QUANTUM AFFINE MODULES FOR NON-TWISTED AFFINE KAC-MOODY ALGEBRAS

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ABSTRACT. We construct new irreducible weight modules over quantum affine algebras of type I with all weight spaces infinite dimensional. These modules are obtained by parabolic induction from irreducible modules over the Heisenberg subalgebra.

INTRODUCTION

One of the distinctive features of affine Kac-Moody algebras, in contrast to the finite dimensional semisimple Lie algebra case, is the existence of Borel subalgebras which are not Weyl group conjugate to the standard ones generated by the positive (resp. negative) simple generators ([11] and [7]). This leads to a very rich module theory, which includes weight modules all of whose weight spaces are infinite dimensional constructed by Benkart, Bekkert, Futorny and Kashuba ([3]). In this paper we construct modules for untwisted affine quantum groups based on this construction which also have all their weight spaces infinite dimensional.

The main tool we will use in this construction is the Poincaré Birkhoff Witt type basis given in terms of the Drinfel'd generators of the quantum affine algebra ([5], see also [1]). These generators contain a copy of the infinite rank quantized Heisenberg algebra, G_q , generated by the imaginary root vectors. We obtain an isomorphism between G_q and an infinite rank Weyl algebra of which it is a trivial deformation. This allows us to apply the classification of Futorny, Grncharov, and Mazorchuk ([6]) to obtain an ample supply of irreducible G_q -modules. We use parabolic induction on these modules to obtain modules of the full quantum affine algebra. If V is an $\mathbb{A}$ -admissible irreducible G_q -module, then we use the $\mathbb{A}$ -form of the induced module to show that it is a deformation of an irreducible module for the affine Kac-Moody algebra (see §1.1 for the definition of $\mathbb{A}$). Moreover, as in Lusztig's result ([13]) we see that the weight multiplicities are the same as in the $q = 1$ case. In the final section, we apply the result of the previous section to the case of φ -imaginary Verma modules for $\varphi : \mathbb{N} \rightarrow \{\pm\}$ an arbitrary function. By [3, Proposition 3.4] and its corollary we obtain conditions for all weight spaces of these modules to be infinite dimensional and irreducible.

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1. PRELIMINARIES

Throughout this paper, $\mathbb{N}$ denotes the set of non-negative integers. Let $I = \{1, \dots, n\}$ and let $\mathfrak{g} = X_n^{(1)}$ be an untwisted affine type Kac-Moody algebra corresponding to a generalized affine Cartan matrix of type 1: $A = (a_{ij})_{0 \leq i, j \leq n}$. The loop algebra construction (see [12]) gives:

$$(1.1) \quad \mathfrak{g} = (\mathring{\mathfrak{g}} \otimes \mathbb{C}[t, t^{-1}]) \oplus \mathbb{C}c \oplus \mathbb{C}d$$

where $\mathring{\mathfrak{g}}$ is a simple finite dimensional Lie algebra over $\mathbb{C}$ of type X_n , c is the central generator and d is the degree derivation. Then $\mathfrak{h} = \mathring{\mathfrak{h}} \oplus \mathbb{C}c \oplus \mathbb{C}d$ is a Cartan subalgebra of $\mathfrak{g}$, where $\mathring{\mathfrak{h}}$ is a Cartan subalgebra of $\mathring{\mathfrak{g}}$. Let $\mathcal{D} = (d_0, \dots, d_n)$ be a diagonal matrix with relatively prime integer entries such that the matrix $\mathcal{D}A$ is symmetric. Let $\mathring{\Delta}$ be the root system of $\mathring{\mathfrak{g}}$ and let $\mathring{\Delta}^\pm$ be the sets of positive and negative roots with respect to some choice of simple roots $\alpha_1, \dots, \alpha_n$. Then, the sets of positive (resp. negative) roots of $\mathfrak{g}$ are:

$$\Delta^\pm = \{\alpha \pm k\delta \mid \alpha \in \mathring{\Delta} \cup \{0\}, k \in \mathbb{Z}_{>0}\} \cup \mathring{\Delta}^\pm.$$

For each $\alpha \in \Delta$ set $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}$. Denote by $\mathring{Q}$ the free abelian group generated by $\alpha_1, \dots, \alpha_n$, a root lattice of $\mathring{\mathfrak{g}}$. Let $Q := \mathring{Q} \oplus \mathbb{Z}\delta$ be the root lattice of $\mathfrak{g}$ and let $P = \{\lambda \in \mathfrak{h}^* \mid \lambda(h_i) \in \mathbb{Z}, i \in I, \lambda(d) \in \mathbb{Z}\}$ be the weight lattice, where $h_1, \dots, h_n$ is a basis of $\mathring{\mathfrak{h}}$.

Unlike in the finite dimensional case, Δ does not consist of a single Weyl group orbit. Instead, $\Delta = \Delta^{re} \cup \Delta^{im}$ where Δ^{re} is the set of all roots that are conjugate to a simple root α_i and $\Delta^{im} = \Delta \setminus \Delta^{re}$, called the sets of real and imaginary roots respectively. As seen in [12], for an untwisted affine type Kac-Moody algebra we have $\Delta^{re} = \{\alpha + k\delta \mid \alpha \in \mathring{\Delta}, k \in \mathbb{Z}\}$ and $\Delta^{im} = \{k\delta \mid k \in \mathbb{Z} \setminus \{0\}\}$ where δ is the indivisible imaginary root.

Consider the partition $\Delta = S \cup -S$ of the root system of $\mathfrak{g}$ where $S = \{\alpha + k\delta \mid \alpha \in \mathring{\Delta}^+, k \in \mathbb{Z}\} \cup (\Delta^{im})^+$. This is a non-standard partition of the root system Δ in the sense that S is not conjugated to the sets $\Delta^\pm$ by the Weyl group. The classification of closed partitions of the root system for affine Kac-Moody algebras was obtained by Jakobsen and Kac [11], and independently by Futorny [7, 8].

The *quantum affine algebra* $U_q(\mathfrak{g})$ is the $\mathbb{C}(q^{1/2})$ -algebra with 1 generated by

$$E_i, \quad F_i, \quad K_\alpha, \quad \gamma^{\pm 1/2}, \quad D^{\pm 1}, \quad i \in I \cup \{0\}, \quad \alpha \in Q,$$

and defining relations:

$$DD^{-1} = D^{-1}D = K_i K_i^{-1} = K_i^{-1} K_i = \gamma^{1/2} \gamma^{-1/2} = 1,$$

$$[\gamma^{\pm 1/2}, U_q(\mathfrak{g})] = [D, K_i^{\pm 1}] = [K_i, K_j] = 0,$$

$$(\gamma^{\pm 1/2})^2 = K_\delta^{\pm 1},$$

$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$

$$K_\alpha E_i K_\alpha^{-1} = q^{(\alpha|\alpha_i)} E_i, \quad K_\alpha F_i K_\alpha^{-1} = q^{-(\alpha|\alpha_i)} F_i,$$

$$D E_i D^{-1} = q^{\delta_{i,0}} E_i, \quad D F_i D^{-1} = q^{-\delta_{i,0}} F_i,$$

$$\sum_{s=0}^{1-a_{ij}} (-1)^s E_i^{(1-a_{ij}-s)} E_j E_i^{(s)} = 0 = \sum_{s=0}^{1-a_{ij}} (-1)^s F_i^{(1-a_{ij}-s)} F_j F_i^{(s)}, \quad i \neq j,$$

where

$$q_i := q^{d_i}, \quad [n]_i = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}, \quad [n]_i! := \prod_{k=1}^n [k]_i$$

and $K_i = K_{\alpha_i}$, $E_i^{(s)} = E_i/([s]_i!)$ and $F_i^{(s)} = F_i/([s]_i!)$. We also define

$$\begin{bmatrix} r \\ s \end{bmatrix}_i := \frac{[r]_i!}{[s]_i! [r-s]_i!}.$$

Let $U_q^+ = U_q^+(\mathfrak{g})$ (resp. $U_q^- = U_q^-(\mathfrak{g})$) be the subalgebra of $U_q(\mathfrak{g})$ generated by E_i (resp. F_i), $i \in I$, and let $U_q^0 = U_q^0(\mathfrak{g})$ denote the subalgebra generated by $K_i^{\pm 1}$ ($i \in I$), $\gamma^{\pm 1/2}$, and $D^{\pm 1}$.

