DICHOTOMY OF GLOBAL CAPACITY DENSITY

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ABSTRACT. Let $1 < p < \infty$ and let $d\mu(x) = w(x)dx$ be a p -admissible weight in $\mathbb{R}^n$, $n \geq 2$. By $\text{Cap}_{p,\mu}(E, D)$ we denote the variational (p, μ) -capacity of condenser (E, D) . We show a dichotomy of the global density with respect to $\text{Cap}_{p,\mu}$. One of our results is as follows: Let $\lambda > 1$ and let $B(x, r)$ stand for the open ball with center at x and radius r . Then

$$\lim_{r \rightarrow \infty} \left(\inf_{x \in \mathbb{R}^n} \frac{\text{Cap}_{p,\mu}(E \cap B(x, r), B(x, \lambda r))}{\text{Cap}_{p,\mu}(B(x, r), B(x, \lambda r))} \right)$$

is equal to either 0 or 1; the first case occurs if and only if

$$\inf_{x \in \mathbb{R}^n} \frac{\text{Cap}_{p,\mu}(E \cap B(x, r_0), B(x, \lambda r))}{\text{Cap}_{p,\mu}(B(x, r), B(x, \lambda r))}$$

is identically equal to 0. This provides a sharp contrast between capacity and Lebesgue measure.

1. INTRODUCTION

Let φ be a nonnegative set function on $\mathbb{R}^n$, $n \geq 2$, such that

- (i) If $E \subset F$, then $\varphi(E) \leq \varphi(F)$.
- (ii) If U is a nonempty bounded open set, then $0 < \varphi(U) < \infty$.

By $B(x, r)$ we denote the open ball with center at x and radius r . In this note we study densities

$$\underline{\mathcal{D}}(\varphi, r, E) = \inf_{x \in \mathbb{R}^n} \frac{\varphi(E \cap B(x, r))}{\varphi(B(x, r))}, \quad \overline{\mathcal{D}}(\varphi, r, E) = \sup_{x \in \mathbb{R}^n} \frac{\varphi(E \cap B(x, r))}{\varphi(B(x, r))}.$$

By definition $0 \leq \underline{\mathcal{D}}(\varphi, r, E) \leq \overline{\mathcal{D}}(\varphi, r, E) \leq 1$. We note that $\underline{\mathcal{D}}(\varphi, r, E) > 0$ means that E is uniformly distributed in $\mathbb{R}^n$ in the scale r with respect to φ . A typical example of φ is the n -dimensional Lebesgue outer measure m . We have the following property.

Proposition 1.1. *If $\underline{\mathcal{D}}(m, r_0, E) > 0$ for some $r_0 > 0$, then $\underline{\mathcal{D}}(m, r, E) > 0$ for $r \geq r_0$. More precisely, there exists a constant $A > 1$ depending only on n such that*

$$(1.1) \quad \underline{\mathcal{D}}(m, R, E) \geq A^{-1} \underline{\mathcal{D}}(m, r, E) \quad \text{for } R \geq r.$$

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This proposition means that if E is uniformly distributed in $\mathbb{R}^n$ in the scale r_0 with respect to m , then so is in the scale r , $r \geq r_0$. We note that the density is not improved no matter how large r is. It is easy to construct a closed set E such that $\lim_{r \rightarrow \infty} \underline{D}(m, r, E) = \lim_{r \rightarrow \infty} \overline{D}(m, r, E) = c$ for each $c \in (0, 1)$.

Proposition 1.2. *Let $0 < \ell < 1$ and let E be the union of all closed cubes with centers running over all $\mathbb{Z}^n$ and sides of length ℓ parallel to the coordinate axes. Then*

$$(1.2) \quad \lim_{r \rightarrow \infty} \underline{D}(m, r, E) = \lim_{r \rightarrow \infty} \overline{D}(m, r, E) = \ell^n.$$

If φ is a capacity, then the situation is very different. By $C_\ell(E)$ we denote the logarithmic capacity of $E \subset \mathbb{R}^2$. Stegenga [5] proved the following theorem.

Theorem A. *Let Ω be an open set in $\mathbb{R}^2$ and put*

$$\phi(r) = \inf_{z \in \Omega} \frac{C_\ell(B(z, r) \setminus \Omega)}{C_\ell(B(z, r))}.$$

If $\phi(r) > 0$ for some $r > 0$, then $\lim_{r \rightarrow \infty} \phi(r) = 1$.

Let $E = \mathbb{R}^2 \setminus \Omega$ in Theorem A. Then $\phi(r) = \underline{D}(C_\ell, r, E)$ since $\inf_{z \in \Omega}$ can be replaced by $\inf_{z \in \mathbb{R}^2}$ in the definition of $\phi(r)$. Hence Theorem A provides a dichotomy of $\lim_{r \rightarrow \infty} \underline{D}(C_\ell, r, E)$, i.e., the limit exists and it is equal to either 0 or 1. Note that $\underline{D}(C_\ell, r, E)$ need not be monotone in r , so that even the existence of $\lim_{r \rightarrow \infty} \underline{D}(C_\ell, r, E)$ is nontrivial.

In this note we shall show that the same phenomenon holds for other capacities. Hereafter, let $1 < p < \infty$ and let w be a p -admissible weight as in [3, Chapter 1]. We write $d\mu(x) = w(x)dx$. Let $D \subset \mathbb{R}^n$ be an open set. For a compact subset K of D we define

$$(1.3) \quad \text{Cap}_{p, \mu}(K, D) = \inf \left\{ \int_D |\nabla u|^p d\mu : u \geq 1 \text{ on } K, u \in C_0^\infty(D) \right\}.$$

In the usual way, $\text{Cap}_{p, \mu}(K, D)$ extends to $\text{Cap}_{p, \mu}(E, D)$ for an arbitrary subset E of D . We call $\text{Cap}_{p, \mu}(E, D)$ the (variational) (p, μ) -capacity of condenser (E, D) or simply the (p, μ) -capacity of E in D . By definition $\text{Cap}_{p, \mu}(E, D)$ is increasing with respect to E and decreasing with respect to D . See [3, Chapter 2].

We shall consider the density over a general set. We write $E(x, r) = \{ry + x \in \mathbb{R}^n : y \in E\}$ for $E \subset \mathbb{R}^n$, $r > 0$ and $x \in \mathbb{R}^n$. That is $E(x, r)$ stands for the set E dilated by factor r and translated by x . If $x = 0$, then we simply write rE for $E(0, r)$. Note that if Q is the unit ball $B(0, 1)$, then $Q(x, r) = B(x, r)$. Another typical example of Q is an open cube. We allow Q to be even disconnected. We consider the density over $Q(x, r)$ with Q satisfying the following interior corkscrew condition.

