_{n-1,1}(\mathcal{T}, q) \\ &= \frac{q}{1-q} a_{n-1,1}(\mathcal{U}, q) + a_{n-1,1}(\mathcal{T}, q) = \sum_{k=1}^{n-1} \frac{q}{1-q} a_{n-k,1}(\mathcal{U}, q) + a_{1,1}(\mathcal{T}, q) \\ (2.4) \quad &= \sum_{k=1}^{n-1} (n-k+1) \frac{F_{n-k}(q)}{(1-q)^{n-k}} \frac{q}{1-q} + \frac{q}{1-q}. \end{aligned}$$

Next we will count $a_{n,m}(\mathcal{T}, q)$ for $m \geq 2$. For a path p , recall that the first U -step of a tower $t \in \mathcal{T}_2(p)$ is named bottom. For a given path $p \in D_n$ with m colored towers labeled by s , let $\mathcal{T}_{2,m}$ denote the subset of these m towers that have height ≥ 2 . Since each U -step has weight $\frac{1}{1-q}$ and each tower in $\mathcal{T}_{2,m}$ has one bottom, the weight on the bottoms of path $p \in D_n$ is $(\frac{1}{1-q})^{|\mathcal{T}_{2,m}|}$, which is equal to

$$(2.5) \quad \left(\frac{1}{1-q}\right)^{|\mathcal{T}_{2,m}|} = \left(\frac{q}{1-q}\right)^{|\mathcal{T}_{2,m}|} + \sum_{r=1}^{|\mathcal{T}_{2,m}|} (-1)^{r-1} \binom{|\mathcal{T}_{2,m}|}{r} \left(\frac{1}{1-q}\right)^{|\mathcal{T}_{2,m}|-r}.$$

We will separate the weight $(\frac{1}{1-q})^{|\mathcal{T}_{2,m}|}$ on the bottoms of path p according to (2.5). The term $(\frac{q}{1-q})^{|\mathcal{T}_{2,m}|}$ on the right-hand side of (2.5) corresponds to the case that each bottom is weighted by $\frac{q}{1-q}$. Accordingly, we set $f_{n,m,0}(\mathcal{T}, q)$ to be the generating function of elevated Dyck paths $p \in D_n$ that have m colored towers labeled by s where each bottom has weight $\frac{q}{1-q}$, each U -step, other than the first U -step and the bottom, has weight $\frac{1}{1-q}$ and each peak has weight q . In the same way as for the counting of $a_{n,1}(\mathcal{T}, q)$, the set $(\mathcal{T}^s, m, D_n)$ is in one-to-one correspondence to the set $(\mathcal{U}^s, m, D_{n-m})$ as shown in Lemma 2.1. Therefore,

$$\begin{aligned} a_{n,m}(\mathcal{T}) &= a_{n-m,m}(\mathcal{U}), \\ (2.6) \quad f_{n,m,0}(\mathcal{T}, q) &= \left(\frac{q}{1-q}\right)^m a_{n-m,m}(\mathcal{U}, q). \end{aligned}$$

The term $\binom{|\mathcal{T}_{2,m}|}{r} (\frac{1}{1-q})^{|\mathcal{T}_{2,m}|-r}$ on the right-hand side of (2.5) represents the number of ways to choose r towers from the set $\mathcal{T}_{2,m}$ where each bottom of these r towers has weight 1 and each bottom of the remaining $(|\mathcal{T}_{2,m}| - r)$ towers has weight $\frac{1}{1-q}$. Accordingly, for $r \geq 1$ we define $f_{n,m,r}(\mathcal{T})$ to be the number of elevated Dyck paths $p \in D_n$ that have m colored towers labeled by s and among these m colored towers

there are exactly r bottoms labeled by w , i.e., $|((\mathcal{T}^s, \mathcal{B}^w), (m, r), D_n)| = f_{n,m,r}(\mathcal{T})$. Furthermore, let $f_{n,m,r}(\mathcal{T}, q)$ be the generating function of $f_{n,m,r}(\mathcal{T})$ where each bottom labeled by w has weight 1, each U -step, other than the bottom labeled by w and the first U -step, has weight $\frac{1}{1-q}$ and each peak has weight q . From Lemma 2.2, we obtain $f_{n,m,r}(\mathcal{T}) = |((\mathcal{T}^s, \mathcal{T}^w), (m, r), D_{n-r})|$ and therefore

$$(2.7) \quad f_{n,m,r}(\mathcal{T}) = \binom{m}{r} a_{n-r,m}(\mathcal{T}), \quad f_{n,m,r}(\mathcal{T}, q) = \binom{m}{r} a_{n-r,m}(\mathcal{T}, q).$$

By multiplying both sides of (2.5) by the weights on the U -steps (except the first U -step and the bottoms) and the weights on the peaks of a path $p \in D_n$, and summing over all the paths in D_n , we get

$$a_{n,m}(\mathcal{T}, q) = f_{n,m,0}(\mathcal{T}, q) + \sum_{r=1}^m (-1)^{r-1} f_{n,m,r}(\mathcal{T}, q).$$