We will also use Drinfel'd realization of $U_q(\mathfrak{g})$ [5]. It can be generated over $\mathbb{C}(q^{1/2})$ by

$$x_{ir}^{\pm}, \quad h_{is}, \quad K_i^{\pm 1}, \quad \gamma^{\pm 1/2}, \quad D^{\pm 1}, \quad i \in I, r, s \in \mathbb{Z}, s \neq 0,$$

subject to the following relations:

$$(1.2) \quad DD^{-1} = D^{-1}D = K_i K_i^{-1} = K_i^{-1} K_i = \gamma^{1/2} \gamma^{-1/2} = 1,$$

$$(1.3) \quad [\gamma^{\pm 1/2}, U_q(\mathfrak{g})] = [D, K_i^{\pm 1}] = [K_i, K_j] = [K_i, h_{jk}] = 0,$$

$$(1.4) \quad Dh_{ir} D^{-1} = q^r h_{ir}, \quad Dx_{ir}^{\pm} D^{-1} = q^r x_{ir}^{\pm},$$

$$(1.5) \quad K_i x_{jr}^{\pm} K_i^{-1} = q_i^{\pm(\alpha_i|\alpha_j)} x_{jr}^{\pm},$$

$$(1.6) \quad [h_{ik}, h_{jl}] = \delta_{k,-l} \frac{1}{k} [ka_{ij}]_i \frac{\gamma^k - \gamma^{-k}}{q_j - q_j^{-1}},$$

$$(1.7) \quad [h_{ik}, x_{jl}^{\pm}] = \pm \frac{1}{k} [ka_{ij}]_i \gamma^{\mp |k|/2} x_{j,k+l}^{\pm},$$

$$(1.8) \quad \begin{aligned} x_{i,k+1}^{\pm} x_{jl}^{\pm} - q^{\pm(\alpha_i|\alpha_j)} x_{jl}^{\pm} x_{i,k+1}^{\pm} \\ = q^{\pm(\alpha_i|\alpha_j)} x_{ik}^{\pm} x_{j,l+1}^{\pm} - x_{j,l+1}^{\pm} x_{ik}^{\pm}, \end{aligned}$$

$$(1.9) \quad [x_{ik}^+, x_{jl}^-] = \delta_{ij} \frac{1}{q_i - q_i^{-1}} \left(\gamma^{\frac{k-l}{2}} \psi_{i,k+l} - \gamma^{\frac{l-k}{2}} \varphi_{i,k+l} \right),$$

$$\text{where } \sum_{k=0}^{\infty} \psi_{ik} z^k = K_i \exp \left((q_i - q_i^{-1}) \sum_{l>0} h_{il} z^l \right), \text{ and}$$

$$(1.10) \quad \sum_{k=0}^{\infty} \varphi_{i,-k} z^{-k} = K_i^{-1} \exp \left(-(q_i - q_i^{-1}) \sum_{l>0} h_{i,-l} z^{-l} \right).$$

$$\text{For } i \neq j, \quad N := 1 - a_{ij}$$

$$(1.11) \quad \text{Sym}_{k_1, k_2, \dots, k_N} \sum_{r=0}^N (-1)^r \begin{bmatrix} N \\ r \end{bmatrix}_i x_{ik_1}^{\pm} \cdots x_{ik_r}^{\pm} x_{j_l}^{\pm} x_{ik_{r+1}}^{\pm} \cdots x_{ik_s}^{\pm} = 0$$

(see [1] and [4]). Here Sym denotes symmetrization with respect to $k_1, \dots, k_N$.

Beck [1] has defined an ordering of Δ and a related PBW-type basis for quantum affine algebras. Let $\rho = (1/2) \sum_{\beta \in \tilde{\Delta}^+} \beta$ and fix a reduced expression $t_{\rho} = r_{i_1} r_{i_2} \cdots r_{i_d}$ for the Weyl group element t_{ρ} corresponding to translation by ρ . For $\mathfrak{g}$ recall the Lusztig automorphisms ([13]) $T_i, 0 \leq i \leq n$, of $U_q(\mathfrak{g})$. To each root $\alpha \in \Delta^{re}$ one assigns a corresponding root vector E_{α} . Let $(i_k)_{k \in \mathbb{Z}}$ be the sequence

of integers such that $i_k = i_k \pmod{d}$ and define the following sequence of real roots:

$$\beta_k = \begin{cases} r_{i_1} r_{i_2} \cdots r_{i_{k-1}}(\alpha_k), & \text{if } k \geq 0, \\ r_{i_0} r_{i_{-1}} \cdots r_{i_{k+1}}(\alpha_k), & \text{otherwise.} \end{cases}$$

Then $(\Delta^{re})^+ = (\beta_k)_{k \in \mathbb{Z}}$ and we put:

$$E_{\beta_k} = \begin{cases} T_{i_1} T_{i_2} \cdots T_{i_{k-1}}(F_{i_k}) & \text{if } k \geq 0, \\ T_{i_0}^{-1} T_{i_{-1}}^{-1} \cdots T_{i_{k+1}}^{-1}(E_{i_k}), & \text{otherwise.} \end{cases}$$

The positive imaginary root vectors are defined by the following functional equation:

$$1 + (q_i - q_i^{-1}) \sum_{k \geq 0} K_i^{-1} [E_i, E_{-\alpha_i + k\delta}] z^k = \exp \left((q_i - q_i^{-1}) \sum_{k=1}^{\infty} E_{k\delta}^{(i)} z^k \right).$$

Recall the standard $\mathbb{C}$ -algebra antiautomorphism (see [13]) $\Omega : U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{g})$ given by $\Omega(E_i) = F_i$, $\Omega(F_i) = E_i$, $\Omega(K) = K^{-1}$ for $K \in U_q^0(\mathfrak{g})$ and $\Omega(q^{1/2}) = q^{-1/2}$. For $\beta \in \Delta^+$, $F_\beta := E_{-\beta}$ is defined to be $\Omega(E_\beta)$.