Definition 1.3. Let U be an open set. We say that U satisfies the interior corkscrew condition if there exist $\rho_0 > 0$ and $0 < \kappa < 1$ such that

$$\xi \in \partial U \text{ and } 0 < r \leq \rho_0 \implies B(\xi, r) \cap U \text{ contains a ball of radius } \kappa r.$$

Theorem 1.4. *Let Q and Q^* be bounded open sets such that $\overline{Q} \subset Q^*$. Assume that Q satisfies the interior corkscrew condition. Let E be a Borel set in $\mathbb{R}^n$. For $r > 0$ define*

$$\underline{D}_{Q, Q^*}(\text{Cap}_{p, \mu}, r, E) = \inf_{x \in \mathbb{R}^n} \frac{\text{Cap}_{p, \mu}(E \cap Q(x, r), Q^*(x, r))}{\text{Cap}_{p, \mu}(Q(x, r), Q^*(x, r))}.$$

Then $\lim_{r \rightarrow \infty} \underline{\mathcal{D}}_{Q, Q^*}(\text{Cap}_{p, \mu}, r, E)$ is equal to either 0 or 1; the first case occurs if and only if $\underline{\mathcal{D}}_{Q, Q^*}(\text{Cap}_{p, \mu}, r, E)$ is identically equal to 0.

Obviously, an open ball enjoys the interior corkscrew condition.

Corollary 1.5. *Let $\lambda > 1$. Let E be a Borel set in $\mathbb{R}^n$. For $r > 0$ define*

$$\underline{\mathcal{D}}_{B, \lambda B}(\text{Cap}_{p, \mu}, r, E) = \inf_{x \in \mathbb{R}^n} \frac{\text{Cap}_{p, \mu}(E \cap B(x, r), B(x, \lambda r))}{\text{Cap}_{p, \mu}(B(x, r), B(x, \lambda r))}.$$

Then $\lim_{r \rightarrow \infty} \underline{\mathcal{D}}_{B, \lambda B}(\text{Cap}_{p, \mu}, r, E)$ is equal to either 0 or 1; the first case occurs if and only if $\underline{\mathcal{D}}_{B, \lambda B}(\text{Cap}_{p, \mu}, r, E)$ is identically equal to 0.

Remark 1.6. We can replace $Q(x, r)$ and $B(x, r)$ by $\overline{Q}(x, r)$ and $\overline{B}(x, r)$ in Theorem 1.4 and Corollary 1.5, respectively. See Lemma 5.5 below.

In the unweighted case $w \equiv 1$, we write $\text{Cap}_p(E, D)$ for $\text{Cap}_{p, \mu}(E, D)$. Moreover if $D = \mathbb{R}^n$, then we simply write $\text{Cap}_p(E)$ for $\text{Cap}_p(E, \mathbb{R}^n)$. It is well known that if $p \geq n$, then $\text{Cap}_p(E) = 0$ for every $E \subset \mathbb{R}^n$; if $1 < p < n$ and $\lambda > 1$, then

$$\text{Cap}_p(E) \leq \text{Cap}_p(E, B(x, \lambda r)) \leq A \text{Cap}_p(E) \quad \text{for } E \subset B(x, r),$$

where A depends only on λ , n and p ([4, Proposition 4]). If $1 < p < n$, then we can take $\lambda = \infty$ in Corollary 1.5, and we obtain a counterpart of Theorem A in the (p, μ) -capacity context.

Corollary 1.7. *Let $1 < p < n$ and let E be a Borel set in $\mathbb{R}^n$. Then we see that $\lim_{r \rightarrow \infty} \underline{\mathcal{D}}(\text{Cap}_p, r, E)$ is equal to either 0 or 1; the first case occurs if and only if $\underline{\mathcal{D}}(\text{Cap}_p, r, E)$ is identically equal to 0.*

Remark 1.8. In this note we study weighted capacity in the Euclidean space along the line of [3]. See [4] for the classical unweighted case. It may be possible to extend our arguments to more general metric measure settings (see [2]).

The plan of the paper is as follows. In Section 2 we give proofs of Propositions 1.1 and 1.2. The rest of the paper is devoted to the proof of Theorem 1.4. First, we restate the condition $\underline{\mathcal{D}}_{Q, Q^*}(\text{Cap}_{p, \mu}, r, E) > 0$ for some $r > 0$ by using cubes. Let N be a positive integer. We decompose $\mathbb{R}^n$ into the union of closed cubes $\{Q_j^N\}_j$ of mutually disjoint interior with sides of length N parallel to the coordinate axes and vertices on $(N\mathbb{Z})^n$. Let $\tilde{Q}_j^N$ be the interior of the double of Q_j^N . Lemma 3.3 will show that the condition $\underline{\mathcal{D}}_{Q, Q^*}(\text{Cap}_{p, \mu}, r, E) > 0$ for some $r > 0$ implies the condition

$$(1.4) \quad \inf_j \frac{\text{Cap}_{p, \mu}(E \cap Q_j^N, \tilde{Q}_j^N)}{\text{Cap}_{p, \mu}(Q_j^N, \tilde{Q}_j^N)} > 0 \quad \text{for some } N \geq 1,$$

and vice versa. We also note that we may assume that E is closed (Corollary 3.4).

Secondly, we consider the limit of $\underline{\mathcal{D}}_{Q, Q^*}(\text{Cap}_{p, \mu}, r, E)$. We employ a relationship between capacity and capacitary potential ([3, Lemma 6.19]), which was crucial in the Wiener criterion for the p -harmonic Dirichlet problem, to obtain an estimate of ratios of capacity in terms of the infimum of the capacitary potential, and then in terms of the supremum of p -harmonic measure. The repeated application of the estimate gives a lower bound of a ratio of capacity similar to that in Theorem 1.4 but with a denominator strictly smaller than $\text{Cap}_{p, \mu}(Q(x, r), Q^*(x, r))$ (Corollary 4.5). This is close to the final conclusion of the theorem. In fact, in the unweighted case

$w \equiv 1$, the explicit calculation of $\text{Cap}_p(B(x, r), B(x, R))$ is available ([3, Example 2.12]), and hence we immediately obtain Corollary 1.5 in the unweighted case. Corollary 1.7 is its easy consequence. In the weighted case, however, no explicit calculation of $\text{Cap}_{p, \mu}(Q(x, r), Q^*(x, R))$ is available even if Q and Q^* are balls.

So, in the third step, we show an approximation of the capacity from inside (Corollary 5.6), which requires the interior corkscrew condition on Q . (However, we do not know whether the condition is necessary or not.) This, together with Corollary 4.5, completes the proof of Theorem 1.4.

We use the following notation. By the symbol A we denote an absolute positive constant whose value is unimportant and may change from one occurrence to the next. If necessary, we use $A_0, A_1, \dots$, to specify them. We say that f and g are comparable and write $f \approx g$ if two positive quantities f and g satisfy $A^{-1} \leq f/g \leq A$ with some constant $A \geq 1$. The constant A is referred to as the constant of comparison. We have to pay attention for the dependency of the constant of comparison.