Together with (2.6) and (2.7), we have

$$(2.8) \quad \begin{aligned} a_{n,m}(\mathcal{T}, q) &= \left(\frac{q}{1-q}\right)^m a_{n-m,m}(\mathcal{U}, q) + \sum_{r=1}^m (-1)^{r-1} \binom{m}{r} a_{n-r,m}(\mathcal{T}, q) \\ &= \left(\frac{q}{1-q}\right)^m \binom{n-m+1}{m} \frac{F_{n-m}(q)}{(1-q)^{n-m}} + \sum_{r=1}^m (-1)^{r-1} \binom{m}{r} a_{n-r,m}(\mathcal{T}, q). \end{aligned}$$

We will employ generating functions to solve (2.8). Let $A_m(x)$ be the generating function for $a_{n,m}(\mathcal{T}, q)$ and $m \geq 2$, i.e., the n -th coefficient of $A_m(x)$ —denoted by $[x^n]A_m(x)$ —is $a_{n,m}(\mathcal{T}, q)$. By multiplying both sides of (2.8) by x^n and summing over all n , we obtain

$$\begin{aligned} A_m(x) &= (1-x)^{-m} \sum_{n \geq m} \binom{n-m+1}{m} \left(\frac{q}{1-q}\right)^m \frac{F_{n-m}(q)}{(1-q)^{n-m}} x^n, \\ a_{n,m}(\mathcal{T}, q) &= [x^n]A_m(x) = \sum_{i \geq 0} \binom{n-m-i+1}{m} \left(\frac{q}{1-q}\right)^m \binom{m+i-1}{m-1} \frac{F_{n-m-i}(q)}{(1-q)^{n-m-i}} \\ &= \sum_{k=m}^{n-m+1} \binom{n-k+1}{m} \binom{k-1}{m-1} \frac{F_{n-k}(q)}{(1-q)^{n-k}} \left(\frac{q}{1-q}\right)^m. \end{aligned}$$

In combination of (2.3) and (2.4), the proof of (2.2) is complete. $\square$

3. GENERALIZE (1.6) TO BALLOT NUMBERS

The Ballot numbers $B_{n,r}$ are defined by $B_{n,r} = \frac{r+1}{2n+r+1} \binom{2n+r+1}{n}$. They count the number of paths p from $(0, 0)$ to $(2n+r, r)$ with allowed U -step and D -step and each path p can be decomposed as $p_1 U p_2 \cdots U p_{r+1}$ with p_i , $1 \leq i \leq r+1$, a Dyck path. Let $\frac{M_{n,r}(q)}{(1-q)^n}$ be the generating function for the path $p_1 p_2 \cdots p_{r+1}$ that has total length $2n$ where each peak of $p_1 p_2 \cdots p_{r+1}$ has weight q and each U -step of $p_1 p_2 \cdots p_{r+1}$ has weight $\frac{1}{1-q}$. Then we can generalize (1.6) to an equation for $M_{n,r}(q)$, i.e.,

$$\frac{M_{n,r}(q)}{(1-q)^n} = \frac{q(r+1)}{1-q} + \sum_{m=1}^{n-1} (-1)^{m-1} \sum_{k \geq m} \frac{q^m M_{n-k,r}(q)}{(1-q)^{n-k+m}} \binom{k-1}{m-1} \binom{n-k+1+r}{m}.$$

The proof follows similarly to that for (1.6) and is omitted here. Next we will prove (1.7) by involution and the inclusion-exclusion method on the partitions and answer Andrews' questions on the property of $T_r(n, q)$ given in (1.8).

4. PROPERTIES OF $T_r(n, q)$

We say a polynomial $f(x) = a_n x^n + \cdots + a_0$ is reciprocal if $f(x) = x^n f(\frac{1}{x})$, i.e., $a_r = a_{n-r}$. A sequence $a_0, a_1, \dots, a_n$ of real numbers is said to be unimodal if for some $0 \leq j \leq n$ we have $a_0 \leq \cdots \leq a_{j-1} \leq a_j \geq a_{j+1} \geq \cdots \geq a_n$. We say a polynomial $f(x) = a_n x^n + \cdots + a_1 x + a_0$ is unimodal if the sequence $a_0, a_1, \dots, a_n$ is unimodal. If $f(x)$ and $g(x)$ are unimodal and reciprocal polynomials with nonnegative coefficients, then $f(x)g(x)$ is also unimodal and reciprocal. Here we say $f(x)$ is a positive polynomial if all the coefficients of $f(x)$ are nonnegative. We say the polynomial $f(x)$ has nonnegative coefficients (resp. nonpositive coefficients) up to x^r if for any $0 \leq i \leq r$, the coefficient of x^i in the polynomial $f(x)$ is nonnegative (resp. nonpositive). $\omega \in \mathbb{C}$ is a primitive k -th root of unity if and only if $\omega^k = 1$ and for any $d \in \mathbb{Z}^+$ and $d < k$, $\omega^d \neq 1$. The k -th cyclotomic polynomial $\Phi_k(x) \in \mathbb{Z}[x]$ is the polynomial whose roots are the primitive k -th roots of unity. Let ω be any primitive k -th root of unity, and the polynomial $f(x)$ satisfies $f(\omega) = 0$. Then $\Phi_k(x) \mid f(x)$. The q -Lucas theorem is the following:

Proposition 1 (q -Lucas theorem). *Let m, k, d be positive integers, and write $m = ad + b$ and $k = rd + s$, where $0 \leq b, s \leq d - 1$. Let ω be any primitive d -th root of unity. Then*

$$\begin{bmatrix} m \\ k \end{bmatrix}_\omega = \begin{bmatrix} a \\ r \end{bmatrix} \begin{bmatrix} b \\ s \end{bmatrix}_\omega.$$

Theorem 4.1. $T_r(n, q)$ is a polynomial in q . If n is even, $T_r(n, q)$ is a positive polynomial. If n is odd, $(1+q)T_r(n, q)$ is a positive polynomial. In case $n = 2r - 1$, $T_r(2r - 1, -q)$ is a positive polynomial.