We remark that the vectors $E_{k\delta}^{(i)}$ are related to the Drinfel'd generators by $E_{k\delta}^{(i)} = \gamma^{-|k|/2} h_{ik}$ for all $k \in \mathbb{Z}_{\neq 0}$ (see [4]). Similarly, the vectors $E_{\pm\alpha_i + k\delta}$, $i \in I$, $k \in \mathbb{Z}$, are related to the Drinfel'd generators $x_{i,k}^\pm$ by multiplication with some $K \in U_q^0$. We recall:

Theorem 1.1 ([1]). *The set of monomials in $E_\alpha, \alpha \in \Delta^+$ (resp. $E_\alpha, \alpha \in \Delta^-$), with respect to any ordering is a basis of U_q^+ (resp. U_q^-).*

The total height of x is defined to be $d_0 = \sum_{j=1}^k (a_j + a'_j) \text{ht}(\gamma_j)$. We will use the following ordering of positive roots:

$$(1.12) \quad \beta_0 > \beta_{-1} > \beta_{-2} > \cdots > \delta > 2\delta > \cdots > \beta_2 > \beta_1.$$

We also set $-\alpha < -\beta$ if and only if $\beta > \alpha$ for all positive roots α, β , providing an ordering on Δ^- . Beck's total ordering of the positive roots can be divided into three sets:

$$\{\alpha + k\delta \mid \alpha \in \mathring{\Delta}^+, k \geq 0\} > \{k\delta \mid k > 0\} > \{-\alpha + k\delta \mid \alpha \in \mathring{\Delta}^+, k > 0\}.$$

Similarly, for the negative roots, we have

$$\{-\alpha - k\delta \mid \alpha \in \mathring{\Delta}^+, k \geq 0\} < \{-k\delta \mid k > 0\} < \{\alpha - k\delta \mid \alpha \in \mathring{\Delta}^+, k > 0\}.$$

For $r \in \mathbb{Z}$, we define

$$\beta_r^\pm = \begin{cases} \pm \beta_r & \text{if } r \leq 0, \\ \mp \beta_r & \text{if } r > 0. \end{cases}$$

Hence, we have $S = \{\beta_r^+ \mid r \in \mathbb{Z}\} \cup (\Delta^{im})^+$. Set

$$X_{\beta_r^+} = \begin{cases} E_{\beta_r} & \text{if } r \leq 0, \\ -F_{\beta_r} K_{\beta_r} & \text{if } r \geq 1, \end{cases} \quad X_{\beta_r^-} = \begin{cases} F_{\beta_r} & \text{if } r \leq 0, \\ -K_{\beta_r}^{-1} E_{\beta_r} & \text{if } r \geq 1. \end{cases}$$

We have the following result which gives a PBW basis associated with a non-standard partition of the root system.

Theorem 1.2 ([4]). Given $\underline{m} : \mathbb{Z} \ni r \mapsto m_r \in \mathbb{N}$ such that $\#\{r \in \mathbb{Z} | m_r \neq 0\} < \infty$ define

$$X^-(\underline{m}) = \prod_{r \in \mathbb{Z}} X_{\beta_r^-}^{m_r}, \quad X^+(\underline{m}) = \prod_{r \in \mathbb{Z}} X_{\beta_r^+}^{m_r}$$

where one chooses a fixed ordering for the products.

Given $\underline{l} : \mathbb{N} \times I \rightarrow \mathbb{N}$ such that $\#\{(r, i) \in \mathbb{N} \times I | l_{(r,i)} \neq 0\} < \infty$ define

$$E^{im}(\underline{l}) = \prod_{(r,i) \in \mathbb{N} \times I} E_{(r\delta, i)}^{l_{(r,i)}}, \quad F^{im}(\underline{l}) = \Omega(E^{im}(\underline{l}))$$

where $E_{(r,i)} = E_{r\delta}^{(i)}$. Then the set

$$(1.13) \quad \{X^-(\underline{m})F^{im}(\underline{l})K_\alpha D^r \gamma^{s/2} E^{im}(\underline{l}') X^+(\underline{m}')\}, \quad r, s \in \mathbb{Z}, \quad \alpha \in \check{Q},$$

is a basis of $U_q(\mathfrak{g})$.

A weight $\mathfrak{g}$ -module $\mathbf{V}$ with respect to $\mathfrak{h}$ has a decomposition $\mathbf{V} = \bigoplus_{\lambda \in \mathfrak{h}^*} \mathbf{V}_\lambda$, where $\mathbf{V}_\lambda = \{v \in \mathbf{V} | hv = \lambda(h)v \text{ for all } h \in \mathfrak{h}\}$, and we say that the set of weights of $\mathbf{V}$ is the support of $\mathbf{V}$ and write $\text{supp}(\mathbf{V}) = \{\lambda \in \mathfrak{h}^* | \mathbf{V}_\lambda \neq 0\}$. Similarly, a weight $U_q(\mathfrak{g})$ -module $\mathbf{V}$ has a decomposition $\mathbf{V} = \bigoplus_{\lambda \in P} \mathbf{V}_\lambda$, where $\mathbf{V}_\lambda = \{v \in \mathbf{V} | K_i v = q^{\lambda(h_i)} v, D^{\pm 1} v = q^{\pm \lambda(d)} v\}$.

1.1. A-forms. Set $\mathbb{A} = \mathbb{C}[q^{1/2}, q^{-1/2}, \frac{1}{[n]_{q_i}}, i \in I, n > 1]$. Following [4] we define the algebra $U_{\mathbb{A}} = U_{\mathbb{A}}(\mathfrak{g})$ to be the $\mathbb{A}$ -subalgebra of $U_q(\mathfrak{g})$ with 1 generated by the elements

$$x_{ir}^{\pm 1}, h_{is}, K_i^{\pm 1}, \gamma^{\pm 1/2}, D^{\pm 1}, \begin{bmatrix} K_i; s \\ r \end{bmatrix}, \begin{bmatrix} D; s \\ r \end{bmatrix}, \begin{bmatrix} \gamma; s \\ 1 \end{bmatrix}_i, \begin{bmatrix} \gamma\psi_i; k, l \\ 1 \end{bmatrix}$$

for each $i \in I$, $s \in \mathbb{Z}$ and $n \in \mathbb{Z}_+$, where following [13], we define the *Lusztig elements* in $U_q(\mathfrak{g})$:

$$(1.14) \quad \begin{bmatrix} \gamma; s \\ 1 \end{bmatrix}_i = \frac{\gamma^s - \gamma^{-s}}{q_i - q_i^{-1}},$$

$$(1.15) \quad \begin{bmatrix} \gamma\psi_i; k, l \\ 1 \end{bmatrix} = \frac{\gamma^{\frac{k-l}{2}} \psi_{i,k+l} - \gamma^{\frac{l-k}{2}} \varphi_{i,k+l}}{q_i - q_i^{-1}},$$

$$(1.16) \quad \begin{bmatrix} K_i; s \\ r \end{bmatrix} = \prod_{j=1}^r \frac{K_i q_i^{s-j+1} - K_i^{-1} q_i^{-(s-j+1)}}{q_i^j - q_i^{-j}}, \quad \text{and}$$

$$(1.17) \quad \begin{bmatrix} D; s \\ r \end{bmatrix} = \prod_{j=1}^r \frac{D q_0^{s-j+1} - D^{-1} q_0^{-(s-j+1)}}{q_0^j - q_0^{-j}}$$

where $q_0 = q^{d_0}$. Let $U_{\mathbb{A}}^+$ (resp. $U_{\mathbb{A}}^-$) denote the subalgebra of $U_{\mathbb{A}}$ generated by the x_{ik}^+ , $k \in \mathbb{Z}$, $i \in I$ (resp. x_{ik}^- , $k \in \mathbb{Z}$, $i \in I$). Also denote by $U_{\mathbb{A}}^0$ the subalgebra of $U_{\mathbb{A}}$ generated by the elements h_{il} , $l \in \mathbb{N} \setminus \{0\}$, $1 \leq i \leq n$, $\gamma^{\pm 1/2}$, $K_i^{\pm 1}$, $\begin{bmatrix} K_i; s \\ r \end{bmatrix}$, $D^{\pm 1}$, $\begin{bmatrix} D; s \\ r \end{bmatrix}$, $\begin{bmatrix} \gamma; s \\ 1 \end{bmatrix}_i$ and $\begin{bmatrix} \gamma\psi_i; k, -k \\ 1 \end{bmatrix}$.