2. PROOF OF PROPOSITIONS 1.1 AND 1.2

Proof of Proposition 1.1. In view of the 5-covering lemma, we have a family of pairwise disjoint balls $B(x_j, r)$ such that $B(x, R) \subset \bigcup_j B(x_j, 5r)$. Without loss of generality we may assume that E is measurable. Then for every $x \in \mathbb{R}^n$

$$\begin{aligned} m(E \cap B(x, R)) &\geq \sum_j m(E \cap B(x_j, r)) \geq \sum_j \underline{D}(m, r, E) m(B(x_j, r)) \\ &= 5^{-n} \underline{D}(m, r, E) \sum_j m(B(x_j, 5r)) \geq 5^{-n} \underline{D}(m, r, E) m(B(x, R)), \end{aligned}$$

which implies (1.1) with $A = 5^n$. $\square$

Proof of Proposition 1.2. First we consider the density of E with respect to a cube. Let $N \geq 2$ be an integer. By geometry we have $m(E \cap [-N, N]^n) = (2N)^n \ell^n$, so that by translation

$$(2.1) \quad \frac{m(E \cap ([-N, N]^n + \xi))}{m([-N, N]^n)} = \ell^n \quad \text{uniformly for } \xi \in (N\mathbb{Z})^n.$$

Let us estimate the density of E over a ball $B(x, R)$. Let $\{\mathcal{Q}_j^{2N}\}_j$ be the enumeration of cubes with sides of length $2N$ parallel to the coordinate axes and vertices on $(2N\mathbb{Z})^n$ as in the introduction. Let S_R be the union of all cubes $\mathcal{Q}_j^{2N}$ contained in $B(x, R)$ and let T_R be the union of all cubes $\mathcal{Q}_j^{2N}$ intersecting $B(x, R)$. We have $m(S_R) \leq m(B(x, R)) \leq m(T_R)$ and

$$\lim_{R/N \rightarrow \infty} \frac{m(S_R)}{m(B(x, R))} = \lim_{R/N \rightarrow \infty} \frac{m(T_R)}{m(B(x, R))} = 1.$$

Let $0 < \varepsilon < 1$ and we can choose R such that

$$(2.2) \quad 1 \leq \frac{m(T_R)}{m(S_R)} \leq 1 + \varepsilon \quad \text{uniformly for } x \in \mathbb{R}^n.$$

Applying (2.1) to each constituent cube of S_R and T_R , and then taking the summation, we obtain

$$\frac{m(E \cap S_R)}{m(S_R)} = \frac{m(E \cap T_R)}{m(T_R)} = \ell^n.$$

This, together with (2.2), yields

$$\begin{aligned} (1 + \varepsilon)^{-1} \ell^n &\leq \frac{m(S_R)}{m(T_R)} \cdot \frac{m(E \cap S_R)}{m(S_R)} = \frac{m(E \cap S_R)}{m(T_R)} \leq \frac{m(E \cap B(x, R))}{m(B(x, R))} \\ &\leq \frac{m(E \cap T_R)}{m(S_R)} = \frac{m(T_R)}{m(S_R)} \cdot \frac{m(E \cap T_R)}{m(T_R)} \leq (1 + \varepsilon) \ell^n. \end{aligned}$$

Since $0 < \varepsilon < 1$ and $x \in \mathbb{R}^n$ are arbitrary, we have (1.2). $\square$

3. RELAXATION OF THE ASSUMPTION OF THEOREM 1.4

Let us begin with some elementary properties of $\text{Cap}_{p, \mu}(E, D)$. In our context the estimates in [3, Lemmas 2.14 and 2.16] read as follows.

Lemma 3.1. *Let Q , Q^* and $\tilde{Q}$ be bounded open sets such that $\overline{Q} \subset Q^*$ and $\overline{Q} \subset \tilde{Q}$. Then the following estimates hold uniformly for $x \in \mathbb{R}^n$ and $r > 0$:*

- (i) $\text{Cap}_{p, \mu}(Q(x, r), Q^*(x, r)) \approx \text{Cap}_{p, \mu}(\overline{Q}(x, r), Q^*(x, r)) \approx r^{-p} \mu(Q(x, r)).$
- (ii) $\text{Cap}_{p, \mu}(E, Q^*(x, r)) \approx \text{Cap}_{p, \mu}(E, \tilde{Q}(x, r))$ for $E \subset \overline{Q}(x, r).$

Here the constant of comparison depends only on n, p, μ, Q, Q^* and $\tilde{Q}$.

We compare the density of capacity over an open set and that over a ball.

Lemma 3.2. *Let Q and Q^* be bounded open sets with $\overline{Q} \subset Q^*$. Suppose $B(x_0, \tau_0) \subset Q \subset B(x_0, \tau_1)$ and $Q^* \subset B(x_0, \lambda \tau_1)$ for some $x_0 \in Q$ with $0 < \tau_0 < \tau_1$ and $\lambda > 1$. Then there exists $A > 1$ depending only on $n, p, \mu, Q, Q^*, \tau_0, \tau_1$ and λ such that*

$$A^{-1} \underline{\mathcal{D}}_{B, \lambda B}(\text{Cap}_{p, \mu}, \tau_0 r, E) \leq \underline{\mathcal{D}}_{Q, Q^*}(\text{Cap}_{p, \mu}, r, E) \leq A \underline{\mathcal{D}}_{B, \lambda B}(\text{Cap}_{p, \mu}, \tau_1 r, E),$$

whenever $E \subset \mathbb{R}^n$ and $r > 0$.

Proof. By translation, we may assume that $x_0 = 0$. By definition $B(x, \tau_0 r) \subset Q(x, r) \subset B(x, \tau_1 r)$ and $Q^*(x, r) \subset B(x, \lambda \tau_1 r)$. Hence

$$\begin{aligned} \text{Cap}_{p, \mu}(E \cap B(x, \tau_0 r), B(x, \lambda \tau_1 r)) &\leq \text{Cap}_{p, \mu}(E \cap Q(x, r), B(x, \lambda \tau_1 r)) \\ &\leq \text{Cap}_{p, \mu}(E \cap B(x, \tau_1 r), B(x, \lambda \tau_1 r)) \end{aligned}$$

by monotonicity, and so

$$\begin{aligned} A^{-1} \frac{\text{Cap}_{p, \mu}(E \cap B(x, \tau_0 r), B(x, \lambda \tau_0 r))}{\text{Cap}_{p, \mu}(B(x, \tau_0 r), B(x, \lambda \tau_0 r))} &\leq \frac{\text{Cap}_{p, \mu}(E \cap Q(x, r), Q^*(x, r))}{\text{Cap}_{p, \mu}(Q(x, r), Q^*(x, r))} \\ &\leq A \frac{\text{Cap}_{p, \mu}(E \cap B(x, \tau_1 r), B(x, \lambda \tau_1 r))}{\text{Cap}_{p, \mu}(B(x, \tau_1 r), B(x, \lambda \tau_1 r))} \end{aligned}$$

by Lemma 3.1 and the doubling property of μ . Taking the infima with respect to $x \in \mathbb{R}^n$, we obtain the required inequalities. $\square$

Lemma 3.3. *Let Q and Q^* be bounded open sets with $\overline{Q} \subset Q^*$. Then the following conditions are equivalent to each other:*

- (i) $\underline{\mathcal{D}}_{Q, Q^*}(\text{Cap}_{p, \mu}, r, E) > 0$ for some $r > 0$.
- (ii) $\underline{\mathcal{D}}_{B, \lambda B}(\text{Cap}_{p, \mu}, r, E) > 0$ for some $r > 0$.
- (iii) (1.4) holds for some $N \geq 1$.