Proof. We can simplify $T_r(n, q)$ defined in (1.8) into

$$(4.1) \quad T_r(n, q) = q^{r^2-r} \frac{1}{[n]_q} \begin{bmatrix} n \\ r \end{bmatrix}_{q^2} \begin{bmatrix} 2n-2r \\ n-1 \end{bmatrix}_q$$

$$(4.2) \quad = q^{r^2-r} \begin{bmatrix} n-1 \\ r \end{bmatrix}_{q^2} \frac{(1+q^n)}{[n-1]_q} \begin{bmatrix} 2n-2r-1 \\ n-2 \end{bmatrix}_q.$$

Consequently from (4.1) and (4.2) we have for $r \geq 1$,

$$(4.3) \quad \begin{aligned} T_r(n, q) &= ([n]_q - q[n-1]_q) T_r(n, q) \\ &= q^{r^2-r} \begin{bmatrix} n \\ r \end{bmatrix}_{q^2} \begin{bmatrix} 2n-2r \\ n-1 \end{bmatrix}_q - q^{r^2-r} \begin{bmatrix} n-1 \\ r \end{bmatrix}_{q^2} (q + q^{n+1}) \begin{bmatrix} 2n-2r-1 \\ n-2 \end{bmatrix}_q, \end{aligned}$$

which implies $T_r(n, q)$ is a polynomial.

We will next show the polynomial $T_r(n, q)$ is positive for even n and the polynomial $(1+q)T_r(n, q)$ is positive for odd n . Let $d = \gcd(n, r)$; then

$$\begin{aligned} T_r(n, q) &= q^{r^2-r} \begin{bmatrix} n \\ r \end{bmatrix}_{q^2} \frac{[d]_{q^2}}{[n]_{q^2}} \frac{[n]_{q^2}}{[d]_{q^2} [2n-2r+1]_q} \begin{bmatrix} 2n-2r+1 \\ n \end{bmatrix}_q \\ &= q^{r^2-r} \begin{bmatrix} n \\ r \end{bmatrix}_{q^2} \frac{[d]_{q^2}}{[n]_{q^2}} \frac{(1+q^n)(1-q)}{1-q^{2d}} \begin{bmatrix} 2n-2r \\ n-1 \end{bmatrix}_q. \end{aligned}$$

It has been proved by Brunetti *et al.* that $\begin{bmatrix} n \\ r \end{bmatrix}_{q^2} \frac{[d]_{q^2}}{[n]_{q^2}}$ is a positive polynomial based on the fact that polynomial $[d]_q \begin{bmatrix} n \\ r \end{bmatrix}_q$ is unimodal and reciprocal [4]. In the same way, let $m = \gcd(n, 2r - 1)$; then $\frac{[m]_q}{[2n-2r+1]_q} \begin{bmatrix} 2n-2r+1 \\ n \end{bmatrix}_q$ is a positive polynomial. It remains to prove $\frac{(1+q^n)(1-q)}{1-q^{2d}} \begin{bmatrix} 2n-2r \\ n-1 \end{bmatrix}_q$ is a positive polynomial for even n and $\frac{(1+q^n)(1-q^2)}{1-q^{2d}} \begin{bmatrix} 2n-2r \\ n-1 \end{bmatrix}_q$ is a positive polynomial for odd n . Recall that $m = \gcd(n, 2r - 1)$. First we observe m must be odd and therefore

$$\frac{(1+q^n)(1-q)}{1-q^{2d}} \begin{bmatrix} 2n-2r \\ n-1 \end{bmatrix}_q = \frac{[m]_q [m]_{-q}}{[2n-2r+1]_q} \begin{bmatrix} 2n-2r+1 \\ n \end{bmatrix}_q \frac{[n]_{q^2}}{[m]_{q^2} [d]_{q^2}}.$$

