CORRIGENDUM TO “ON A DISCRETE VERSION OF TANAKA’S THEOREM FOR MAXIMAL FUNCTIONS”

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ABSTRACT. In this note we present a brief fix for an oversight in the proof of Lemma 3(iii) in a 2012 paper by Bober, Carneiro, Hughes and Pierce.

There is an oversight in the proof of Lemma 3 (iii) of [1], in which it is claimed that

$$(0.1) \quad \widetilde{M}f(n) = \max\{M_L f(n), M_R f(n)\}, \quad \forall n \in \mathbb{Z}.$$

This *is not true* in general (as can be seen by the simple example of f being a delta function at the origin). The right-hand side of (0.1) is in fact equal to a slightly different discrete non-centered Hardy-Littlewood maximal function, namely

$$\widetilde{\widetilde{M}}f(n) = \sup_{r,s \in \mathbb{Z}^+} \frac{1}{(r+s+\lambda_{rs})} \left\{ \left(\sum_{k=-r}^{k=-1} |f(n+k)| \right) + \lambda_{rs}|f(n)| + \left(\sum_{k=1}^{k=s} |f(n+k)| \right) \right\},$$

where $\lambda_{rs} = 1$ if $rs > 0$ and $\lambda_{rs} = \frac{1}{2}$ if $rs = 0$ (i.e., when we take an average over an interval that has n as an endpoint, we consider its contribution with weight $\frac{1}{2}$). As it is, the proof in [1] works to prove Lemma 3 (iii) and hence Theorem 1 for the maximal operator $\widetilde{\widetilde{M}}$. In order to prove Lemma 3 (iii) and hence Theorem 1 for $\widetilde{M}$, which was the original claim, we make a brief adjustment in the argument. We recall our convention that a point n is a local maximum of a function g on $\mathbb{Z}$ if $g(n-1) \leq g(n)$ and $g(n) > g(n+1)$. In particular, it is always possible to specify a sequence as in (2.3) of [1] so that 2.4 of [1] holds.

Lemma 3. (iii) *Let $f : \mathbb{Z} \rightarrow \mathbb{R}$ be a function of bounded variation. If n is a local maximum of $\widetilde{M}f$, then $\widetilde{M}f(n) = |f(n)|$.*

Proof. Without loss of generality we may assume that f is non-negative. Let $n \in \mathbb{Z}$ be a point such that $\widetilde{M}f(n) > f(n)$, and assume that n is a local maximum of $\widetilde{M}f$. Let us arrive at a contradiction. Recall that

$$\widetilde{M}f(n) = \sup_{r,s \in \mathbb{Z}^+} \frac{1}{(r+s+1)} \sum_{k=-r}^{k=s} f(n+k).$$

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Case 1. $\widetilde{M}f(n)$ is attained for some $r, s \in \mathbb{Z}^+$. Let $I = [n - r, n + s]$ be such an interval. If $s \geq 1$, then we may take an average over the same interval I in order to bound $\widetilde{M}f(n + 1)$, i.e.,

$$\widetilde{M}f(n + 1) \geq \frac{1}{(r + s + 1)} \sum_{k=-r}^{k=s} f(n + k) = \widetilde{M}f(n),$$

a contradiction. If $s = 0$, since $\widetilde{M}f(n) > f(n)$, we must have $r \geq 1$. We then have

$$\begin{aligned} \widetilde{M}f(n) &= \frac{1}{(r + 1)} \sum_{k=-r}^{k=0} f(n + k), \\ \widetilde{M}f(n - 1) &\geq \frac{1}{r} \sum_{k=-r}^{k=-1} f(n + k). \end{aligned}$$

This also leads to a contradiction by observing that

$$\widetilde{M}f(n) \leq (r + 1)\widetilde{M}f(n) - r\widetilde{M}f(n - 1) \leq f(n).$$

Case 2. $\widetilde{M}f(n)$ is not attained for any $r, s \in \mathbb{Z}^+$. In this case we see that $\widetilde{M}f(m) \geq \widetilde{M}f(n)$ for any $m \in \mathbb{Z}$, which is a contradiction. This argument was already used in our original proof, and we replicate it here for convenience. Given $\varepsilon > 0$, we may take an interval $I = [n - r, n + s]$ big enough so that

$$\widetilde{M}f(n) - \varepsilon \leq \frac{1}{(r + s + 1)} \sum_{k=-r}^{k=s} f(n + k).$$

For a fixed $m \in \mathbb{Z}$, we consider the average over the interval $I' = [m - r, m + s]$. Letting $C = \|f\|_{\ell^\infty(\mathbb{Z})}$, we obtain

$$\begin{aligned} \widetilde{M}f(m) &\geq \frac{1}{(r + s + 1)} \sum_{k=-r}^{k=s} f(m + k) \\ &\geq (\widetilde{M}f(n) - \varepsilon) - \frac{2C|m - n|}{r + s + 1}. \end{aligned}$$

Letting $(r + s) \rightarrow \infty$, we get

$$\widetilde{M}f(m) \geq \widetilde{M}f(n) - \varepsilon.$$

Letting $\varepsilon \rightarrow 0$, we arrive at the desired conclusion.

Note that one can modify this non-centered maximal operator, for instance, taking weight $\frac{1}{2}$ at one or both endpoints of the averaging intervals. Another way is to consider only intervals containing an odd number of points, because such intervals can be identified with the balls in $\mathbb{Z}$ with the natural metric. We remark that it is possible to obtain the analogous result for these operators using the same method with slightly modified proofs of all auxiliary properties.

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REFERENCES

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