Proof. Lemma 3.2 gives (i) $\iff$ (ii).

(ii) $\implies$ (iii): Suppose $\underline{\mathcal{D}}_{B,\lambda B}(\text{Cap}_{p,\mu}(r, E)) > 0$. Let $N = 2r$. Take an arbitrary closed cube $\mathcal{Q}_j^N$ as in the introduction. Observe that if x is the center of $\mathcal{Q}_j^N$, then $B(x, r) \subset \mathcal{Q}_j^N \subset \overline{B}(x, r\sqrt{n})$, so that

$$\begin{aligned} & \text{Cap}_{p,\mu}(E \cap B(x, r), B(x, \lambda r)) \\ & \approx \text{Cap}_{p,\mu}(E \cap B(x, r), B(x, \lambda r\sqrt{n})) \\ & \leq \text{Cap}_{p,\mu}(E \cap \mathcal{Q}_j^N, B(x, \lambda r\sqrt{n})) \approx \text{Cap}_{p,\mu}(E \cap \mathcal{Q}_j^N, \tilde{\mathcal{Q}}_j^N) \end{aligned}$$

by monotonicity and by Lemma 3.1 (ii). By Lemma 3.1 (i)

$$\underline{\mathcal{D}}_{B,\lambda B}(\text{Cap}_{p,\mu}(r, E)) \leq \frac{\text{Cap}_{p,\mu}(E \cap B(x, r), B(x, \lambda r))}{\text{Cap}_{p,\mu}(B(x, r), B(x, \lambda r))} \leq A \frac{\text{Cap}_{p,\mu}(E \cap \mathcal{Q}_j^N, \tilde{\mathcal{Q}}_j^N)}{\text{Cap}_{p,\mu}(\mathcal{Q}_j^N, \tilde{\mathcal{Q}}_j^N)}.$$

Since j is arbitrary, we have (ii).

(iii) $\implies$ (ii): Suppose (1.4) holds with $N \geq 1$. Choose $r > N\sqrt{n}$, say $r = 2N\sqrt{n}$. We claim that, for every $x \in \mathbb{R}^n$, $B(x, r)$ must contain at least one $\mathcal{Q}_j^N$. In fact, write $x = (x_1, \dots, x_n)$. Then $\prod_{i=1}^n [x_i - N, x_i + N] \subset B(x, r)$. Let $\xi_i \in N\mathbb{Z}$ be such that $\xi_i \leq x_i < \xi_i + N$. Then $[\xi_i, \xi_i + N] \subset [x_i - N, x_i + N]$, so that we have the claim with $\mathcal{Q}_j^N = \prod_{i=1}^n [\xi_i, \xi_i + N]$. It follows from Lemma 3.1 (ii) and monotonicity that

$$\text{Cap}_{p,\mu}(E \cap \mathcal{Q}_j^N, \tilde{\mathcal{Q}}_j^N) \approx \text{Cap}_{p,\mu}(E \cap \mathcal{Q}_j^N, B(x, \lambda r)) \leq \text{Cap}_{p,\mu}(E \cap B(x, r), B(x, \lambda r)),$$

so that

$$\frac{\text{Cap}_{p,\mu}(E \cap \mathcal{Q}_j^N, \tilde{\mathcal{Q}}_j^N)}{\text{Cap}_{p,\mu}(\mathcal{Q}_j^N, \tilde{\mathcal{Q}}_j^N)} \leq A \frac{\text{Cap}_{p,\mu}(E \cap B(x, r), B(x, \lambda r))}{\text{Cap}_{p,\mu}(B(x, r), B(x, \lambda r))}$$

by Lemma 3.1 (i). Taking the infimum of the left hand side with respect to j , and then taking the infimum of the right hand side with respect to $x \in \mathbb{R}^n$, we obtain (ii). $\square$

We note that we may assume that E is a closed set in Theorem 1.4.

Corollary 3.4. *Let Q and Q^* be bounded open sets with $\overline{Q} \subset Q^*$. Let $E \subset \mathbb{R}^n$ be a Borel set. If $\underline{\mathcal{D}}_{Q,Q^*}(\text{Cap}_{p,\mu}(r, E)) > 0$ for some $r > 0$, then E has a closed subset F such that $\underline{\mathcal{D}}_{Q,Q^*}(\text{Cap}_{p,\mu}(R, F)) > 0$ for some $R > 0$.*

Proof. Lemma 3.3 asserts that (1.4) holds for some $N \geq 1$. Apply the capacitability to each $E \cap \mathcal{Q}_j^N$ ([3, Theorem 2.5]). We find a closed subset $F_j \subset E \cap \mathcal{Q}_j^N$ such that $\text{Cap}_{p,\mu}(F_j, \tilde{\mathcal{Q}}_j^N) \geq \frac{1}{2} \text{Cap}_{p,\mu}(E \cap \mathcal{Q}_j^N, \tilde{\mathcal{Q}}_j^N)$. We observe that $F = \bigcup_j F_j$ is a closed subset of E satisfying (1.4) with F in place of E . Lemma 3.3 again asserts $\underline{\mathcal{D}}_{Q,Q^*}(\text{Cap}_{p,\mu}(R, F)) > 0$ for some $R > 0$. $\square$

Hereafter, we assume that E in Theorem 1.4 is a closed set.

4. PROOF OF COROLLARY 1.5 IN THE UNWEIGHTED CASE

In this section we give a proof of Corollary 1.5 in the unweighted case. The arguments until Corollary 4.5 work for the general case. On the other hand, Lemma 4.6 is specific for the unweighted case, so the proof of Corollary 1.5 is restricted in the unweighted case. Although it is superfluous, it may clarify the arguments.

Let us study a capacitary potential of a compact set. Let K be a compact subset of D . The minimizer of the infimum in (1.3) belongs to the Sobolev space $W_0^{1,p}(D)$. It is referred to as the (p, μ) -capacitary potential of K in D and is denoted by $\mathfrak{R}(K, D)$. We see that $0 \leq \mathfrak{R}(K, D) \leq 1$ in D , $\mathfrak{R}(K, D) = 1$ on K outside a set of null capacity, and $\mathfrak{R}(K, D)$ is p -harmonic in $D \setminus K$. By $\omega_p^x(E, D)$ we denote the p -harmonic measure of $E \subset \partial D$ in D evaluated at x , i.e., $\omega_p(E, D)$ is the Perron solution of the boundary function χ_E . We see that if K is a compact subset of D , then

$$(4.1) \quad \omega_p(\partial D, D \setminus K) = 1 - \mathfrak{R}(K, D) \quad \text{on } D.$$

Strictly speaking, the p -harmonic measure is extended by 0 on K , which coincides with the right hand side outside a set of null (p, μ) -capacity.