Here $\frac{[n]_q}{[m]_q [d]_q}$ is a polynomial since $\gcd(m, d) = 1$ and $m|n, d|n$. Together with the fact that $\frac{[m]_q}{[2n-2r+1]_q} \begin{bmatrix} 2n-2r+1 \\ n \end{bmatrix}_q$ is a polynomial, it follows that $\frac{(1+q^n)(1-q)}{1-q^{2d}} \begin{bmatrix} 2n-2r \\ n-1 \end{bmatrix}_q$ is a polynomial. We next prove $\frac{(1+q^n)(1-q)}{1-q^{2d}} \begin{bmatrix} 2n-2r \\ n-1 \end{bmatrix}_q$ is a positive polynomial if n is even. If n is even, then $\frac{1-q}{1-q^{2d}} \begin{bmatrix} 2n-2r \\ n-1 \end{bmatrix}_q$ is a polynomial since for any $x | (2d)$ and $x > 1$, $x | (2n-2r)$ and $x \nmid (n-1)$. If not, $x | (n-1)$, and then $x | \gcd(n-1, 2) = 1$, contradicting the assumption. Thus by using the q -Lucas theorem, $\frac{1-q}{1-q^{2d}} \begin{bmatrix} 2n-2r \\ n-1 \end{bmatrix}_q$ is a polynomial. To be precise, for any $x \in \mathbb{Z}^+$ such that $x|(2d)$ and $x > 1$, suppose $2n-2r = c_1x$, $n-1 = c_2x+t_1$ for some $t_1 \neq 0$, and let ρ be any primitive x -th root of unity. Then by applying the q -Lucas theorem, we have $\begin{bmatrix} 2n-2r \\ n-1 \end{bmatrix}_\rho = \begin{bmatrix} c_1x \\ c_2x+t_1 \end{bmatrix}_\rho = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \begin{bmatrix} 0 \\ t_1 \end{bmatrix}_\rho = 0$. This shows for any $x | (2d)$ and $x > 1$, $\Phi_x(q) | \begin{bmatrix} 2n-2r \\ n-1 \end{bmatrix}_q$. Since $[2d]_q = \prod_{\substack{x|(2d) \\ x>1}} \Phi_x(q)$ and any two cyclotomic polynomials are relatively prime, we can conclude that $\prod_{\substack{x|(2d) \\ x>1}} \Phi_x(q) | \begin{bmatrix} 2n-2r \\ n-1 \end{bmatrix}_q$ and therefore $\frac{1-q}{1-q^{2d}} \begin{bmatrix} 2n-2r \\ n-1 \end{bmatrix}_q = \frac{1}{[2d]_q} \begin{bmatrix} 2n-2r \\ n-1 \end{bmatrix}_q$ is a polynomial. In view of the unimodality of polynomial $\begin{bmatrix} 2n-2r \\ n-1 \end{bmatrix}_q$, $(1-q) \begin{bmatrix} 2n-2r \\ n-1 \end{bmatrix}_q$ has nonnegative coefficients up to $q^{\lfloor \frac{(n-1)(n-2r+1)}{2} \rfloor}$. Therefore we expand $\frac{1}{1-q^{2d}}$ as a power series at $q = 0$ and obtain

$$\frac{(1-q)}{1-q^{2d}} \begin{bmatrix} 2n-2r \\ n-1 \end{bmatrix}_q = (1-q) \begin{bmatrix} 2n-2r \\ n-1 \end{bmatrix}_q + q^{2d}(1-q) \begin{bmatrix} 2n-2r \\ n-1 \end{bmatrix}_q + \cdots,$$

which implies that the polynomial $\frac{(1-q)}{1-q^{2d}} \begin{bmatrix} 2n-2r \\ n-1 \end{bmatrix}_q$ has nonnegative coefficients up to $q^{\lfloor \frac{(n-1)(n-2r+1)}{2} \rfloor}$. Together with the reciprocity of $\frac{(1-q)}{1-q^{2d}} \begin{bmatrix} 2n-2r \\ n-1 \end{bmatrix}_q$, we conclude that $\frac{(1-q)}{1-q^{2d}} \begin{bmatrix} 2n-2r \\ n-1 \end{bmatrix}_q$ has nonnegative coefficients and therefore $T_r(n, q)$ is a positive polynomial if n is even.

In the same way, we can prove that $\frac{(1-q^2)}{1-q^{2d}} \begin{bmatrix} 2n-2r \\ n-1 \end{bmatrix}_q$ is a positive polynomial if n is odd. This implies $(1+q)T_r(n, q)$ is a positive polynomial if n is odd.

By setting $q \rightarrow -q$ and $n \rightarrow 2r - 1$, we can obtain

$$T_r(2r-1, -q) = q^{r^2-r} \begin{bmatrix} 2r-1 \\ r \end{bmatrix}_{q^2} \frac{1+q}{1+q^{2r-1}} = q^{r^2-r} [2r-1]_q C_{r-1}(q^2),$$

which indicates $T_r(2r-1, -q)$ has nonnegative coefficients. $\square$

5. PROOF OF (1.7)

We transform $T_r(n, q)$ from (4.3) into

$$(5.1) \quad T_r(n, q) = q^{r^2-r} \begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_{q^2} \begin{bmatrix} 2n-2r+1 \\ n \end{bmatrix}_q - q^{r^2-r} \begin{bmatrix} n-1 \\ r \end{bmatrix}_{q^2} q \begin{bmatrix} 2n-2r-1 \\ n-2 \end{bmatrix}_q \\ + q^{r^2+r} \begin{bmatrix} n-1 \\ r \end{bmatrix}_{q^2} \begin{bmatrix} 2n-2r-1 \\ n \end{bmatrix}_q \text{ valid for } n \geq 2r+1, \text{ and} \\ T_r(n, q) = q^{r^2-r} \begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_{q^2} \begin{bmatrix} 2n-2r+1 \\ n \end{bmatrix}_q - q^{r^2-r} \begin{bmatrix} n-1 \\ r \end{bmatrix}_{q^2} q \begin{bmatrix} 2n-2r-1 \\ n-2 \end{bmatrix}_q \\ \text{valid for } 2r-1 \leq n \leq 2r.$$