We recall:

Lemma 1.3 ([1, Proposition 3.10, Lemma 3.15]). *Set $a = q_i^2 \gamma^{1/2}$, $b = q_i^2 \gamma^{-1/2}$, $c = -q_i^{a_{ij}} \gamma^{1/2}$, $d = -q_i^{a_{ij}} \gamma^{-1/2}$ for $i \neq j$, $r > 0$, $m \in \mathbb{Z}$. Then*

$$\begin{aligned} [\psi_{ir}, x_{im}^-] &= -\gamma^{1/2} [2]_i \left(\sum_{k=1}^{r-1} a^{k-1} (q_i - q_i^{-1}) \psi_{i,r-k} x_{i,m+k}^- + a^{r-1} x_{i,m+r}^- \right), \\ [\psi_{ir}, x_{im}^+] &= \gamma^{-1/2} [2]_i \left(\sum_{k=1}^{r-1} b^{k-1} (q_i - q_i^{-1}) x_{i,m+k}^+ \psi_{i,r-k} + b^{r-1} x_{i,m+r}^+ \right), \\ [\psi_{ir}, x_{jm}^-] &= \gamma^{1/2} [a_{ij}]_i \left(\sum_{k=1}^{r-1} c^{k-1} (q_i - q_i^{-1}) \psi_{i,r-k} x_{j,m+k}^- + c^{r-1} x_{j,m+r}^- \right), \\ [\psi_{ir}, x_{jm}^+] &= -\gamma^{-1/2} [a_{ij}]_i \left(\sum_{k=1}^{r-1} d^{k-1} (q_i - q_i^{-1}) x_{j,m+k}^+ \psi_{i,r-k} + d^{r-1} x_{j,m+r}^+ \right). \end{aligned}$$

A similar set of formulas can be obtained for φ_{ir} by applying the antiautomorphism Ω .

The next result follows from direct calculations; see [4, Proposition 3.4.2].

Proposition 1.4. *The following commutation relations hold between the generators of $U_{\mathbb{A}}$. For $k \in \mathbb{Z}$, $l \in \mathbb{Z} \setminus \{0\}$, $1 \leq i, j \leq n$,*

$$\begin{aligned} x_{ik}^+ \begin{bmatrix} K_j; s \\ r \end{bmatrix} &= \begin{bmatrix} K_j; s - a_{ji} \\ r \end{bmatrix} x_{ik}^+, \\ x_{ik}^+ \begin{bmatrix} D; s \\ r \end{bmatrix} &= \begin{bmatrix} D; s - k \\ r \end{bmatrix} x_{ik}^+, \\ \begin{bmatrix} K_j; s \\ r \end{bmatrix} x_{ik}^- &= x_{ik}^- \begin{bmatrix} K_j; s - a_{ji} \\ r \end{bmatrix}, \\ \begin{bmatrix} D; s \\ r \end{bmatrix} x_{ik}^- &= x_{ik}^- \begin{bmatrix} D; s - k \\ r \end{bmatrix}, \\ \begin{bmatrix} D; s \\ r \end{bmatrix} h_{ik} &= h_{ik} \begin{bmatrix} D; s + k \\ r \end{bmatrix}, \\ [\gamma^{\pm 1/2}, U_{\mathbb{A}}] &= [D, K_i^{\pm 1}] = [K_i, K_j] = [K_i, h_{jk}] = 0, \\ Dh_{ik} &= q^k h_{ik} D, \quad Dx_{ik}^{\pm} = q^k x_{ik}^{\pm} D, \\ K_i x_{jr}^{\pm} &= q_i^{\pm(\alpha_i | \alpha_j)} x_{jr}^{\pm} K_i, \\ [h_{ik}, h_{jl}] &= \delta_{k,-l} \frac{1}{k} [ka_{ij}]_i \begin{bmatrix} \gamma; k \\ 1 \end{bmatrix}_i, \\ [h_{ik}, x_{jl}^{\pm}] &= \pm \frac{1}{k} [ka_{ij}]_i \gamma^{\mp |k|/2} x_{j,k+l}^{\pm}, \\ [x_{ik}^+, x_{jl}^-] &= \delta_{ij} \begin{bmatrix} \gamma \psi_i; k, l \\ 1 \end{bmatrix}. \end{aligned}$$

Applying Proposition 1.4 and following the proof of Corollary 3.4.3 in [4] we obtain:

Corollary 1.5. *The algebra $U_{\mathbb{A}}$ inherits the triangular decomposition of $U_q(\mathfrak{g})$: $U_{\mathbb{A}} = U_{\mathbb{A}}^{-} U_{\mathbb{A}}^0 U_{\mathbb{A}}^{+}$. In particular, any element u of $U_{\mathbb{A}}$ can be written as an $\mathbb{A}$ -linear combination of monomials of the form $u^{-} u^0 u^{+}$ where $u^{\pm} \in U_{\mathbb{A}}^{\pm}$ and $u^0 \in U_{\mathbb{A}}^0$.*

2. IRREDUCIBLE WEIGHT MODULES OVER QUANTUM HEISENBERG ALGEBRAS

In the Drinfel'd presentation ([5]) of the quantized enveloping algebra $U_q(\mathfrak{g})$, the quantum Heisenberg subalgebra G_q of $U_q(\mathfrak{g})$ is generated over $\mathbb{C}(q^{1/2})$ by h_{ik} , $i \in I$, $k \in \mathbb{Z}_{\neq 0}$, and $\gamma^{\pm 1/2}$, where $\gamma^{\pm 1/2}$ is central and

$$(2.1) \quad h_{ik} h_{jl} - h_{jl} h_{ik} = \delta_{k+l,0} \cdot a_{ij;q}^k \cdot \frac{\gamma^k - \gamma^{-k}}{q - q^{-1}} \quad \forall i, j \in I, k, l \in \mathbb{Z}_{\neq 0}$$

where

$$(2.2) \quad a_{ij;q}^k = \frac{[k \cdot a_{ij}]_{q_i}}{k \cdot [d_j]_{q_j}}.$$

We have

$$(2.3) \quad \lim_{q \rightarrow 1} a_{ij;q}^k = \frac{(\alpha_i | \alpha_j)}{d_i d_j}$$

where $\lim_{q \rightarrow 1} a_{ij;q}^k$ means we expand any expression $[n]_{q_i}$ in powers of q_i and then specialize q to 1. By (2.3) and the non-degeneracy of the form $(\cdot | \cdot)$, the matrix $(a_{ij;q}^k)_{i,j \in I}$ is invertible for any $k \in \mathbb{Z}_{\neq 0}$.

