PROPER HOLOMORPHIC POLYNOMIAL MAPS BETWEEN BOUNDED SYMMETRIC DOMAINS OF CLASSICAL TYPE

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ABSTRACT. We prove that two proper holomorphic polynomial maps between bounded symmetric domains of classical type which preserve the origin are equivalent if and only if they are isotropically equivalent. Using this property we show that each member of a one-parameter family of maps from a 2015 paper by the author is inequivalent.

1. INTRODUCTION

Let Ω_1, Ω_2 be domains in $\mathbb{C}^n$, and let $\mathbb{C}^N$ and $f, g : \Omega_1 \rightarrow \Omega_2$ be holomorphic maps. We say that f is *proper* if $f^{-1}(K)$ is compact for every compact subset $K \subset \Omega_2$. We say that f and g are *equivalent* if and only if $f = A \circ g \circ B$ for some $B \in \text{Aut}(\Omega_1)$ and $A \in \text{Aut}(\Omega_2)$. For a domain Ω , denote the group of automorphisms fixing $p \in \Omega$ by $\text{Isot}_p(\Omega)$. Suppose that for fixed $p \in \Omega_1$, $f(p) = g(p)$. Then we say that f and g are *isotropically equivalent at p* if there are $U \in \text{Isot}_p(\Omega_1)$ and $V \in \text{Isot}_{g(p)}(\Omega_2)$ such that $f = V \circ g \circ U$. The notion of isotropic equivalence coincides with that unitary equivalence of [3] defined when Ω_1 and Ω_2 are balls. The following domains are called bounded symmetric domains of classical type:

- (1) $\Omega_{r,s}^I = \{Z \in M_{r,s}^{\mathbb{C}} : I_r - ZZ^* > 0\}$, where $s \geq r = \text{rank}(\Omega_{r,s}^I)$.
- (2) $\Omega_n^{II} = \{Z \in M_{n,n}^{\mathbb{C}} : I_n - ZZ^* > 0, Z^t = -Z\}$, $\text{rank}(\Omega_n^{II}) = \lfloor \frac{n}{2} \rfloor$.
- (3) $\Omega_n^{III} = \{Z \in M_{n,n}^{\mathbb{C}} : I_n - ZZ^* > 0, Z^t = Z\}$, $\text{rank}(\Omega_n^{III}) = n$.
- (4) $\Omega_n^{IV} = \{Z = (z_1, \dots, z_n) \in \mathbb{C}^n : ZZ^* < 1, 0 < 1 - 2ZZ^* + |ZZ^t|^2\}$, $\text{rank}(\Omega_n^{IV}) = 2$.

Here we denote by $M > 0$ positive definiteness of the square matrix M , by $M_{r,s}^{\mathbb{C}}$ the set of $r \times s$ complex matrices, and by I_r the $r \times r$ identity matrix.

The aim of this paper is to prove the following theorems as a generalization of the results in [3] which are concerned with the proper holomorphic polynomial maps between balls.

Theorem 1.1. *Let Ω_1, Ω_2 be bounded symmetric domains of classical type, and let $f, g : \Omega_1 \rightarrow \Omega_2$ be proper holomorphic polynomial maps such that $f(0) = g(0) = 0$. Then f and g are equivalent if and only if they are isotropically equivalent at 0.*

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Consider a proper holomorphic map i_h from $\Omega_{r,s}^I$ to $\Omega_{r',s'}^I$ which is defined by

$$Z \mapsto \begin{pmatrix} Z & 0 \\ 0 & h(Z) \end{pmatrix}$$

with matrix-valued map h satisfying $I_{r'-r} - h(Z)h(Z)^* > 0$ for every $Z \in \Omega_{r,s}^I$. If a proper holomorphic map f is equivalent to i_h for some h , we call f a *standard map*.

Theorem 1.2. *There are uncountably many inequivalent proper holomorphic non-standard maps from $\Omega_{r,s}^I$ to $\Omega_{2r-1,2s}^I$ for $r \geq 2$, $s \geq 2$.*

The motivation of this paper comes from generalizing the study on proper holomorphic maps between balls to maps between bounded symmetric domains of rank greater than or equal to two. Proper holomorphic maps between balls have been studied since Alexander ([1]) proved that every proper holomorphic self-map of the n -dimensional unit ball $\mathbb{B}^n$ with $n \geq 2$ is a holomorphic automorphism. For proper holomorphic maps between balls with different dimensions, much work has been done, relating the maximum degree of proper holomorphic maps to the difference of dimensions between the domain ball and the target ball. For detail, see [3, 5, 7–11, 13–15, 17, 25].

