SLICING INEQUALITIES FOR SUBSPACES OF L_p

ALEXANDER KOLDOBSKY

(Communicated by Thomas Schlumprecht)

ABSTRACT. We prove slicing inequalities for measures of the unit balls of subspaces of L_p , $-\infty < p < \infty$. For example, for every $k \in \mathbb{N}$ there exists a constant $C(k)$ such that for every $n \in \mathbb{N}$, $k < n$, every convex k -intersection body (unit ball of a normed subspace of L_{-k}) L in $\mathbb{R}^n$ and every measure μ with non-negative even continuous density in $\mathbb{R}^n$,

$$\mu(L) \leq C(k) \max_{\xi \in S^{n-1}} \mu(L \cap \xi^\perp) |L|^{1/n},$$

where $\xi^\perp$ is the central hyperplane in $\mathbb{R}^n$ perpendicular to ξ , and $|L|$ is the volume of L . This and other results are in the spirit of the hyperplane problem of Bourgain. The proofs are based on stability inequalities for intersection bodies and estimates for the Banach-Mazur distance from the unit ball of a subspace of L_p to the class of intersection bodies.

1. INTRODUCTION

The hyperplane problem [1, 3, 4, 31] asks whether there exists an absolute constant C so that for any origin-symmetric convex body K in $\mathbb{R}^n$

$$(1) \quad |K|^{\frac{n-1}{n}} \leq C \max_{\xi \in S^{n-1}} |K \cap \xi^\perp|,$$

where $\xi^\perp$ is the central hyperplane in $\mathbb{R}^n$ perpendicular to ξ , and $|K|$ stands for volume of proper dimension. The problem is still open. Bourgain [5] proved that $C \leq O(n^{1/4} \log n)$, and Klartag [13] removed the logarithmic term and established the best-to-date estimate $C \leq O(n^{1/4})$. We refer the reader to [6] for the history and partial results.

Inequality (1) has been proved for several special classes of bodies. For instance, if K is an intersection body, the inequality is known with the best possible constant (see [7, Th. 9.4.11]):

$$(2) \quad |K|^{\frac{n-1}{n}} \leq c_n \max_{\xi \in S^{n-1}} |K \cap \xi^\perp|,$$

where $c_n = |B_2^n|^{\frac{n-1}{n}} / |B_2^{n-1}| < 1$, and B_2^n is the unit Euclidean ball. One possible path to attack the hyperplane problem is through the scale of finite dimensional subspaces of L_p , because one approximately gets all origin-symmetric convex bodies, as $p \rightarrow +\infty$. It was shown by Ball [2] that the hyperplane problem has an affirmative answer for the unit balls of finite dimensional subspaces of L_p , $1 \leq p \leq 2$. Junge [9] extended this result to every $p \in (1, \infty)$, with the constant C depending on p and

Received by the editors July 20, 2014 and, in revised form, December 26, 2014.

2010 *Mathematics Subject Classification.* Primary 52A20; Secondary 46B07.

This work was partially supported by the US National Science Foundation, grant DMS-1265155.

going to infinity when $p \rightarrow \infty$. Milman [28] gave a different proof for subspaces of L_p , $2 \leq p < \infty$, with the constant $C \leq O(\sqrt{p})$. Another proof of this estimate can be found in [23].

The paper [23] also explored a different path, through n -dimensional spaces that embed in L_{-p} , $0 < p < n$. The latter concept was introduced in [15, 17]; see definition and basic facts below. In particular, a normed space embeds in L_{-k} if and only if its unit ball belongs to the class of k -intersection bodies introduced in [16, 17] in connection with generalizations of the Busemann-Petty problem. The path through L_{-p} presents some hope because every n -dimensional normed space embeds in L_{-p} with $n - 3 \leq p < n$. However, the paper [23] did not go very far, as the hyperplane conjecture was confirmed there only for the unit balls of spaces that embed in L_{-p} , $0 < p < 2$, with the constant going to infinity as $p \rightarrow -2$.

In this note we prove the hyperplane conjecture for the unit balls of normed spaces that embed in L_{-p} for arbitrary $p > 0$, and, in particular, for k -intersection bodies for all $k \in \mathbb{N}$. Moreover, we prove the result with arbitrary measures in place of volume.