For simplicity, we set $a_n^{(r)} = q^{r^2+r} \begin{bmatrix} n-1 \\ r \end{bmatrix}_{q^2} \begin{bmatrix} 2n-2r-1 \\ n \end{bmatrix}_q$. Then we can write $T_r(n, q) = a_n^{(r-1)} + a_n^{(r)} - q^{r^2-r} \begin{bmatrix} n-1 \\ r \end{bmatrix}_{q^2} q \begin{bmatrix} 2n-2r-1 \\ n-2 \end{bmatrix}_q$ for $n \geq 2r+1$ and $T_r(n, q) = a_n^{(r-1)} - q^{r^2-r} \begin{bmatrix} n-1 \\ r \end{bmatrix}_{q^2} q \begin{bmatrix} 2n-2r-1 \\ n-2 \end{bmatrix}_q$ for $2r-1 \leq n \leq 2r$. First we observe

$$\sum_{r=1}^{\lfloor \frac{n+1}{2} \rfloor - 1} (-1)^{r-1} (a_n^{(r-1)} + a_n^{(r)}) + (-1)^{\lfloor \frac{n+1}{2} \rfloor - 1} a_n^{(\lfloor \frac{n+1}{2} \rfloor - 1)} \\ = a_n^{(0)} = \begin{bmatrix} 2n-1 \\ n \end{bmatrix}_q = C_n(q) + q \begin{bmatrix} 2n-1 \\ n-2 \end{bmatrix}_q$$

and it remains to prove

$$(5.2) \quad \begin{bmatrix} 2n-1 \\ n-2 \end{bmatrix}_q = \sum_{r=1}^{\lfloor \frac{n+1}{2} \rfloor} (-1)^{r-1} q^{r^2-r} \begin{bmatrix} n-1 \\ r \end{bmatrix}_{q^2} \begin{bmatrix} 2n-2r-1 \\ n-2 \end{bmatrix}_q.$$

We interpret (5.2) in terms of partitions. A partition λ is defined as a finite sequence of nonnegative integers $(\lambda_1, \lambda_2, \dots, \lambda_m)$ in the weakly decreasing order $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m$. Each $\lambda_i \neq 0$ is called a part of λ . The number and the sum of parts of λ are denoted by $\ell(\lambda)$ and $|\lambda|$, respectively. The partition-theoretic interpretation of the q -binomial $\begin{bmatrix} n+k \\ k \end{bmatrix}_q$ is $\begin{bmatrix} n+k \\ k \end{bmatrix}_q = \sum_{\substack{\ell(\lambda) \leq k \\ \lambda_1 \leq n}} q^{|\lambda|}$ as given in [1]. Therefore we have

$$(5.3) \quad q^{n-2r+1} \begin{bmatrix} 2n-2r-1 \\ n-2 \end{bmatrix}_q = \sum_{\substack{\ell(\nu)=n-2r+1 \\ \nu_1 \leq n-1}} q^{|\nu|}, \quad q^{r^2+r} \begin{bmatrix} n-1 \\ r \end{bmatrix}_{q^2} = \sum_{\substack{\mu_1 > \mu_2 > \dots > \mu_r \geq 1 \\ \mu_i \leq n-i}} q^{2|\mu|}.$$

Given two partitions μ and ν , say $\mu = (\mu_1, \dots, \mu_m)$ and $\nu = (\nu_1, \dots, \nu_n)$, let $\mu^2 \cup \nu$ be the partition whose parts are $\mu_1, \mu_1, \dots, \mu_m, \mu_m, \nu_1, \dots, \nu_n$ in decreasing order. Let $\mu \cup \nu$ be the partition whose parts are $\mu_1, \dots, \mu_m, \nu_1, \dots, \nu_n$ in decreasing order, and let $\nu \setminus (\mu_1, \dots, \mu_m)$ be the partition obtained from ν by removing the parts equal to $\mu_1, \dots, \mu_m$.

For any pair (μ, ν) such that $n-1 \geq \mu_1 > \dots > \mu_r \geq 1$ where $\mu_i \leq n-i$ and $n-1 \geq \nu_1 \geq \dots \geq \nu_{n-2r+1} \geq 1$, let x be the smallest part in the partition $\mu^2 \cup \nu$ with repetition. We shall construct a new pair (μ', ν') as follows:

- (1) If $\mu_r = x$, then we choose $\mu' = (\mu_1, \dots, \mu_{r-1})$ and $\nu' = \nu \cup (x, x)$.
- (2) Otherwise, $\nu_{n-2r} = \nu_{n-2r+1} = x$, and we choose $\mu' = (\mu_1, \dots, \mu_r, x)$ and $\nu' = \nu \setminus (x, x)$.

Consequently $2|\mu| + |\nu| = 2|\mu'| + |\nu'|$ but the lengths of μ and μ' differ by 1. Indeed the map $f : (\mu, \nu) \mapsto (\mu', \nu')$ is an involution, since for any pair (μ', ν') , we can define the inverse map $g : (\mu', \nu') \mapsto (\mu, \nu)$ as follows. Let y be the smallest part in the partition $\mu'^2 \cup \nu'$ with repetition. If $y \notin \mu'$, then we choose $\mu = \mu' \cup (y)$ and $\nu = \nu' \setminus (y, y)$. Otherwise we choose $\mu = \mu' \setminus (y)$ and $\nu = \nu' \cup (y, y)$. Clearly, g is the inverse of f . This involution leads to (5.2).