In case of bounded symmetric domains of rank greater than or equal to two, there are several rigidity theorems on proper holomorphic maps. In contrast with the case of balls, the difference of the ranks between the domains is more crucial than that of the dimensions. The first result on bounded symmetric domains along these lines is the following, which is due to Tsai ([22]). Let $f : \Omega_1 \rightarrow \Omega_2$ be a proper holomorphic map between irreducible bounded symmetric domains Ω_1 and Ω_2 . If $\text{rank}(\Omega_1) \geq \text{rank}(\Omega_2)$, then $\text{rank}(\Omega_1) = \text{rank}(\Omega_2)$ and f should be a totally geodesic isometric embedding with respect to the Bergman metrics on the domains. For more detail, see [16, 19, 20, 23, 24].

If the difference of the ranks of the bounded symmetric domains gets bigger, then it is expected that there are lots of inequivalent proper holomorphic maps. In [21], one way of finding proper holomorphic non-standard maps between bounded symmetric domains of type I is suggested, and several proper holomorphic maps are constructed. For example, for $r' = 2r - 1$ and $s' = 2s - 1$, there is a proper holomorphic map defined by

$$(1.1) \quad \begin{pmatrix} z_{11} & \cdots & z_{1s} \\ \vdots & \ddots & \vdots \\ z_{r1} & \cdots & z_{rs} \end{pmatrix} \mapsto \begin{pmatrix} z_{11}^2 & z_{11}z_{12} & \cdots & z_{11}z_{1s} & z_{12} & \cdots & z_{1s} \\ z_{11}z_{21} & z_{21}z_{12} & \cdots & z_{21}z_{1s} & z_{22} & \cdots & z_{2s} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ z_{11}z_{r1} & z_{r1}z_{12} & \cdots & z_{r1}z_{1s} & z_{r2} & \cdots & z_{rs} \\ z_{21} & z_{22} & \cdots & z_{2s} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ z_{r1} & z_{r2} & \cdots & z_{rs} & 0 & \cdots & 0 \end{pmatrix}.$$

This map is a generalized version of the Whitney map which is a proper holomorphic map from $\mathbb{B}^n$ to $\mathbb{B}^{2n-1}$ defined by

$$f_w(z_1, \dots, z_n) = (z_1, \dots, z_{n-1}, z_n z_1, z_n z_2, \dots, z_n^2).$$

Interestingly, Huang and Ji ([13]) proved that any proper rational map from $\mathbb{B}^n$ to $\mathbb{B}^{2n-1}$ with $n \geq 3$ is equivalent to $z \mapsto (z, 0)$ or f_w , and any proper holomorphic

embedding which is C^2 -smooth up to the boundary is equivalent to $z \mapsto (z, 0)$. If the dimension of the target domain is larger than $2n$, there are infinitely many inequivalent proper holomorphic maps. For example, $f_\theta : \mathbb{B}^n \rightarrow \mathbb{B}^{2n}$ given by

$$(1.2) \quad f_\theta(z) = (z_1, \dots, z_{n-1}, \cos \theta z_n, \sin \theta z_1 z_n, \dots, \sin \theta z_n z_n)$$

with $0 \leq \theta \leq \frac{\pi}{2}$ are found by D'Angelo ([3]). In [3], D'Angelo showed that any two proper holomorphic polynomial maps from $\mathbb{B}^n$ to $\mathbb{B}^N$ preserving the origin are equivalent if and only if they are isotropically equivalent at the origin, and as a consequence, (1.2) are inequivalent for all $0 \leq \theta \leq \frac{\pi}{2}$. Furthermore, Hamada ([11]) showed that any proper rational map from $\mathbb{B}^n$ to $\mathbb{B}^{2n}$ with $n \geq 4$ is equivalent to f_θ for some θ , $0 \leq \theta \leq \pi/2$. Similarly there are infinitely many proper holomorphic non-standard maps $G_t : \Omega_{r,s}^I \rightarrow \Omega_{2r-1,2s}^I$ with $0 \leq t \leq 1$ which are defined by