The following relationship between capacity and capacitary potential is useful.

Lemma 4.1 ([3, Lemma 6.19 and its remark]). *Let $u_K = \mathfrak{R}(K, D)$, where D is a bounded open set and K is a compact subset of D . If $0 < t \leq 1$, then*

$$\text{Cap}_{p,\mu}(\{u_K \geq t\}, D) = t^{1-p} \text{Cap}_{p,\mu}(K, D).$$

As an easy corollary, we have an upper estimate of capacitary potential, which can be restated with p -harmonic measure with the aid of (4.1).

Lemma 4.2. *Let Q and Q^* be bounded open sets such that $\overline{Q} \subset Q^*$. Let $r > 0$ and $x \in \mathbb{R}^n$. If compact subsets K and F of $\overline{Q}(x, r)$ have positive (p, μ) -capacity, then*

$$\inf_F \mathfrak{R}(K, Q^*(x, r)) \leq \left\{ \frac{\text{Cap}_{p,\mu}(K, Q^*(x, r))}{\text{Cap}_{p,\mu}(F, Q^*(x, r))} \right\}^{1/(p-1)}.$$

In other words,

$$1 - \left\{ \frac{\text{Cap}_{p,\mu}(K, Q^*(x, r))}{\text{Cap}_{p,\mu}(F, Q^*(x, r))} \right\}^{1/(p-1)} \leq \sup_F \omega_p(\partial Q^*(x, r), Q^*(x, r) \setminus K).$$

Proof. Let $u_K = \mathfrak{R}(K, Q^*(x, r))$ and let $t = \inf_F u_K$. Observe that $F \subset \{u_K \geq t\}$. By Lemma 4.1 we have

$$\text{Cap}_{p,\mu}(F, Q^*(x, r)) \leq \text{Cap}_{p,\mu}(\{u_K \geq t\}, Q^*(x, r)) = t^{1-p} \text{Cap}_{p,\mu}(K, Q^*(x, r)),$$

so that taking the $1/(p-1)$ -power gives the first inequality. The second inequality follows from (4.1). $\square$

Lemma 4.1 also yields a lower estimate of capacitary potential. We state the estimate only for concentric balls since it is sufficient in the sequel; it may be possible to give a generalization with Q and Q^* under some conditions on Q and Q^* .

Lemma 4.3 ([3, Lemma 6.21]). *For $\lambda > 1$ there exists a constant $A_0 > 1$ depending only on n, p, μ and λ with the following property: Let $r > 0$ and $x \in \mathbb{R}^n$. If K is a compact subset of $\overline{B}(x, r)$, then*

$$\inf_{\overline{B}(x, r)} \mathfrak{R}(K, B(x, \lambda r)) \geq A_0^{-1} \left\{ \frac{\text{Cap}_{p,\mu}(K, B(x, \lambda r))}{\text{Cap}_{p,\mu}(B(x, r), B(x, \lambda r))} \right\}^{1/(p-1)}.$$

In other words,
(4.2)

$$\sup_{\overline{B}(x,r)} \omega_p(\partial B(x, \lambda r), B(x, \lambda r) \setminus K) \leq 1 - A_0^{-1} \left\{ \frac{\text{Cap}_{p,\mu}(K, B(x, \lambda r))}{\text{Cap}_{p,\mu}(B(x, r), B(x, \lambda r))} \right\}^{1/(p-1)}.$$

Using Lemma 4.3 repeatedly, we obtain the following lemma. The basic idea comes from [1, Lemma 12.6]. Write $\delta_U(x) = \text{dist}(x, \partial U)$ for an open set U .

Lemma 4.4. *Let $0 < r < \lambda r < R$ and $x \in \mathbb{R}^n$. Let Q and Q^* be bounded open sets such that $\overline{Q} \subset Q^*$. For a positive integer k let*

$$Q_k(x, R) = \{y \in Q(x, R) : \delta_{Q(x, R)}(y) \geq k\lambda r\}.$$

Suppose a closed set $E \subset \mathbb{R}^n$ satisfies

$$(4.3) \quad \frac{\text{Cap}_{p,\mu}(E \cap \overline{B}(y, r), B(y, \lambda r))}{\text{Cap}_{p,\mu}(B(y, r), B(y, \lambda r))} \geq \eta$$

for $y \in Q_1(x, R)$ with some $0 < \eta < 1$. If $Q_k(x, R) \neq \emptyset$, then
(4.4)

$$\omega_p(\partial Q^*(x, R), Q^*(x, R) \setminus (E \cap Q_1(x, R))) \leq (1 - A_0^{-1} \eta^{1/(p-1)})^{k-1} \quad \text{on } Q_k(x, R),$$

where $A_0 > 1$ is as in (4.2).

Proof. Let us prove (4.4) by induction on k . For simplicity we put

$$\Omega = \omega_p(\partial Q^*(x, R), Q^*(x, R) \setminus (E \cap Q_1(x, R))).$$

Then Ω is a p -subharmonic function in $Q^*(x, R)$ with $0 \leq \Omega \leq 1$. Since (4.4) trivially holds for $k = 1$, we assume that $k \geq 2$ and (4.4) holds for $k - 1$. In view of the comparison principle, it is sufficient to show that

$$(4.5) \quad \Omega \leq (1 - A_0^{-1} \eta^{1/(p-1)})^{k-1} \quad \text{on } \partial Q_k(x, R).$$

Take an arbitrary point $y \in \partial Q_k(x, R)$. Observe $B(y, \lambda r) \subset Q_{k-1}(x, R)$. Let $K = E \cap \overline{B}(y, r)$. Note $K \subset E \cap Q_1(x, R)$. In view of (4.2) and (4.3), we obtain

$$\omega_p^y(\partial B(y, \lambda r), B(y, \lambda r) \setminus K) \leq 1 - A_0^{-1} \eta^{1/(p-1)}.$$

It follows from the comparison principle and the induction hypothesis that

$$\Omega(y) \leq (1 - A_0^{-1} \eta^{1/(p-1)})^{k-2} \omega_p^y(\partial B(y, \lambda r), B(y, \lambda r) \setminus K) \leq (1 - A_0^{-1} \eta^{1/(p-1)})^{k-1}.$$

Since $y \in \partial Q_k(x, R)$ is arbitrary, we have (4.5). The lemma is proved. $\square$

Corollary 4.5. *Let $\lambda, r, R, \eta, E, Q, Q^*, Q_k(x, R)$ and $A_0 > 1$ be as in Lemma 4.4. Let $0 < c < 1$ and choose a positive integer k such that*

$$(1 - A_0^{-1} \eta^{1/(p-1)})^{k-1} \leq 1 - c^{1/(p-1)}.$$

If $\text{Cap}_{p,\mu}(Q_k(x, R), Q^*(x, R)) > 0$, then

$$(4.6) \quad \frac{\text{Cap}_{p,\mu}(E \cap Q_1(x, R), Q^*(x, R))}{\text{Cap}_{p,\mu}(Q_k(x, R), Q^*(x, R))} \geq c.$$