The first result on the hyperplane problem for arbitrary measures was proved in [19]. If K is an intersection body in $\mathbb{R}^n$ and μ is any measure with even continuous density f in $\mathbb{R}^n$ (so that $\mu(B) = \int_B f$ for every compact set $B \subset \mathbb{R}^n$), then

$$(3) \quad \mu(K) \leq \frac{n}{n-1} c_n \max_{\xi \in S^{n-1}} \mu(K \cap \xi^\perp) |K|^{1/n},$$

where c_n is the same constant as in (2). The constant in (3) is sharp and differs from the case of volume in (2) by a small factor of $n/(n-1)$. Recall that $c_n < 1$.

The condition that K is an intersection body in inequality (3) was removed in [20] at the expense of extra $\sqrt{n}$ in the right-hand side. For any origin-symmetric convex body K in $\mathbb{R}^n$ and any measure μ with even continuous density in $\mathbb{R}^n$,

$$(4) \quad \mu(K) \leq \sqrt{n} \frac{n}{n-1} c_n \max_{\xi \in S^{n-1}} \mu(K \cap \xi^\perp) |K|^{1/n}.$$

Analogues of (4) for sections of lower dimensions and for complex convex bodies were proved in [21].

Our main result is the following inequality. For any $p > 0$, there exists a constant $C(p)$, such that for any $n \in \mathbb{N}$, $n > p$, any convex body L in $\mathbb{R}^n$ that is the unit ball of a normed space embedding in L_{-p} , any $1 \leq m \leq n-1$, and any measure μ with even continuous non-negative density in $\mathbb{R}^n$,

$$(5) \quad \mu(L) \leq (C(p))^m \frac{n}{n-m} \max_{H \in Gr_{n-m}} \mu(L \cap H) |L|^{m/n},$$

where Gr_{n-m} is the Grassmannian of $(n-m)$ -dimensional subspaces of $\mathbb{R}^n$. In the case where $p = k \in \mathbb{N}$, inequality (5) applies to any convex k -intersection body L .

We also show that the constant $\sqrt{n}$ in (4) can be replaced by $n^{1/2-1/p}$ when K is the unit ball of an n -dimensional subspace of L_p , $p > 2$. The proofs are based on a stability result for intersection bodies, which was proved in [21] and continued the study of stability in volume comparison problems initiated in [18]; see also [22].

2. EMBEDDING IN L_{-p} AND k -INTERSECTION BODIES

We need several definitions and facts. A closed bounded set K in $\mathbb{R}^n$ is called a *star body* if every straight line passing through the origin crosses the boundary of

K at exactly two points different from the origin, the origin is an interior point of K , and the *Minkowski functional* of K defined by

$$\|x\|_K = \min\{a \geq 0 : x \in aK\}$$

is a continuous function on $\mathbb{R}^n$.

The *radial function* of a star body K is defined by

$$\rho_K(x) = \|x\|_K^{-1}, \quad x \in \mathbb{R}^n, \ x \neq 0.$$

If $x \in S^{n-1}$, then $\rho_K(x)$ is the radius of K in the direction of x .

The class of intersection bodies was introduced by Lutwak [26]. Let K, L be origin-symmetric star bodies in $\mathbb{R}^n$. We say that K is the intersection body of L if the radius of K in every direction is equal to the $(n-1)$ -dimensional volume of the section of L by the central hyperplane orthogonal to this direction, i.e., for every $\xi \in S^{n-1}$,

$$(6) \quad \rho_K(\xi) = \|\xi\|_K^{-1} = |L \cap \xi^\perp|.$$

All bodies K that appear as intersection bodies of different star bodies form *the class of intersection bodies of star bodies*. The class of *intersection bodies* can be defined as the closure of the class of intersection bodies of star bodies in the radial metric

$$\rho(K, L) = \sup_{\xi \in S^{n-1}} |\rho_K(\xi) - \rho_L(\xi)|.$$

A more general concept of a k -intersection body was introduced in [16, 17]. For an integer k , $1 \leq k < n$ and star bodies D, L in $\mathbb{R}^n$, we say that D is the k -intersection body of L if for every $(n-k)$ -dimensional subspace H of $\mathbb{R}^n$,

$$|D \cap H^\perp| = |L \cap H|,$$

where $H^\perp$ is the k -dimensional subspace orthogonal to H . Taking the closure in the radial metric of the class of all D 's that appear as k -intersection bodies of star bodies, we define the class of *k -intersection bodies* (the original definition in [16, 17] was different; the equivalence of definitions was proved by Milman [29]). If $k = 1$, one gets the usual intersection bodies. Intersection bodies played a crucial role in the solution of the Busemann-Petty problem and its generalizations; see [14, Ch. 5].