Let $(b_{ij;q}^k)_{i,j \in I}$ be the inverse matrix of $(a_{ij;q}^k)_{i,j \in I}$. Make the following change of variables among the negative generators of G_q :

$$(2.4) \quad h'_{j,-k} = \sum_{m=1}^n b_{mj;q}^k h_{m,-k}, \quad j \in I, k \in \mathbb{Z}_{>0}.$$

Then (2.1) is equivalent to $(\forall i, j \in I, \forall k, l \in \mathbb{Z}_{>0})$

$$(2.5a) \quad h_{ik} h'_{j,-l} - h'_{j,-l} h_{ik} = \delta_{kl} \delta_{ij} \cdot \frac{\gamma^k - \gamma^{-k}}{q - q^{-1}},$$

$$(2.5b) \quad h_{ik} h_{jl} - h_{jl} h_{ik} = 0,$$

$$(2.5c) \quad h'_{i,-k} h'_{j,-l} - h'_{j,-l} h'_{i,-k} = 0.$$

Let $A_{n \cdot \infty}$ be the countably infinite rank Weyl algebra with generators X_{ik}, ∂_{ik} , ($i \in I, k \in \mathbb{Z}_{>0}$) and relations

$$(2.6) \quad \partial_{ik} X_{ik} - X_{ik} \partial_{ik} = 1,$$

$$(2.7) \quad X_{ik} X_{jl} - X_{jl} X_{ik} = \partial_{ik} X_{jl} - X_{jl} \partial_{ik} = \partial_{ik} \partial_{jl} - \partial_{jl} \partial_{ik} = 0, \quad (i, k) \neq (j, l).$$

Proposition 2.1. *For any $\ell \in \mathbb{Z} \setminus \{0\}$, we have an isomorphism of $\mathbb{C}(q^{1/2})$ -algebras*

$$\begin{aligned} \psi : G_q / \langle \gamma - q^\ell \rangle &\longrightarrow A_{n \cdot \infty}, \\ h_{ik} &\longmapsto [k\ell]_q \cdot \partial_{ik}, \\ h'_{i,-k} &\longmapsto X_{ik} \end{aligned}$$

for all $i \in I, k \in \mathbb{Z}_{>0}$.

Proof. Using (2.5) one checks that ψ is well-defined. It is clearly surjective. Then using the relations of the Weyl algebra one checks that the inverse assignments also define a surjective homomorphism. $\square$

Hence, we can identify $G_q/\langle\gamma - q^\ell\rangle$ with a countably infinite rank Weyl algebra. In [6] the authors classified all simple weight (when all $\partial_{ik}X_{ik}$ are simultaneously diagonalizable) modules over countably infinite rank Weyl algebras. Through the isomorphism in Proposition 2.1 we obtain an ample supply of irreducible weight modules over the quantum Heisenberg algebra G_q .

2.1. φ -imaginary Verma modules for the quantum Heisenberg algebra.

We construct a particular class of modules over the quantum Heisenberg algebra G_q quantizing the so-called φ -imaginary Verma modules for classical Heisenberg algebras considered in [3]. Denote by H_q the associative algebra over $\mathbb{C}(q^{1/2})$ generated by $a_i, i \in \mathbb{Z}_{\neq 0}, \gamma^{\pm 1/2}$ with $\gamma^{\pm 1/2}$ central and the following relations:

$$(2.8) \quad a_i a_j - a_j a_i = \delta_{i+j,0} \frac{[2i]_q}{i} \frac{\gamma^i - \gamma^{-i}}{q - q^{-1}}, i \in \mathbb{Z}_{\neq 0}.$$

H_q is $\mathbb{Z}$ -graded with $\deg(a_i) = i$ and $\deg(\gamma^{\pm 1/2}) = 0$. Due to (2.5) we have an isomorphism between G_q and the tensor product of n copies of H_q . Therefore, we can use representations of H_q to construct modules over G_q . Again H_q can be viewed as a certain Weyl algebra by Proposition 2.1.

Let $\varphi : \mathbb{Z}_{>0} \rightarrow \{\pm\}$ and define $H_\varphi^\pm$ to be the subalgebra of H_q generated by $a_{\pm\varphi(i)i}, i \in \mathbb{N}$. Let B_φ be the subalgebra of H_q generated by H_φ^+ and $\gamma^{\pm 1/2}$. Let $\mathbb{C}(q^{1/2})v_\ell, \ell \in \mathbb{Z}$, be the representation of B_φ such that $\gamma \cdot v_\ell = q^\ell v_\ell, H_\varphi^+ \cdot v_\ell = 0$. Then the φ -imaginary Verma module for H_q, ℓ is defined to be:

$$(2.9) \quad M_\varphi^q(\ell) = H_q \otimes_{B_\varphi} \mathbb{C}(q^{1/2})v_\ell.$$

We denote by H the infinite Heisenberg Lie algebra generated by $a_i, i \in \mathbb{Z}_{\neq 0}, c$ with c central and the following relations:

$$[a_i, a_j] = \delta_{i+j,0} c, i, j \in \mathbb{Z}_{\neq 0}.$$

φ -imaginary Verma modules for H were studied in [3].

H_q has a natural grading in which $a_i, i \in \mathbb{Z}_{\neq 0}$, is given degree i and c is given degree 0. We define $(H_q)_n$ to be the homogeneous component of H_q of degree n and set $(H_\varphi^-)_n = (H_q)_n \cap H_\varphi^-$. We have the following analogous properties of φ -imaginary Verma modules for a quantum Heisenberg algebra, H_q :

Proposition 2.2. (1) $M_\varphi^q(\ell)$ is a $\mathbb{Z}$ -graded H_q -module, where

$$M_\varphi^q(\ell) = \bigoplus_{n \in \mathbb{Z}} M_\varphi^q(\ell)_n,$$

and $M_\varphi^q(\ell)_n = (H_\varphi^-)_n \cdot v_\ell$. If $\varphi(k) \neq \varphi(l)$ for some $k, l \in \mathbb{N}$, then $M_\varphi^q(\ell)_n$ is infinite dimensional for any $n \in \mathbb{Z}$,

(2) $M_\varphi^q(\ell)$ is irreducible if and only if $\ell \neq 0$.

Proof. By Proposition 2.1 we have $H_q/\langle\gamma - q^\ell\rangle \cong U(H)/\langle c - \ell\rangle$. Therefore, there is an equivalence between the category of weight modules of H_q on which γ acts as q^ℓ and the category of modules of $U(H)$ on which c acts as ℓ . The result then follows from Propositions 3.2 and 3.3 of [3]. $\square$

Of course, the construction of φ -imaginary Verma modules and their properties are easily extendible to the case of G_q .

3. GENERALIZED LOOP MODULES

Here we review a construction of generalized loop modules of the Heisenberg subalgebra of an untwisted loop algebra from [3], [9] and [10]. They are defined by inducing from certain parabolic submodules based on a non-standard partition of the roots of $\mathfrak{g}$.

Consider a Heisenberg Lie subalgebra $G := \mathbb{C}c \oplus \bigoplus_{n \in \mathbb{Z} \setminus \{0\}} \mathfrak{g}_{n\delta}$ of the affine algebra $\mathfrak{g}$. Thus, $[x, y] = (x, y)c$ for all $x \in \mathfrak{g}_{m\delta}, y \in \mathfrak{g}_{n\delta}$, where (x, y) is a skew-symmetric bilinear form such that $(\mathfrak{g}_{m\delta}, \mathfrak{g}_{n\delta}) = 0$ if $n + m \neq 0$ and its restriction to $\mathfrak{g}_{m\delta} \times \mathfrak{g}_{-m\delta}$ is non-degenerate for all $m \neq 0$. Set $G^\pm = \bigoplus_{n \in \mathbb{N}} \mathfrak{g}_{\pm n\delta}$.

We will denote by $\mathcal{K}$ the category of all $\mathbb{Z}$ -graded G -modules with the grading compatible with the above grading of G . If V is a G -module (resp. $\mathfrak{g}$ -module) with a scalar action of c , then this scalar is called the *level* of V .