Proof. Let $K = E \cap Q_1(x, R)$. By Lemma 4.4 we have

$$\sup_{Q_k(x, R)} \omega_p(\partial Q^*(x, R), Q^*(x, R) \setminus K) \leq (1 - A_0^{-1} \eta^{1/(p-1)})^{k-1} \leq 1 - c^{1/(p-1)}.$$

Hence Lemma 4.2 with $F = Q_1(x, R)$ implies the required inequality. $\square$

Since $0 < c < 1$ is arbitrary, we are very close to the conclusion of Theorem 1.4. The problem is that $Q_k(x, R)$ in Corollary 4.5 is a proper subset of $Q(x, R)$, so that the denominator $\text{Cap}_{p,\mu}(Q_k(x, R), Q^*(x, R))$ in (4.6) may be strictly less than $\text{Cap}_{p,\mu}(Q(x, R), Q^*(x, R))$. We have to show that $\text{Cap}_{p,\mu}(Q_k(x, R), Q^*(x, R))$ is close to $\text{Cap}_{p,\mu}(Q(x, R), Q^*(x, R))$, provided R is large. In the unweighted case $w \equiv 1$, we can obtain such an approximation by the explicit calculation of the p -capacity of a ring of concentric balls.

Lemma 4.6 ([3, Example 2.12]). *Let $w \equiv 1$ and write Cap_p for $\text{Cap}_{p,\mu}$. If $0 < r < R < \infty$, then $\text{Cap}_p(B(x, r), B(x, R)) = \text{Cap}_p(\overline{B}(x, r), B(x, R))$ which is explicitly calculated as*

$$\begin{cases} \sigma_n \left(\frac{|n-p|}{p-1} \right)^{p-1} \left| R^{\frac{p-n}{p-1}} - r^{\frac{p-n}{p-1}} \right|^{1-p} & \text{if } p \neq n, \\ \sigma_n \left(\log \frac{R}{r} \right)^{1-p} & \text{if } p = n, \end{cases}$$

where σ_n is the area of the unit sphere $\partial B(0, 1)$. If $1 < p < n$, then $\text{Cap}_p(B(x, r), \mathbb{R}^n) = A_1 r^{n-p}$ with

$$A_1 = \sigma_n \left(\frac{n-p}{p-1} \right)^{p-1}.$$

If $p \geq n$, then $\text{Cap}_p(E, \mathbb{R}^n) = 0$ for every set $E \subset \mathbb{R}^n$.

We can now prove Corollary 1.5 in the unweighted case.

Proof of Corollary 1.5 in the unweighted case. It is sufficient to show that if $\underline{D}_{B,\lambda B}(\text{Cap}_p, r, E) > 0$, then $\lim_{r \rightarrow \infty} \underline{D}_{B,\lambda B}(\text{Cap}_p, r, E) = 1$. We assume that (4.3) holds for all $y \in \mathbb{R}^n$ with some $r > 0$, $\lambda > 1$ and $\eta > 0$. Let $0 < c < 1$ and choose a positive integer k such that

$$(4.7) \quad (1 - A_0^{-1} \eta^{1/(p-1)})^{k-1} \leq 1 - c^{1/2(p-1)}.$$

Let $R > k\lambda r$. Observe that $\{y \in B(x, R) : \delta_{B(x, R)}(y) \geq k\lambda r\} = \overline{B}(x, R - k\lambda r)$. In view of Lemma 4.6 we find $R_0 > 0$ such that if $R \geq R_0$, then

$$\frac{\text{Cap}_p(\overline{B}(x, R - k\lambda r), B(x, \lambda R))}{\text{Cap}_p(B(x, R), B(x, \lambda R))} \geq \sqrt{c}.$$

By (4.7) and Corollary 4.5 with $Q = B(0, 1)$, $Q^* = B(0, \lambda)$ and $\sqrt{c}$ in place of c we have

$$\frac{\text{Cap}_p(E \cap B(x, R), B(x, \lambda R))}{\text{Cap}_p(\overline{B}(x, R - k\lambda r), B(x, \lambda R))} \geq \frac{\text{Cap}_p(E \cap \overline{B}(x, R - \lambda r), B(x, \lambda R))}{\text{Cap}_p(\overline{B}(x, R - k\lambda r), B(x, \lambda R))} \geq \sqrt{c}.$$

Multiplying the inequalities, we obtain the corollary, since $x \in \mathbb{R}^n$ and $0 < c < 1$ are arbitrary. $\square$

5. APPROXIMATION OF CAPACITY FROM INSIDE AND PROOF OF THEOREM 1.4

In this section we complete the proof of Theorem 1.4 in the general case by showing that, provided R is large, $\text{Cap}_{p,\mu}(Q_k(x, R), Q^*(x, R))$ in Corollary 4.5 is close to $\text{Cap}_{p,\mu}(Q(x, R), Q^*(x, R))$. Under the interior corkscrew condition on Q , we shall show an estimate from inside uniform with respect to $x \in \mathbb{R}^n$ and $R > 0$ (Corollary 5.6). We note that there is a big difference between $B(x, r)$ and $Q(x, r)$.

Remark 5.1. For a general open set Q the inclusion $\overline{Q}(x, r) \subset Q(x, R)$ for $r < R$ need not hold, even if Q is star-shaped with respect to the origin. For instance, if $Q = \{x \in B(0, 2) : x_1 > 0\} \cup \{x \in B(0, 1) : x_1 \leq 0\}$, then Q is an open set star-shaped with respect to the origin and $Q(x, r) \subset Q(x, R)$ for $r < R$; and yet $\overline{Q}(x, r) \not\subset Q(x, R)$ for $r < R < 2r$.

Let us begin with an approximation from outside. Let D be an open set and let K be a compact subset of D . For $\varepsilon > 0$ the closed ε -neighborhood of K is denoted by

$$K[\varepsilon] = \{x \in \mathbb{R}^n : \text{dist}(x, K) \leq \varepsilon\}.$$

By definition $\text{Cap}_{p, \mu}(K, D) \leq \text{Cap}_{p, \mu}(K[\varepsilon], D)$ whenever $K[\varepsilon] \subset D$. Let us estimate $\text{Cap}_{p, \mu}(K[\varepsilon], D)$ in case K is the closure of an open set U satisfying a condition slightly weaker than the interior corkscrew condition.