$$(1.3) \quad Z \mapsto \begin{pmatrix} \sqrt{t}z_{11}^2 & \sqrt{t}z_{11}z_{12} & \dots & \sqrt{t}z_{11}z_{1s} & \sqrt{1-t}z_{11} & z_{12} & \dots & z_{1s} \\ \sqrt{t}z_{11}z_{21} & \sqrt{t}z_{21}z_{12} & \dots & \sqrt{t}z_{21}z_{1s} & \sqrt{1-t}z_{21} & z_{22} & \dots & z_{2s} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \sqrt{t}z_{11}z_{r1} & \sqrt{t}z_{r1}z_{12} & \dots & \sqrt{t}z_{r1}z_{1s} & \sqrt{1-t}z_{r1} & z_{r2} & \dots & z_{rs} \\ z_{21} & z_{22} & \dots & z_{2s} & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ z_{r1} & z_{r2} & \dots & z_{rs} & 0 & 0 & \dots & 0 \end{pmatrix}$$

for $Z = (z_{ij})_{1 \leq i \leq r, 1 \leq j \leq s} \in \Omega_{r,s}^I$. In this paper, as a first step in studying analogous phenomena on proper holomorphic maps between bounded symmetric domains of rank greater than or equal to two, we generalize the result of D'Angelo in [3] to the domains of classical type and show that (1.3) are inequivalent for every t .

2. PRELIMINARIES

In this section, we introduce terminology and some basic background. A bounded domain Ω is called *symmetric* if for each $p \in \Omega$, there is a holomorphic automorphism i_p such that i_p^2 is the identity map of Ω which has p as an isolated fixed point. All bounded symmetric domains are of homogeneous domain, i.e., the automorphism group acts transitively on the domain. In 1920s, Cartan classified all irreducible bounded symmetric domains. There are four classical types and two exceptional types. The four classical types are given by items (1), (2), (3) and (4) in the Introduction. Note that $\Omega_{m,1}^I$ is the m -dimensional unit ball, and Ω_1^{III} is the unit disc.

From now on, we will use the notation $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL(r+s, \mathbb{C})$ to split M into four block matrices with $A \in M_{r,r}^{\mathbb{C}}$, $B \in M_{r,s}^{\mathbb{C}}$, $C \in M_{s,r}^{\mathbb{C}}$, and $D \in M_{s,s}^{\mathbb{C}}$. We will denote by $ASM_{n,n}^{\mathbb{C}}$ the set of anti-symmetric complex $n \times n$ matrices and by $SM_{n,n}^{\mathbb{C}}$ the set of symmetric complex $n \times n$ matrices.

Let $U(r, s)$ be the subgroup of $GL(r+s, \mathbb{C})$ consisting of $M \in GL(r+s, \mathbb{C})$ satisfying

$$M \begin{pmatrix} -I_r & 0 \\ 0 & I_s \end{pmatrix} M^* = \begin{pmatrix} -I_r & 0 \\ 0 & I_s \end{pmatrix}.$$

Let $SU(r, s)$ be the subset of $U(r, s)$ which consists of the matrices with determinant one. Explicitly,

$$(2.1) \quad SU(r, s) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SL(r+s, \mathbb{C}) : AA^* - BB^* = I_r, \right. \\ \left. AC^* = BD^*, CC^* - DD^* = -I_s \right\}.$$

Let $O(n+2, \mathbb{C})$ be the complex orthogonal group of $(n+2) \times (n+2)$ matrices.