Let $P = P_0 \oplus \mathfrak{N}$ be a parabolic subalgebra of $\mathfrak{g}$ with the Levi subalgebra $P_0 = G + \mathfrak{h}$. If N is a module from $\mathcal{K}$, then one defines on it a structure of a P_0 -module by choosing any $\lambda \in \mathfrak{h}^*$ and setting $hv = \lambda(h)v$ for any $h \in \mathfrak{h}$ and any $v \in N$. Moreover, N can be viewed as a P -module with a trivial action of $\mathfrak{N}$.

Following [3] and [9], [10] consider the induced $\mathfrak{g}$ -module

$$\text{ind}_\lambda(P, \mathfrak{g}; N) = U(\mathfrak{g}) \otimes_{U(P)} N.$$

Hence we obtain a functor

$$\text{ind}_\lambda(P, \mathfrak{g}) : N \longmapsto \text{ind}_\lambda(P, \mathfrak{g}; N)$$

from the category $\mathcal{K}$ of G -modules to the category of weight $\mathfrak{g}$ -modules. Let $\tilde{\mathcal{K}}$ be the full subcategory of $\mathcal{K}$ consisting of modules on which c acts injectively.

The following theorem shows that the functor $\text{ind}_\lambda(P, \mathfrak{g})$ preserves irreducibility when applied to any irreducible G -module in $\tilde{\mathcal{K}}$. A particular case of this result was established in [3].

Theorem 3.1 ([3], [9], [10]). *Let $P = P_0 \oplus \mathfrak{N}$ be a parabolic subalgebra of $\mathfrak{g}$, where $P_0 = G + \mathfrak{h}$. Let $\lambda \in \mathfrak{h}^*$ such that $\lambda(c) \neq 0$. Then the functor $\text{ind}_\lambda(P, \mathfrak{g})$ preserves the irreducibles.*

In particular, let $P = (\mathfrak{h} + G) \oplus \mathfrak{g}_R$ be the parabolic subalgebra of $\mathfrak{g}$ with Levi factor $\mathfrak{h} + G$, where $\mathfrak{g}_R = \bigoplus_{\beta \in R} \mathfrak{g}_\beta$, where $R = \{\alpha + k\delta \mid \alpha \in \check{\Delta}^+, k \in \mathbb{Z}\}$.

Let $V \in \tilde{\mathcal{K}}$ of level $\ell \neq 0$, $V = \bigoplus_{k \in \mathbb{Z}} V_k$. Consider $\lambda \in \mathfrak{h}^*$ such that $\lambda(c) = \ell$ and define a P -module structure on V by setting $\mathfrak{g}_R V = 0$ and $hw = (\lambda + k\delta)(h)w$ for any $w \in V_k$ and any $h \in \mathfrak{h}$. Then the *generalized loop module* is defined as

$$(3.1) \quad M(\lambda, V) = U(\mathfrak{g}) \otimes_{U(P)} V$$

and we have:

Corollary 3.2. *If V is an irreducible G -module in $\tilde{\mathcal{K}}$ of level $\lambda(c) \neq 0$, then $M(\lambda, V)$ is an irreducible $\mathfrak{g}$ -module.*

Note that $\text{supp } M(\lambda, V) = \bigcup_{\beta \in \check{Q}_+} \{\lambda - \beta + k\delta \mid k \in \mathbb{Z}\}$. Moreover, $\dim M(\lambda, V)_\mu = \infty$ for any μ of the form $\mu = \lambda - \beta + k\delta$ where $\beta \neq 0$ and $k \in \mathbb{Z}$.

4. GENERALIZED LOOP MODULES—THE QUANTUM CASE

Now we define generalized loop modules for the quantized algebra. Let $\mathcal{K}_q$ denote the category of $\mathbb{Z}$ -graded G_q -modules. Let $U_q^d(\pm)$ be the subalgebra of $U_q(\mathfrak{g})$ generated by the Drinfel'd generators $x_{ik}^\pm$ ($i \in I, k \in \mathbb{Z}$) and let G_q^d be the subalgebra of $U_q(\mathfrak{g})$ generated by G_q and U_q^0 . Set B_q^d to be the subalgebra generated by $U_q^d(+)\cup G_q^d$. Let $\mathbf{V}$ be a G_q -module on which c acts as the scalar $\ell \in \mathbb{Z}$ and let $\lambda \in P$ be a weight such that $\lambda(c) = \ell$. Extend the action of G_q on $\mathbf{V}$ to B_q^d by setting $x_{ik}^+ \cdot \mathbf{V} = 0$, $K_i^{\pm 1} \cdot v = q^{\pm \lambda(h_i)}v$, and $D^{\pm 1} \cdot v = q^{\pm \lambda(d)}v$ for all $i \in I, k \in \mathbb{Z}$, and $v \in \mathbf{V}$. Define $\mathbf{M}_q(\lambda, \mathbf{V}) = U_q(\mathfrak{g}) \otimes_{B_q^d} \mathbf{V}$. Then $\mathbf{M}_q(\lambda, \mathbf{V})$ is called the *quantum generalized loop module* associated with $\mathbf{V}$ and λ .

We can also define quantum generalized loop modules using the first presentation of $U_q(\mathfrak{g})$. Let $U_q^r(\pm)$ be the subalgebra of $U_q(\mathfrak{g})$ generated by $\{X_{\beta_k^\pm} \mid k \in \mathbb{Z}\}$, and let G_q^r be the subalgebra generated by $\{E_{k\delta}^{(i)} \mid i \in I, k \in \mathbb{Z}\} \cup \{\gamma^{\pm 1/2}\}$ and U_q^0 . Note that $G_q^r = G_q^d$. Denote by B_q^r the subalgebra of $U_q(\mathfrak{g})$ generated by $U_q^r(+)\cup G_q^r$. Let $\lambda \in P$ and let $\mathbf{V}$ be an irreducible G_q^r -module such that $K_i^{\pm 1} \cdot v = q^{\pm \lambda(h_i)}v$ and $D^{\pm 1} \cdot v = q^{\pm \lambda(d)}v$ for $v \in \mathbf{V}$ and $i \in I \cup \{0\}$. We give $\mathbf{V}$ the structure of a B_q^r -module by setting $X_{\beta_k^+} \mathbf{V} = 0$. We have:

Proposition 4.1. $\mathbf{M}_q(\lambda, \mathbf{V})$ is isomorphic to $U_q(\mathfrak{g}) \otimes_{B_q^r} \mathbf{V}$ as U_q -modules.

Proof. The proof is similar to the proof of Corollary 4.1.3 in [4], which concerns the case where $\mathbf{V} = \mathbb{C}v$. $\square$

Applying Theorem 1.2 we immediately obtain:

Theorem 4.2. Suppose $\mathbf{V}$ is an irreducible module in $\mathcal{K}_q$. As a vector space over $\mathbb{C}(q^{1/2})$, $\mathbf{M}_q(\lambda, \mathbf{V})$ is isomorphic to $\bigoplus_i U_q(-) \otimes v_i$ where $\{v_i\}$ is a basis of $\mathbf{V}$, hence is free as a $U_q(-)$ -module.