In exactly the same way as for the proof of (1.1) and (1.6), we can prove (5.2) by counting the partitions that have exactly k different parts with repetition labeled by s . Let $g_{n,k}(q^{n+1} \left[\begin{smallmatrix} 2n-1 \\ n-2 \end{smallmatrix} \right]_q)$ count the partitions λ with $\lambda_1 \leq n-1$ and $\ell(\lambda) = n+1$ such that there are exactly k different parts with repetition. Then the number of partitions λ with $\lambda_1 \leq n-1$ and $\ell(\lambda) = n+1$, such that there are exactly r different parts with repetition labeled by s , is

$$(5.4) \quad \sum_{k \geq r} \binom{k}{r} g_{n,k}(q^{n+1} \left[\begin{smallmatrix} 2n-1 \\ n-2 \end{smallmatrix} \right]_q) = q^{r^2+r} \left[\begin{smallmatrix} n-1 \\ r \end{smallmatrix} \right]_{q^2} q^{n-2r+1} \left[\begin{smallmatrix} 2n-2r-1 \\ n-2 \end{smallmatrix} \right]_q.$$

Again, by employing the generating function of $g_{n,k}(q^{n+1} \left[\begin{smallmatrix} 2n-1 \\ n-2 \end{smallmatrix} \right]_q)$, we have

$$\begin{aligned} G_n(x, q) &= \sum_{k \geq 0} g_{n,k}(q^{n+1} \left[\begin{smallmatrix} 2n-1 \\ n-2 \end{smallmatrix} \right]_q) x^k = \sum_{r \geq 0} \frac{G_n^{(r)}(1, q)}{r!} (x-1)^r \\ &= \sum_{r \geq 0} q^{r^2+r} \left[\begin{smallmatrix} n-1 \\ r \end{smallmatrix} \right]_{q^2} q^{n-2r+1} \left[\begin{smallmatrix} 2n-2r-1 \\ n-2 \end{smallmatrix} \right]_q (x-1)^r. \end{aligned}$$

By setting $x = 0$ on both sides, we get (5.2) and therefore (1.7) follows. From the proof we see that the expression of $T_r(n, q)$ given in (5.1) implies that the sieving process works on the partitions counted by $\left[\begin{smallmatrix} 2n-1 \\ n-2 \end{smallmatrix} \right]_q$.

6. GENERALIZE (1.7) TO q -BALLOT NUMBERS

The q -Ballot numbers $B_j(n, q)$ are defined by

$$B_j(n, q) = \frac{[j]_q}{[2n+j]_q} \left[\begin{smallmatrix} 2n+j \\ n \end{smallmatrix} \right]_q = \frac{[j]_q}{[n]_q} \left[\begin{smallmatrix} 2n+j-1 \\ n-1 \end{smallmatrix} \right]_q.$$

They count the major index of lattice paths from $(0, 0)$ to $(2n+j-1, -j+1)$ with allowed U -step and D -step that never go below $y = -j+1$; see [9]. In particular, $B_1(n, q) = C_n(q)$ and $B_r(n, 1) = B_{n,r-1}$. Furthermore,

$$B_j(n, q) = \left[\begin{smallmatrix} 2n+j-2 \\ n \end{smallmatrix} \right]_q - q^j \left[\begin{smallmatrix} 2n+j-2 \\ n-2 \end{smallmatrix} \right]_q$$

also counts the partitions λ with $\lambda_1 \leq n+j-2$ and $\ell(\lambda) \leq n$ whose successive ranks are all $< j-1$; see [2]. We generalize (1.7) to an equation for q -Ballot numbers as follows.

Theorem 6.1. *The q -Ballot numbers $B_j(n, q)$ satisfy*

$$(6.1) \quad B_j(n, q) = \sum_{r=1}^n (-1)^{r-1} T_r^{(j)}(n, q),$$

$$(6.2) \quad T_r^{(j)}(n, q) = q^{r^2-r} \left[\begin{smallmatrix} n \\ r \end{smallmatrix} \right]_{q^2} \left[\begin{smallmatrix} 2n+j-1-2r \\ n-1 \end{smallmatrix} \right]_q \frac{[j]_q}{[n]_q},$$

where $T_r^{(j)}(n, q)$ is a polynomial for $n \geq 2r - j$. In particular $T_r^{(j)}(2r - j, -q)$ is a positive polynomial if $j \leq r$.

Proof. First we shall show $T_r^{(j)}(n, q)$ is a polynomial for $n \geq 2r - j$. By setting $d = \gcd(n, r)$ and $m = \gcd(n, 2r - j)$ we express

$$\begin{aligned} T_r^{(j)}(n, q) &= q^{r^2-r} \begin{bmatrix} n \\ r \end{bmatrix}_{q^2} \frac{[d]_{q^2}}{[n]_{q^2}} \begin{bmatrix} 2n+j-1-2r \\ n-1 \end{bmatrix}_q \frac{[n]_{q^2}[j]_q}{[d]_{q^2}[n]_q} \\ &= q^{r^2-r} \begin{bmatrix} n \\ r \end{bmatrix}_{q^2} \frac{[d]_{q^2}}{[n]_{q^2}} \frac{[n]_{q^2}[j]_q}{[d]_{q^2}[m]_q} \frac{[m]_q}{[2n+j-2r]_q} \begin{bmatrix} 2n+j-2r \\ n \end{bmatrix}_q, \end{aligned}$$