Lemma 5.2. *Let U be a bounded open set with closure $K = \overline{U} \subset D$. Suppose there exist r_1, r_2 and $\kappa > 0$ such that $0 < r_1 < r_2 < \frac{1}{2} \text{dist}(K, \partial D)$ and*

$$(5.1) \quad \xi \in \partial U \text{ and } r_1 \leq r \leq r_2 \implies B(\xi, r) \cap U \text{ contains a ball of radius } \kappa r.$$

Then there exists a constant κ_0 , $0 < \kappa_0 < 1$, depending only on κ, p, μ and n such that

$$\frac{\text{Cap}_{p, \mu}(K[2^{-j}r_2], D)}{\text{Cap}_{p, \mu}(K, D)} \leq (1 - \kappa_0^j)^{-p}, \quad \text{whenever } 2^{-j}r_2 \geq r_1.$$

In particular, if U satisfies the interior corkscrew condition, then, for each $b > 1$, there exists $\varepsilon > 0$ depending only on $b, \kappa, \rho_0, p, \mu$ and n such that

$$\frac{\text{Cap}_{p, \mu}(K[\varepsilon], D)}{\text{Cap}_{p, \mu}(K, D)} \leq b.$$

Proof. In view of (5.1) we find $\kappa' > 0$ depending only on κ, p, μ and n such that if $\xi \in \partial U$ and $r_1 \leq r \leq r_2$, then

$$\frac{\text{Cap}_{p, \mu}(\overline{B}(\xi, r) \cap K, B(\xi, 2r))}{\text{Cap}_{p, \mu}(B(\xi, r), B(\xi, 2r))} \geq \kappa',$$

which, together with (4.2), yields

$$(5.2) \quad \omega_p(\partial B(\xi, 2r), B(\xi, 2r) \setminus (\overline{B}(\xi, r) \cap K)) \leq \kappa_0 \quad \text{on } \overline{B}(\xi, r)$$

with $0 < \kappa_0 < 1$.

Now let $u_K = \mathfrak{R}(K, D)$. Since $K[2r_2] \subset D$ by assumption, it follows from (4.1) and the comparison principle that if $\xi \in \partial U$ and $r_1 \leq r \leq r_2$, then

$$1 - u_K \leq \omega_p(\partial B(\xi, 2r), B(\xi, 2r) \setminus (\overline{B}(\xi, r) \cap K)) \quad \text{on } B(\xi, 2r),$$

and hence

$$1 - u_K \leq \kappa_0 \quad \text{on } \overline{B}(\xi, r)$$

by (5.2). Replace r by $2^{-j}r_2$ with a nonnegative integer j such that $2^{-j}r_2 \geq r_1$. By induction we have

$$1 - u_K \leq \kappa_0^j \quad \text{on } \overline{B}(\xi, 2^{-j}r_2),$$

whenever $2^{-j}r_2 \geq r_1$. Since $\xi \in \partial U$ is arbitrary, it follows that

$$u_K \geq 1 - \kappa_0^j \quad \text{on } \overline{U}[2^{-j}r_2] = K[2^{-j}r_2].$$

By definition

$$\text{Cap}_{p, \mu}(K[2^{-j}r_2], D) \leq (1 - \kappa_0^j)^{-p} \text{Cap}_{p, \mu}(K, D),$$

as required. Suppose now U satisfies the interior corkscrew condition. Without loss of generality we may assume that $\rho_0 < \frac{1}{2} \text{dist}(K, \partial D)$. Hence we can take $r_2 = \rho_0$ and j arbitrarily large. Letting j be so large $(1 - \kappa_0^j)^{-p} \leq b$, we obtain the second assertion with $\varepsilon = 2^{-j} \rho_0$. $\square$

Next we consider an approximation of capacity from inside. For a bounded open set U and $\varepsilon > 0$ we write $U_\varepsilon = \{x \in U : \delta_U(x) > \varepsilon\}$. We observe the following geometric properties of U_ε .

Lemma 5.3. *Let U be a bounded open set with closure $\overline{U}$ contained in D . Suppose U satisfies the interior corkscrew condition with $0 < \kappa < 1$ and $0 < \rho_0 < \frac{1}{2} \text{dist}(U, \partial D)$. If $0 < \varepsilon < \kappa \rho_0/2$, then*

- (i) $\overline{U} \subset \overline{U}_\varepsilon[(1 + 2/\kappa)\varepsilon]$;
- (ii) U_ε enjoys (5.1) with $r_1 = 2\varepsilon/\kappa$, $r_2 = \rho_0$ and $\frac{1}{2}\kappa(1 - \kappa)$ in place of κ , i.e.,

$\xi \in \partial U_\varepsilon$ and $2\varepsilon/\kappa \leq r \leq \rho_0 \implies B(\xi, r) \cap U_\varepsilon$ contains a ball of radius $\frac{1}{2}\kappa(1 - \kappa)r$.

Proof. Let us begin with the proof of (i). Suppose $x \in \overline{U} \setminus U_\varepsilon$. By definition we find $\xi \in \partial U$ such that $|x - \xi| \leq \varepsilon$. By the interior corkscrew condition with $r = 2\varepsilon/\kappa$ we find a ball $B(y, 2\varepsilon) \subset U \cap B(\xi, 2\varepsilon/\kappa)$. This means $\delta_U(y) \geq 2\varepsilon$ and hence $y \in U_\varepsilon$, so that $\text{dist}(x, \overline{U}_\varepsilon) \leq |x - \xi| + |\xi - y| \leq \varepsilon + 2\varepsilon/\kappa$. Hence $\overline{U} \subset \overline{U}_\varepsilon[\varepsilon + 2\varepsilon/\kappa]$. Thus (i) is proved.

Next we show (ii). Let $\xi \in \partial U_\varepsilon$ and $2\varepsilon/\kappa \leq r \leq \rho_0$. By definition we find $\xi^* \in \partial U$ with $|\xi - \xi^*| = \varepsilon$. Since $\varepsilon \leq \kappa r/2$, it follows that

$$B(\xi, r) \supset B(\xi^*, r - \varepsilon) \supset B(\xi^*, (1 - 2^{-1}\kappa)r),$$

so that we find a ball $B(y, \kappa(1 - 2^{-1}\kappa)r)$ lying in

$$U \cap B(\xi^*, (1 - 2^{-1}\kappa)r) \subset U \cap B(\xi, r)$$

by the interior corkscrew condition. By definition

$$\delta_{U_\varepsilon}(y) \geq \delta_U(y) - \varepsilon \geq \kappa(1 - 2^{-1}\kappa)r - \varepsilon \geq \frac{\kappa(1 - \kappa)}{2}r,$$

which implies (ii). $\square$

Remark 5.4. In general, U_ε need not satisfy the interior corkscrew condition, i.e., $2\varepsilon/\kappa$ in (ii) cannot be replaced by 0 even if $\frac{1}{2}\kappa(1 - \kappa)$ is changed by a smaller value.

Now we show an approximation of capacity from inside, which may be of independent interest.

Lemma 5.5. *Let U be a bounded open set with closure $\overline{U}$ contained in D . Suppose U satisfies the interior corkscrew condition with $0 < \kappa < 1$ and $0 < \rho_0 < \frac{1}{2} \text{dist}(U, \partial D)$. Then, for each $b > 1$, there exists $\varepsilon > 0$ depending only on $b, \kappa, \rho_0, p, \mu$, and n such that*

$$\frac{\text{Cap}_{p, \mu}(\overline{U}_\varepsilon, D)}{\text{Cap}_{p, \mu}(\overline{U}, D)} \geq \frac{1}{b}.$$

In particular, $\text{Cap}_{p, \mu}(U, D) = \text{Cap}_{p, \mu}(\overline{U}, D)$.