Denote by $G_{\mathbb{A}}^d$ the $\mathbb{A}$ -subalgebra of $U_{\mathbb{A}}(\mathfrak{g})$ generated by

$$h_{is}, K_i^{\pm 1}, \gamma^{\pm 1/2}, D^{\pm 1}, \begin{bmatrix} Ki; s \\ r \end{bmatrix}, \begin{bmatrix} D; s \\ r \end{bmatrix}, \begin{bmatrix} \gamma; s \\ 1 \end{bmatrix}_i, \begin{bmatrix} \gamma\psi_i; k, l \\ 1 \end{bmatrix}$$

for $i \in I, r, s \in \mathbb{Z}, s \neq 0$. Using Proposition 2.2 one constructs a large supply of irreducible $G_{\mathbb{A}}^d$ -modules. We have $G_{\mathbb{A}}^d/\langle q^{1/2} - 1 \rangle \simeq U(G + \mathfrak{h})$ and $G_{\mathbb{A}}^d \otimes_{\mathbb{A}} \mathbb{C}(q) \simeq G_q^d$.

Let $\mathbf{V}$ be an irreducible G_q -module in $\mathcal{K}_q$. Extend it to a G_q^d -module through some fixed $\lambda \in P$. Suppose there exists a basis $\{v_j\}$, $j \in J$, of $\mathbf{V}$ such that $V_{\mathbb{A}} = \sum_j \mathbb{A}v_j$ is a $G_{\mathbb{A}}^d$ -submodule such that its classical limit $V_{\mathbb{A}}/(q-1)V_{\mathbb{A}}$ is an irreducible $G + \mathfrak{h}$ -module in the category $\mathcal{K}$. We will call such module $\mathbb{A}$ -admissible. Again using Proposition 2.2 $\mathbb{A}$ -admissible irreducible G_q -modules can be constructed from irreducible modules of a certain Weyl algebra.

Now fix an irreducible module $\mathbf{V} \in \mathcal{K}_q$ which is $\mathbb{A}$ -admissible with respect to a suitable basis $\{v_j\}$, $j \in J$. We define the $\mathbb{A}$ -form of $\mathbf{M}_q(\lambda, \mathbf{V})$ to be the $U_{\mathbb{A}}(\mathfrak{g})$ -submodule $\mathbf{M}^{\mathbb{A}}(\lambda, \mathbf{V}) = \sum_j U_{\mathbb{A}}(\mathfrak{g}) \otimes_{B_q^d} v_j$. We immediately have the following:

Lemma 4.3. As a module over $\mathbb{A}$, $\mathbf{M}^{\mathbb{A}}(\lambda, \mathbf{V})$ is spanned by $U_{\mathbb{A}}^- \otimes v_j$, and is free as a $U_{\mathbb{A}}^-$ -module.

Consider the map $\xi : \mathbb{C}(q^{1/2}) \otimes_{\mathbb{A}} M^{\mathbb{A}}(\lambda, V) \rightarrow M_q(\lambda, V)$ given by $\xi(f \otimes v) = fv$, $f \in \mathbb{C}(q^{1/2})$, $v \in M^{\mathbb{A}}(\lambda, V)$. This is clearly surjective, and has inverse $\zeta : M_q(\lambda, V) \rightarrow \mathbb{C}(q^{1/2}) \otimes_{\mathbb{A}} M^{\mathbb{A}}(\lambda, V)$ given by $\zeta(X^-(\underline{m})v_j) = 1 \otimes X^-(\underline{m})v_j$, where $\{X^-(\underline{m})v_j\}$ is an $\mathbb{A}$ -basis of $M_q(\lambda, V)$. Therefore, we have the following:

Proposition 4.4. *For any $\lambda \in P$ and any $\mathbb{A}$ -admissible $V \in \mathcal{K}_q$, $\mathbb{C}(q^{1/2}) \otimes_{\mathbb{A}} M^{\mathbb{A}}(\lambda, V) \cong M_q(\lambda, V)$ as $\mathbb{C}(q^{1/2})$ vector spaces.*

From Lemmas 4.2 and 4.3 we have:

Proposition 4.5. *$M^{\mathbb{A}}(\lambda, V)$ is a weight module with the weight decomposition $M^{\mathbb{A}}(\lambda, V) = \bigoplus_{\mu \in P} M^{\mathbb{A}}(\lambda, V)_{\mu}$, where $M^{\mathbb{A}}(\lambda, V)_{\mu} = M^{\mathbb{A}}(\lambda, V) \cap M_q(\lambda, V)_{\mu}$.*

The isomorphism given in Proposition 4.4 restricts to the weight spaces so that for each $\mu \in P$, $M(\lambda, V)_{\mu}$ is a free $\mathbb{A}$ -module. Also, when $\dim_{\mathbb{C}(q^{1/2})}(M_q(\lambda, V)_{\mu}) < \infty$, we have $\text{rank}_{\mathbb{A}}(M^{\mathbb{A}}(\lambda, V)_{\mu}) = \dim_{\mathbb{C}(q^{1/2})}(M_q(\lambda, V)_{\mu})$.

We now give the classical limits of $U_q(\mathfrak{g})$ and $M_q(\lambda, V)$. We set $U' = \mathbb{A}/\mathbb{J} \otimes_{\mathbb{A}} U_{\mathbb{A}}$, where $\mathbb{J}$ is the ideal of $\mathbb{A}$ generated by $q^{1/2} - 1$. Now, define $\overline{U} = U'/K'$ where K' is the ideal of U' generated by $K_i - 1$, $D - 1$, and $\gamma^{1/2} - 1$. We denote by $\overline{u}$ the image of $u \in U'$ under this isomorphism. Thus, $\overline{U}$ is the $q^{1/2} = 1$ limit of $U_q(\mathfrak{g})$, hence $\overline{U} \cong U(\mathfrak{g})$, with $\overline{E}_i, \overline{F}_i, \overline{\left[\begin{smallmatrix} K_i; r \\ s \end{smallmatrix} \right]}, i \in I \cup \{0\}$, and $\overline{\left[\begin{smallmatrix} D; r \\ s \end{smallmatrix} \right]}$ getting sent to $e_i, f_i, \binom{h_i+r}{s}, i \in I \cup \{0\}$, and $\binom{d+r}{s}$ respectively. Also, we have that the root generators of $U_{\mathbb{A}}$, namely $\{x_{ik}, h_{il} | i \in I, k \in \mathbb{Z}, l \in \mathbb{Z}_{\neq 0}\}$, are sent to the corresponding root generators of $U(\mathfrak{g})$ under this isomorphism (see [1, Theorem 4.7]).