Another generalization of intersection bodies was introduced by Zhang [36] in connection with the lower dimensional Busemann-Petty problem. For $1 \leq m \leq n-1$, denote by Gr_{n-m} the Grassmanian of $(n-m)$ -dimensional subspaces of $\mathbb{R}^n$. The $(n-m)$ -dimensional spherical Radon transform $R_{n-m} : C(S^{n-1}) \mapsto C(Gr_{n-m})$ is a linear operator defined by

$$R_{n-m}g(H) = \int_{S^{n-1} \cap H} g(x) \, dx, \quad \forall H \in Gr_{n-m}$$

for every function $g \in C(S^{n-1})$. Denote by

$$R_{n-m}(C(S^{n-1})) = X \subset C(Gr_{n-m}).$$

Let $M^+(X)$ be the space of linear positive continuous functionals on X , i.e., for every $\nu \in M^+(X)$ and non-negative function $f \in X$, we have $\nu(f) \geq 0$.

An origin-symmetric star body K in $\mathbb{R}^n$ is called a *generalized m -intersection body* if there exists a functional $\nu \in M^+(X)$, so that for every $g \in C(S^{n-1})$,

$$(7) \quad \int_{S^{n-1}} \|x\|_K^{-m} g(x) \, dx = \nu(R_{n-m}g).$$

It was shown in [17] (see [14, p. 92] or [29] for different proofs) that every generalized m -intersection body is an m -intersection body (it sounds counterintuitive, but the class of m -intersection bodies is larger). Note that generalized 1-intersection bodies coincide with the original intersection bodies of Lutwak. We denote the class of generalized m -intersection bodies by BP_m .

The concept of embedding of finite dimensional normed spaces in L_{-p} with $p > 0$ was introduced in [15, 17] as an analytic extension of embedding of normed spaces into L_p with $p > 0$. It is a well-known fact going back to P. Levy (see for example [14, Lemma 6.4]) that an n -dimensional normed space $(\mathbb{R}^n, \|\cdot\|)$ embeds in $L_p([0, 1])$, $p > -1$ if and only if there exists a finite Borel measure μ on the sphere S^{n-1} in $\mathbb{R}^n$ so that for every $x \in \mathbb{R}^n$,

$$(8) \quad \|x\|^p = \int_{S^{n-1}} |(x, \xi)|^p d\mu(\xi).$$

We say that a normed space embeds in L_p if its norm satisfies (8), without associating L_p with a measure space. Embedding in L_{-p} is defined as an analytic extension of (8).

Definition 1. Let X be an n -dimensional normed space, and $0 < p < n$. We say that X embeds in L_{-p} if there exists a finite Borel measure μ on S^{n-1} so that, for every even Schwartz test function $\phi \in \mathcal{S}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \|x\|_X^{-p} \phi(x) \, dx = \int_{S^{n-1}} \left(\int_{\mathbb{R}} |t|^{p-1} \hat{\phi}(t\theta) \, dt \right) d\mu(\theta),$$

where $\hat{\phi}$ stands for the Fourier transform.

A connection between intersection bodies and embedding in L_p was established in [17, Th. 3].

Proposition 1. *Let $1 \leq k < n$. The following are equivalent:*

- (i) *An origin symmetric star body D in $\mathbb{R}^n$ is a k -intersection body;*
- (ii) *$\|\cdot\|_D^{-k}$ represents a positive definite distribution;*
- (iii) *the space $(\mathbb{R}^n, \|\cdot\|_D)$ embeds in L_{-k} .*

The advantage of this connection (and, consequently, of introducing embeddings in L_{-p}) is that now one can try to extend to negative values of p different results about usual L_p -spaces. Every such extension gives new information about intersection bodies. Let us show several examples of this approach.

A well-known simple fact is that every two-dimensional normed space embeds in L_1 . It can be proved (see [14, Theorem 4.13]) that for every symmetric convex body K in $\mathbb{R}^n$ and every $p \in [n-3, n)$, the function $\|\cdot\|_K^{-p}$ represents a positive definite distribution, so by Proposition 1 every n -dimensional Banach space embeds in L_{-n+3} . Putting $n = 2$ we get the property of two-dimensional spaces mentioned above. Putting $n = 4$ we see that every four-dimensional normed space embeds in L_{-1} . By Proposition 1, every four-dimensional symmetric convex body is a 1-intersection body, which solves in the affirmative the critical four-dimensional case of the Busemann-Petty problem; see [14, Ch. 5].

Another well-known property of L_p -spaces is that, for any $0 < p < q \leq 2$, the space L_q embeds isometrically in L_p (see for example [14, Corollary 6.7]), so L_p -spaces become larger when p decreases from 2. This result can be extended to negative p as follows.