Proof. Let $\varepsilon > 0$ be determined later. In view of Lemma 5.3 (ii), we let κ_1 be the constant κ_0 in Lemma 5.2 with $\kappa(1 - \kappa)/2$ in place of κ . Note that κ_1 depends only on κ, p, μ and n . For each $b > 1$ we can choose j such that $(1 - \kappa_1^j)^{-p} < b$.

Let $\varepsilon = 2^{-j-2}\rho_0\kappa$. Observe that $2\varepsilon/\kappa < (1 + 2/\kappa)\varepsilon \leq 2^{-j}\rho_0$. By Lemma 5.3 (i), (ii) and Lemma 5.2 with $\overline{U}_\varepsilon$ in place of K we have

$$\frac{\text{Cap}_{p,\mu}(\overline{U}, D)}{\text{Cap}_{p,\mu}(\overline{U}_\varepsilon, D)} \leq \frac{\text{Cap}_{p,\mu}(\overline{U}_\varepsilon[(1 + 2/\kappa)\varepsilon], D)}{\text{Cap}_{p,\mu}(\overline{U}_\varepsilon, D)} \leq \frac{\text{Cap}_{p,\mu}(\overline{U}_\varepsilon[2^{-j}\rho_0], D)}{\text{Cap}_{p,\mu}(\overline{U}_\varepsilon, D)} \leq b,$$

which gives the required estimate. The last assertion follows since $\overline{U}_\varepsilon \subset U$ and $b > 1$ is arbitrary. $\square$

Since Lemma 5.5 and its proof are scale- and translation-invariant, we obtain the following uniform approximation.

Corollary 5.6. *Let Q and Q^* be bounded open sets such that $\overline{Q} \subset Q^*$. Suppose Q satisfies the interior corkscrew condition. Then for each $b > 1$ there exists an open set Q' such that $\overline{Q'} \subset Q$ and*

$$\frac{\text{Cap}_{p,\mu}(\overline{Q'}(x, R), Q^*(x, R))}{\text{Cap}_{p,\mu}(Q(x, R), Q^*(x, R))} \geq \frac{1}{b}$$

uniformly for $x \in \mathbb{R}^n$ and $R > 0$.

Proof. Let Q and Q^* be bounded open sets such that $\overline{Q} \subset Q^*$. Suppose Q satisfies the interior corkscrew condition with $0 < \kappa < 1$ and $0 < \rho_0 < \frac{1}{2} \text{dist}(Q, \partial Q^*)$. Then, for every $x \in \mathbb{R}^n$ and $R > 0$, the open set $Q(x, R)$ satisfies the interior corkscrew condition with the same κ and $\rho_0 R$. Note that $0 < \rho_0 R < \frac{1}{2} \text{dist}(Q(x, R), \partial Q^*(x, R))$. Let κ_1 be as in the proof of Lemma 5.5. For each $b > 1$ we choose j such that $(1 - \kappa_1^j)^{-p} < b$. Let $\varepsilon = 2^{-j-2}\rho_0\kappa$ and $Q' = Q_\varepsilon$. Since $(1 + 2/\kappa)\varepsilon R \leq 2^{-j}\rho_0 R$ and

$$Q'(x, R) = \{y \in Q(x, R) : \delta_{Q(x, R)}(y) > \varepsilon R\}$$

by dilation and translation, the required estimate follows from the proof of Lemma 5.5 with $U = Q(x, R)$ and $D = Q^*(x, R)$. $\square$

Proof of Theorem 1.4. It is sufficient to show that if $\underline{D}_{Q, Q^*}(\text{Cap}_{p,\mu}, r, E) > 0$ for some $r > 0$, then $\lim_{r \rightarrow \infty} \underline{D}_{Q, Q^*}(\text{Cap}_{p,\mu}, r, E) = 1$. The main idea is the same as in the proof of Corollary 1.5. In view of Lemma 3.2, we may assume that (4.3) holds for all $y \in \mathbb{R}^n$ with some $r > 0$, $\lambda > 1$ and $\eta > 0$. Let $0 < c < 1$ and choose a positive integer k satisfying (4.7). Let Q' be as in Corollary 5.6 with $b = 1/\sqrt{c}$. Choose $R > 0$ such that $R \text{dist}(\partial Q', \partial Q) > k\lambda r$. For an arbitrary point $x \in \mathbb{R}^n$ we have

$$\text{dist}(\partial Q'(x, R), \partial Q(x, R)) = R \text{dist}(\partial Q', \partial Q) > k\lambda r,$$

so that $\overline{Q'}(x, R) \subset Q_k(x, R) = \{y \in Q(x, R) : \delta_{Q(x, R)}(y) \geq k\lambda r\}$. Hence

$$\frac{\text{Cap}_{p,\mu}(Q_k(x, R), Q^*(x, R))}{\text{Cap}_{p,\mu}(Q(x, R), Q^*(x, R))} \geq \sqrt{c}$$

by Corollary 5.6. On the other hand, Corollary 4.5 with $\sqrt{c}$ in place of c gives

$$\frac{\text{Cap}_{p,\mu}(E \cap Q(x, R), Q^*(x, R))}{\text{Cap}_{p,\mu}(Q_k(x, R), Q^*(x, R))} \geq \frac{\text{Cap}_{p,\mu}(E \cap Q_1(x, R), Q^*(x, R))}{\text{Cap}_{p,\mu}(Q_k(x, R), Q^*(x, R))} \geq \sqrt{c}.$$

Multiplying the inequalities, we obtain the theorem, since $x \in \mathbb{R}^n$ and $0 < c < 1$ are arbitrary. $\square$

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REFERENCES

- [1] Hiroaki Aikawa, *Intrinsic ultracontractivity via capacity width*, to appear in Rev. Mat. Iberoamericana.
- [2] Jana Björn, Paul MacManus, and Nageswari Shanmugalingam, *Fat sets and pointwise boundary estimates for p -harmonic functions in metric spaces*, J. Anal. Math. **85** (2001), 339–369, DOI 10.1007/BF02788087. MR1869615 (2002j:31017)
- [3] Juha Heinonen, Tero Kilpeläinen, and Olli Martio, *Nonlinear potential theory of degenerate elliptic equations*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1993. Oxford Science Publications. MR1207810 (94e:31003)
- [4] Vladimir G. Maz'ja, *The continuity at a boundary point of the solutions of quasi-linear elliptic equations* (Russian, with English summary), Vestnik Leningrad. Univ. **25** (1970), no. 13, 42–55. MR0274948 (43 #706)
- [5] David A. Stegenga, *A geometric condition which implies BMOA*, Michigan Math. J. **27** (1980), no. 2, 247–252. MR568645 (81h:30039)

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