For $\lambda \in P$ and V an $\mathbb{A}$ -admissible irreducible module in $\mathcal{K}_q$ let $\overline{M}(\lambda, V) = \mathbb{A}/\mathbb{J} \otimes_{\mathbb{A}} M^{\mathbb{A}}(\lambda, V)$. After taking this classical limit, V becomes an irreducible G -module $\overline{V}$ in $\mathcal{K}$, since it is assumed to be $\mathbb{A}$ -admissible. For $\mu \in P$, let $\overline{M}(\lambda, V)_{\mu} = \mathbb{A}/\mathbb{J} \otimes_{\mathbb{A}} M^{\mathbb{A}}(\lambda, V)_{\mu}$. Thus, $\overline{M}(\lambda, V)$ is a U' -module and the weight space decomposition of $M^{\mathbb{A}}(\lambda, V)$ gives $\overline{M}(\lambda, V) = \bigoplus_{\mu \in P} \overline{M}(\lambda, V)_{\mu}$. Also $\dim_{\mathbb{C}}(\overline{M}(\lambda, V)_{\mu}) = \text{rank}_{\mathbb{A}}(M^{\mathbb{A}}(\lambda, V)_{\mu})$. We have that K_i, D , and $\gamma^{1/2}$ act as the identity on $\overline{M}(\lambda, V)$, since they act as integral powers of $q^{1/2}$ on weight vectors of $M^{\mathbb{A}}(\lambda, V)$, which evaluate to 1 in the classical limit. Therefore we have that $\overline{M}(\lambda, V)$ is also a well-defined $\overline{U}$ -module, called the classical limit of $M^{\mathbb{A}}(\lambda, V)$. Using this identification one may easily verify the following:

Proposition 4.6. *Let $V \in \mathcal{K}_q$ be an irreducible $\mathbb{A}$ -admissible G_q -module. Then as a $U(\mathfrak{g})$ -module, $\overline{M}(\lambda, V)$ is a weight module generated by $\overline{V}$ such that, for any $\mu \in P$, $\overline{M}(\lambda, V)_{\mu}$ is the μ -weight space of $\overline{M}(\lambda, V)$.*

Finally, we prove:

Proposition 4.7. *For any $\mathbb{A}$ -admissible irreducible G_q -module in $\mathcal{K}_q$ and any $\lambda \in P$, $\overline{M}(\lambda, V)$ is a free $U(\mathfrak{g}_{-R})$ -module generated by $\overline{V}$, where $R = \{\alpha + k\delta \mid \Delta, k \in \mathbb{Z}\}$.*

Proof. Let $\{v_j\}$, $j \in J$, be a suitable basis of V . Then $M^{\mathbb{A}}(\lambda, V)$ is the $\mathbb{A}$ -span of $u^- \otimes_{\mathbb{A}} v_j$, for $u^- \in U_{\mathbb{A}}^-$, by Lemma 4.3. Therefore, by the previous proposition, $\overline{M}(\lambda, V)$ is spanned over $\mathbb{C}$ by the images $\overline{u^-} \otimes_{\mathbb{C}} \overline{v_j}$. But the images $\overline{u^-} \in \overline{U}$ correspond to basis monomials in $U(\mathfrak{g}_{-R})$, and the $\overline{v_j}$ are a basis of $\overline{V}$ by the admissibility of V . Therefore, $\overline{M}(\lambda, V)$ is a free $U(\mathfrak{g}_{-R})$ -module, generated by $\overline{v_j}$, $j \in J$. $\square$

The conclusion is that $\overline{M}(\lambda, V) \cong M(\lambda, \overline{V})$. By Propositions 4.6 and 4.7 we see that $M_q(\lambda, V)$ is a quantum deformation of $M(\lambda, \overline{V})$. Hence, applying Corollary 3.2

we obtain our main result which allows us to construct irreducible $U_q(\mathfrak{g})$ -modules from irreducible G_q -modules.

Theorem 4.8. *If $V \in \mathcal{K}_q$ is an irreducible $\mathbb{A}$ -admissible G_q -module of level $\lambda(c) \neq 0$, then $M_q(\lambda, V)$ is an irreducible $U_q(\mathfrak{g})$ -module.*

Proof. Let W be a proper submodule of $M_q(\lambda, V)$. Then $W^\mathbb{A} = W \cap M^\mathbb{A}(\lambda, V)$ is a submodule of $M^\mathbb{A}(\lambda, V)$ and the classical limit of $W^\mathbb{A}$ is a proper submodule of $M(\lambda, V)$. But, $M(\lambda, V)$ is irreducible by [3] (Theorem 5.6), [10]; therefore, so is $M_q(\lambda, V)$. $\square$

We believe the restriction on V to be $\mathbb{A}$ -admissible can be lifted.

5. φ -IMAGINARY VERMA MODULES FOR AFFINE QUANTUM ALGEBRAS

In this section we consider a particular class of quantum generalized loop modules. In the $q = 1$ case they were introduced in [3]. Let $\varphi : \mathbb{N} \rightarrow \{+, -\}$ be an arbitrary function. Denote $S_\varphi = (S \cap \Delta^{\text{re}}) \cup \{n\delta | n \in \mathbb{Z}_{>0}, \varphi(n) = +\} \cup \{-m\delta | m \in \mathbb{Z}_{>0}, \varphi(m) = -\}$. Let $\lambda \in P$ and let B_q^φ be the subalgebra of $U_q(\mathfrak{g})$ generated by $\{E_\beta | \beta \in S_\varphi\} \cup U_q^0(\mathfrak{g})$. Let $\mathbb{C}(q^{1/2})v_\lambda$ be the representation of B_q^φ such that $K_i \cdot v_\lambda = q^{\lambda(h_i)}v_\lambda, i \in I, D \cdot v_\lambda = q^{\lambda(d)}v_\lambda, U_q(S_\varphi) \cdot v_\lambda = 0$. Then the φ -imaginary Verma module for $U_q(\mathfrak{g})$ is defined to be:

$$(5.1) \quad M_q^\varphi(\lambda) = U_q(\mathfrak{g}) \otimes_{B_q^\varphi} \mathbb{C}(q^{1/2})v_\lambda.$$

We have the following property of $M_q^\varphi(\lambda)$, which is an analogue of the $U(\mathfrak{g})$ case in [3]:

Proposition 5.1 (Analogous to Proposition 3.4 in [3]). *If $\varphi(k) \neq \varphi(l)$ for some $k, l \in \mathbb{Z}_{>0}$, then $\dim M_q^\varphi(\lambda)_\mu = \infty$ for all weights μ .*

Consider the G_q -submodule of $M_q^\varphi(\lambda)$ generated by v_λ : $N_q = G_q v_\lambda$. Then:

Proposition 5.2. *The module N_q is irreducible in $\mathcal{K}_q$ if and only if $\lambda(c) \neq 0$. Moreover, it is an $\mathbb{A}$ -admissible module.*

Proof. Note that N_q is a φ -highest weight module, i.e. for any $n \in \mathbb{N}$ either $\mathfrak{g}_{n\delta}v_\lambda = 0$ or $\mathfrak{g}_{-n\delta}v_\lambda = 0$ depending on the sign of $\varphi(n)$. Then a standard argument shows that N_q is irreducible when $\lambda(c) \neq 0$. If $\lambda(c) = 0$, then the maximal submodule of N_q is the vector space direct sum complement of $\mathbb{C}(q^{1/2})v_\lambda$. Choose a φ -highest weight vector $v \in N_q$ and let $N_\mathbb{A} = G_\mathbb{A}^d v$. Then the classical limit of $N_\mathbb{A}$ is irreducible because the universal φ -highest weight module is irreducible itself. Therefore N_q is $\mathbb{A}$ -admissible. $\square$

We have an isomorphism

$$M_q^\varphi(\lambda) \simeq M_q(\lambda, N_q).$$

Theorem 5.3. *Let $\varphi : \mathbb{N} \rightarrow \{+, -\}$, $\lambda \in P$. Then $M_q^\varphi(\lambda)$ is irreducible if and only if $\lambda(c) \neq 0$.*

Proof. Indeed, if $\lambda(c) \neq 0$, then N_q is irreducible by Proposition 5.2. Applying Theorem 4.8 we immediately obtain that $M_q^\varphi(\lambda)$ is irreducible. Conversely, suppose $M_q^\varphi(\lambda)$ is irreducible but $\lambda(c) = 0$. Then N_q contains a non-trivial submodule by Proposition 5.2, which in turn generates a non-trivial submodule of $M_q^\varphi(\lambda)$. $\square$

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