where $\begin{bmatrix} n \\ r \end{bmatrix}_{q^2} \frac{[d]_{q^2}}{[n]_{q^2}}$ and $\frac{[m]_q}{[2n+j-2r]_q} \begin{bmatrix} 2n+j-2r \\ n \end{bmatrix}_q$ are positive polynomials as proved in Theorem 4.1 and it remains to show $\frac{[n]_{q^2}[j]_q}{[d]_{q^2}[m]_q}$ is a polynomial. First we can simplify

$$\frac{[n]_{q^2}[j]_q}{[d]_{q^2}[m]_q} = \frac{[2n]_q[j]_q}{[2d]_q[m]_q}.$$

For any $m_1 \mid m$ and $m_1 > 1$, if $m_1 \mid j$, then $\Phi_{m_1}(q) \mid [j]_q$ and therefore $\Phi_{m_1}(q) \mid \frac{[2n]_q}{[2d]_q} [j]_q$. Otherwise, $m_1 \nmid j$ and therefore $m_1 \nmid d$. In fact, $m_1 \nmid (2d)$. If not, then $m_1 \mid (2r)$. Since $m_1 \mid m$, hence $m_1 \mid (2r - j)$. This leads to $m_1 \mid j$, contradicting the assumption. Let ω be any m_1 -th primitive root of unity, i.e., $\omega^{m_1} = 1$. Then we have $[2n]_\omega = 0$ and $[2d]_\omega \neq 0$ which implies $\Phi_{m_1}(q) \nmid \frac{[2n]_q}{[2d]_q}$. It follows that $\Phi_{m_1}(q) \mid \frac{[2n]_q}{[2d]_q} [j]_q$ for any $m_1 \mid m$ and $m_1 > 1$. Since any two cyclotomic polynomials are relatively prime, we have that $\frac{[2n]_q[j]_q}{[2d]_q[m]_q}$ is a polynomial. Therefore

$$(6.3) \quad \frac{[n]_{q^2}[j]_q}{[d]_{q^2}[m]_q} \frac{[m]_q}{[2n+j-2r]_q} \begin{bmatrix} 2n+j-2r \\ n \end{bmatrix}_q = \frac{(1+q^n)(1-q)}{(1-q^{2d})} [j]_q \begin{bmatrix} 2n+j-2r-1 \\ n-1 \end{bmatrix}_q$$

and $T_r^{(j)}(n, q)$ are polynomials.

In particular, $T_r^{(j)}(2r - j, q)$ is a polynomial. We next prove $T_r^{(j)}(2r - j, -q)$ is a positive polynomial if $j \leq r$. If j is even, then

$$T_r^{(j)}(2r - j, -q) = q^{r^2-r} \begin{bmatrix} 2r-j \\ r \end{bmatrix}_{q^2} \frac{1-q^j}{1-q^{2r-j}}$$

holds. Here $q^{r^2-r} \begin{bmatrix} 2r-j \\ r \end{bmatrix}_{q^2} (1-q^j)$ has nonnegative coefficients up to $q^{\lfloor \frac{3r^2-r-2rj}{2} \rfloor}$ since $q^{\frac{r^2-r}{2}} \begin{bmatrix} 2r-j \\ r \end{bmatrix}_q$ is unimodal and j is even. In combination with the expansion of $\frac{1}{1-q^{2r-j}}$ at $q = 0$ where $j \leq r$, we conclude that $T_r^{(j)}(2r - j, -q)$ is a positive polynomial for even j . However, if j is odd, we cannot prove the claim in the same way as before. If j is odd, we have

$$(6.4) \quad T_r^{(j)}(2r - j, -q) = q^{r^2-r} \begin{bmatrix} 2r-j \\ r \end{bmatrix}_{q^2} \frac{1+q^j}{1+q^{2r-j}}.$$

Since $q^{\frac{r^2-r}{2}} \begin{bmatrix} 2r-j \\ r \end{bmatrix}_q$ is a positive polynomial and since j is odd, we find that $(1+q^j)q^{r^2-r} \begin{bmatrix} 2r-j \\ r \end{bmatrix}_{q^2}$ is a positive polynomial. But in view of the expansion of $\frac{1}{1+q^{2r-j}}$ at $q=0$, we get

$$T_r^{(j)}(2r-j, -q) = (1+q^j)q^{r^2-r} \begin{bmatrix} 2r-j \\ r \end{bmatrix}_{q^2} - q^{2r-j}(1+q^j)q^{r^2-r} \begin{bmatrix} 2r-j \\ r \end{bmatrix}_{q^2} + \cdots,$$

where the polynomial $-q^{2r-j}(1+q^j)q^{r^2-r} \begin{bmatrix} 2r-j \\ r \end{bmatrix}_{q^2}$ could make the first half of the coefficients of $(1+q^j)q^{r^2-r} \begin{bmatrix} 2r-j \\ r \end{bmatrix}_{q^2}$ become negative. Therefore, we choose to prove $T_r^{(j)}(2r-j, -q)$ in (6.4) to be a positive polynomial by considering

$$qT_r^{(j)}(2r-j, q) = q^{r^2-r+1} \begin{bmatrix} 2r-j \\ r \end{bmatrix}_{q^2} \frac{1-q^j}{1-q^{2r-j}}.$$