Proposition 2 ([14, Th. 6.17]). *Every n -dimensional subspace of L_q , $0 < q \leq 2$ embeds in L_{-p} for every $p \in (0, n)$. Hence, the unit ball of every n -dimensional subspace of L_q , $0 < q \leq 2$ is a k -intersection body for every $1 \leq k < n$.*

The factorization theorem of Maurey and Nikišin [27, 32] implies that, for $0 < p < q < 1$, every Banach subspace of L_p is isomorphic to a subspace of L_q . The following extension to negative p was proved in [10] (see also [14, Section 6.3]). Recall that the geometric distance between two star bodies K and L , $d_G(K, L)$, is defined as the infimum of positive numbers r such that there exists some $a > 0$ so that

$$K \subset aL \subset rK,$$

and the Banach-Mazur distance is defined by

$$d_{BM}(K, L) = \inf_{T \in GL_n} d_G(K, TL).$$

Proposition 3 ([10]). *If $-\infty < p < q < 1$, $p \neq 0$ and $q > 0$, then there exists a constant $C = C(p, q)$ so that, for any n with $-n < p$, if X is an n -dimensional normed space that embeds into L_p , then there is a subspace Y of L_q so that $d_{BM}(K, L) \leq C$, where K and L are the unit balls of X and Y , respectively.*

We refer the reader to [12, 14, 24, 29, 30, 34–36] for more about k -intersection bodies, embedding in L_{-p} and their applications.

3. SLICING INEQUALITIES

We use the following stability result for generalized intersection bodies proved in [21].

Proposition 4 ([21]). *Suppose that $1 \leq m \leq n - 1$, K is a generalized m -intersection body in $\mathbb{R}^n$, f is an even continuous function on K , $f \geq 1$ everywhere on K , and $\varepsilon > 0$. If*

$$\int_{K \cap H} f \leq |K \cap H| + \varepsilon, \quad \forall H \in Gr_{n-m},$$

then

$$\int_K f \leq |K| + \frac{n}{n-m} c_{n,m} |K|^{m/n} \varepsilon,$$

where $c_{n,m} = |B_2^n|^{\frac{n-m}{n}} / |B_2^{n-m}|$; note that $c_{n,m} \in ((\frac{1}{e})^{m/2}, 1)$ and that $c_{n,1}$ equals the constant c_n from (2).

We define the Banach-Mazur distance from a star body L in $\mathbb{R}^n$ to the class BP_m of generalized m -intersection bodies by

$$d_{BM}(L, BP_m) = \inf_{K \in BP_m} d_{BM}(L, K).$$

Theorem 1. *Suppose that $1 \leq m \leq n-1$ and L is a star body in $\mathbb{R}^n$ such that $d_{BM}(L, BP_m) < d$ for some $d > 0$. Then for any measure μ with even continuous non-negative density on L*

$$(9) \quad \mu(L) \leq d^m \frac{n}{n-m} c_{n,m} \max_{H \in Gr_{n-m}} \mu(L \cap H) |L|^{m/n}.$$

Proof. Since any linear transformation of a generalized m -intersection body is again a generalized m -intersection body (see [8, Corollary 6.8]), there exists $K \in BP_m$ such that

$$\frac{1}{d} K \subset L \subset K, \quad \text{and then} \quad \frac{1}{d} K \subset -L \subset K.$$

Let g be the density of μ , and let $f = \chi_K + (1/2)g(\chi_L + \chi_{-L})$, where $\chi_K, \chi_L, \chi_{-L}$ are the indicator functions of K, L and $-L$. Then $f \geq 1$ everywhere on K . Put

$$\varepsilon = \max_{H \in Gr_{n-m}} \left(\int_{K \cap H} f - |K \cap H| \right) = \max_{H \in Gr_{n-m}} \int_{L \cap H} g.$$

Now we can apply Proposition 4 to f, K, ε (the function f is not necessarily continuous on K , but the result holds by a simple approximation argument). We get

$$\begin{aligned} \mu(L) &= \int_L g = \int_K f - |K| \\ &\leq \frac{n}{n-m} c_{n,m} |K|^{m/n} \max_{H \in Gr_{n-m}} \int_{L \cap H} g \\ &\leq d^m \frac{n}{n-m} c_{n,m} |L|^{m/n} \max_{H \in Gr_{n-m}} \mu(L \cap H), \end{aligned}$$

because $K \subset dL$, so $|K| \leq d^n |L|$. □

Combining Theorem 1 with Proposition 3, we prove a slicing inequality for subspaces of L_{-p} .