The claim that polynomial $T_r^{(j)}(2r-j, -q)$ has nonnegative coefficients is equivalent to the claim that polynomial $qT_r^{(j)}(2r-j, q)$ has nonnegative coefficients for the odd powers of q , and nonpositive coefficients for the even powers of q . Now for $qT_r^{(j)}(2r-j, q)$ the unimodality of $q^{\frac{r^2-r}{2}} \begin{bmatrix} 2r-j \\ r \end{bmatrix}_q$ implies that $q^{r^2-r+1} \begin{bmatrix} 2r-j \\ r \end{bmatrix}_{q^2} (1-q^j)$ has nonnegative coefficients for the odd powers of q up to $q^{\lfloor \frac{3r^2-r-2rj+1}{2} \rfloor}$. By expanding $\frac{1}{1-q^{2r-j}}$ as a power series at $q=0$, we have that the odd powers of q in the polynomial $qT_r^{(j)}(2r-j, q)$, come from

$$(1+q^{4r-2j}+q^{8r-4j}+\cdots)(1-q^{2r})q^{r^2-r+1} \begin{bmatrix} 2r-j \\ r \end{bmatrix}_{q^2},$$

from which we can conclude that the coefficients of the odd powers of q , up to $q^{\lfloor \frac{3r^2-r-2rj+1}{2} \rfloor}$ in the polynomial $qT_r^{(j)}(2r-j, q)$ are nonnegative. Furthermore, we observe that the maximal degree $d_{\max}$ of q in the polynomial $qT_r^{(j)}(2r-j, q)$ is $3r^2-3r-2rj+2j+1$ and the minimal degree $d_{\min}$ of q in the polynomial $qT_r^{(j)}(2r-j, q)$ is r^2-r+1 . Let a_i be the coefficient of q^i in the polynomial $qT_r^{(j)}(2r-j, q)$. Then from the reciprocity of the polynomial $qT_r^{(j)}(2r-j, q)$ we get $a_i = a_j$ if $j+i = d_{\max} + d_{\min} = 4r^2-2rj-4r+2j+2$. This implies for i odd and $a_i \geq 0$, j is odd and $a_j \geq 0$. Therefore we can conclude that the coefficients of all the odd powers of q in the polynomial $qT_r^{(j)}(2r-j, q)$ are nonnegative. This is equivalent to say that the coefficients of all the even powers of q in the polynomial $T_r^{(j)}(2r-j, q)$ are nonnegative. By expanding $\frac{1-q^j}{1-q^{2r-j}}$ at $q=0$, we have that the even powers of q in the polynomial $qT_r^{(j)}(2r-j, q)$ come from

$$-q^j(1+q^{4r-2j}+q^{8r-4j}+\cdots)(1-q^{2r-2j})q^{r^2-r+1} \begin{bmatrix} 2r-j \\ r \end{bmatrix}_{q^2},$$

where $-q^{r^2-r+1+j} \begin{bmatrix} 2r-j \\ r \end{bmatrix}_{q^2} (1-q^{2r-2j})$ is a polynomial that has nonpositive coefficients for the even powers of q , up to $q^{\lfloor \frac{3r^2-2rj-r+j+1}{2} \rfloor}$. Again from the reciprocity of polynomial $qT_r^{(j)}(2r-j, q)$, we conclude that the coefficients of all the even powers of q in the polynomial $qT_r^{(j)}(2r-j, q)$ are nonpositive. That is to say, the coefficients of all the odd powers of q in the polynomial $T_r^{(j)}(2r-j, -q)$ are nonnegative. Now it remains to prove (6.1). By similar techniques as used to prove (1.7), we

consider the partitions counted by $q^{n+j} \begin{bmatrix} 2n+j-1 \\ n-1 \end{bmatrix}_q$ and let $g_{n,k}(q^{n+j} \begin{bmatrix} 2n+j-1 \\ n-1 \end{bmatrix}_q)$ count the partitions λ with $\lambda_1 \leq n$ and $\ell(\lambda) = n+j$ such that there are exactly k different parts with repetition. By following the same techniques used in the proof of (1.7) of Section 5 (see the proof of (5.4)), we can derive the identity

$$q^{n+j} \begin{bmatrix} 2n+j-1 \\ n-1 \end{bmatrix}_q = \sum_{r=1}^n (-1)^{r-1} q^{r^2+r} \begin{bmatrix} n \\ r \end{bmatrix}_{q^2} q^{n+j-2r} \begin{bmatrix} 2n+j-2r-1 \\ n-1 \end{bmatrix}_q,$$

which is equivalent to (6.1). $\square$

Conjecture. *Here we conjecture that if n is odd, then for $m \geq n \geq 1$ the polynomial $(1+q^n) \begin{bmatrix} m \\ n-1 \end{bmatrix}_q$ is unimodal. If n is even, then for any even $j \neq 0$ and $m \geq n \geq 1$, the polynomial $(1+q^n)[j]_q \begin{bmatrix} m \\ n-1 \end{bmatrix}_q$ is unimodal. This, in combination with (6.3), implies that $T_r(n, q)$ is a positive polynomial for odd n and $n \geq 2r+1$, and $T_r^{(j)}(n, q)$ given in (6.2) is a positive polynomial for $n \geq 2r-j+1$.*

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