Theorem 2. *For any $p > 0$ there exists a constant $C(p)$ such that for any $n \in \mathbb{N}, n > p$, any convex body L in $\mathbb{R}^n$ that is the unit ball of a normed space embedding in L_{-p} , any $1 \leq m \leq n-1$, and any measure μ with even continuous non-negative density in $\mathbb{R}^n$,*

$$(10) \quad \mu(L) \leq (C(p))^m \frac{n}{n-m} c_{n,m} \max_{H \in Gr_{n-m}} \mu(L \cap H) |L|^{m/n}.$$

Recall that $c_{n,m} < 1$.

Proof. By Proposition 3, there exists a body K which is the unit ball of a normed space that embeds in $L_{1/2}$ such that $d_{BM}(K, L) < C(p, 1/2) =: C(p)$. By Proposition 2, K is an intersection body. By [8, Lemma 6.1], every intersection body is a generalized m -intersection body for every $1 \leq m \leq n-1$, so $K \in BP_m$. The result follows from Theorem 1. □

The author is unable to estimate the behavior of the constant $C(p)$ as $p \rightarrow \infty$, because the proof of Proposition 3 is rather involved. For $p = k \in \mathbb{N}$, Theorem 2 can be reformulated using Proposition 1.

Corollary 1. *For any $k \in \mathbb{N}$ there exists a constant $C(k)$ such that for any $n \in \mathbb{N}$, $n > k$, any convex k -intersection body L in $\mathbb{R}^n$, any $1 \leq m \leq n-1$, and any measure μ with even continuous density in $\mathbb{R}^n$,*

$$(11) \quad \mu(L) \leq (C(k))^m \frac{n}{n-m} c_{n,m} \max_{H \in Gr_{n-m}} \mu(L \cap H) |L|^{m/n}.$$

Let us formulate the hyperplane cases ($m = 1$) of Theorem 2 and Corollary 1 separately, as they relate to the hyperplane problem.

Corollary 2. *For any $p > 0$ there exists a constant $D(p)$ such that for any $n \in \mathbb{N}$, $n > p$, any convex body L in $\mathbb{R}^n$ that is the unit ball of a normed space embedding in L_{-p} and any measure μ with even continuous non-negative density in $\mathbb{R}^n$,*

$$(12) \quad \mu(L) \leq D(p) \max_{\xi \in S^{n-1}} \mu(L \cap \xi^\perp) |L|^{1/n}.$$

When $p = k \in \mathbb{N}$, $1 \leq k \leq n-1$ the latter inequality applies to any convex k -intersection body L in $\mathbb{R}^n$; see the formulation in the abstract.

Finally, let us prove a slicing inequality for subspaces of L_p , $p > 2$ with arbitrary measures.

Theorem 3. *Suppose that $p > 2$, L is the unit ball of an n -dimensional subspace of L_p , $1 \leq m \leq n-1$, and μ is a measure with even continuous density in $\mathbb{R}^n$. Then*

$$\mu(L) \leq n^{m/2-m/p} \frac{n}{n-m} c_{n,m} \max_{H \in Gr_{n-m}} \mu(L \cap H) |L|^{m/n}.$$

Proof. By a result of Lewis [25] (see also [33] for a different proof), $d_{BM}(L, B_2^n) \leq n^{1/2-1/p}$. Since $B_2^n \in BP_m$ for every m , the result follows from Theorem 1. $\square$

The hyperplane case is as follows.

Corollary 3. *If L is the unit ball of an n -dimensional subspace of L_p , $p > 2$, then for any measure μ with even continuous density in $\mathbb{R}^n$,*

$$\mu(L) \leq n^{1/2-1/p} \frac{n}{n-1} c_n \max_{\xi \in S^{n-1}} \mu(L \cap \xi^\perp) |L|^{1/n}.$$

Remark. In Theorems 2 and 3 we consider L_p -spaces with $p < 0$ and $p > 2$. As follows from Proposition 2, the unit balls of subspaces of L_p with $0 < p \leq 2$ are intersection bodies and satisfy inequalities (2) and (3). In the remaining case $p = 0$, the concept of embedding in L_0 was introduced and studied in [11]. As proved in [11, Th. 6.3], the unit balls of spaces that embed in L_0 are intersection bodies and also satisfy inequalities (2) and (3).

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI, COLUMBIA, MISSOURI 65211

E-mail address: koldobskiya@missouri.edu