Since every bounded symmetric domain is a Hermitian symmetric space of non-compact type, the domain can be canonically embedded into the corresponding compact dual (Borel embedding), and every holomorphic automorphism of the domain can be extended to the automorphism of its compact dual. For example, let $G_{r,s}$ be the Grassmannian of r -planes in $r+s$ dimensional complex vector space $\mathbb{C}^{r+s}$ which is the compact dual of $\Omega_{r,s}^I$. For $X \in M_{r,r+s}^{\mathbb{C}}$ of rank r , denote $[X]$ the r -plane in $\mathbb{C}^{r+s}$ which is generated by row vectors of X . For $M \in GL(r+s, \mathbb{C})$, M acts on $G_{r,s}$ by $[X] \in G_{r,s} \mapsto [XM]$. Then the Borel embedding ξ^I of $\Omega_{r,s}^I$ is given by

$$\xi^I(Z) = [I_r, Z] \in G_{r,s},$$

and $M \in U(r, s)$ acts on $\Omega_{r,s}^I$ by $g_M(Z) := \xi^{I^{-1}}(\xi(Z)M)$. Explicitly, for $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in U(r, s)$, M acts on $\Omega_{r,s}^I$ by

$$(2.2) \quad Z \mapsto (A + ZC)^{-1}(B + ZD).$$

Similarly, $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in U(n, n)$ acts on Ω_n^{II} and Ω_n^{III} by (2.2).

In the case of Ω_n^{IV} , the explicit expression of the holomorphic automorphism is little more messy. The compact dual of Ω_n^{IV} is the hyperquadric H_n in $\mathbb{P}^{n+1}$ which is defined by $H_n := \{[z_1, \dots, z_{n+2}] \in \mathbb{P}^{n+1} : z_1^2 + \dots + z_n^2 - z_{n+1}^2 - z_{n+2}^2 = 0\}$. Then the Borel embedding ξ^{IV} is

$$(2.3) \quad \xi^{IV}(Z) = [-2iZ, 1 + ZZ^t, i(1 - ZZ^t)] \in H_n.$$

For $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in O(n+2, \mathbb{C}) \cap U(n, 2)$, M acts on Ω_n^{IV} by

$$Z = (z_1, \dots, z_n) \mapsto \frac{1}{(-2iZB + Z'D)(i, 1)^t} (2iZA - Z'C),$$

where $Z' = (1 + ZZ^t, i - iZZ^t)$.

The automorphism groups of classical domains and their isotropy groups at the origin are given by the following:

$$(1) \quad \text{Aut}(\Omega_{r,s}^I) = U(r, s), \quad \text{Isot}(\Omega_{r,s}^I) = \left\{ \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} : U \in U(r), V \in U(s) \right\},$$

$$(2) \quad \text{Aut}(\Omega_n^{II}) = \left\{ M \in U(n, n) : M^t \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} M = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \right\},$$

$$\text{Isot}(\Omega_n^{II}) = \left\{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} : A \in U(n) \right\},$$

$$\begin{aligned}
 (3) \quad \text{Aut}(\Omega_n^{III}) &= \left\{ M \in U(n, n) : M^t \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} M = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \right\}, \\
 \text{Isot}(\Omega_n^{III}) &= \left\{ \begin{pmatrix} A & 0 \\ 0 & \bar{A} \end{pmatrix} : A \in U(n) \right\}, \\
 (4) \quad \text{Aut}(\Omega_n^{IV}) &= O(n+2, \mathbb{C}) \cap U(n, 2), \quad \text{Isot}(\Omega^{IV}) = O(n) \times O(2).
 \end{aligned}$$

3. ISOTROPICALLY EQUIVALENT PROPER HOLOMORPHIC POLYNOMIAL MAPS

In this section we will prove Theorem 1.1. Define a polynomial function $S_{r,s}^I : \Omega_{r,s}^I \rightarrow \mathbb{R}$ by

$$S_{r,s}^I(Z) = \det(I_r - ZZ^*)$$

for $Z = (z_{ij}) \in M_{r,s}^{\mathbb{C}}$, as a real polynomial in $\text{Re}(z_{ij})$, $\text{Im}(z_{ij})$ where $1 \leq i \leq r$, $1 \leq j \leq s$. $S_{r,s}^I$ is a polynomial of degree 2 in each $\text{Re}(z_{ij})$, $\text{Im}(z_{ij})$. In the case of Ω_n^{II} , it is known that $\det(I_r - ZZ^*) = s_n^{II}(Z)^2$ for some polynomial $s_n^{II}(Z)$ (cf. [18]). Define $S_n^{II} : \Omega_n^{II} \rightarrow \mathbb{R}$ and $S_n^{III} : \Omega_n^{III} \rightarrow \mathbb{R}$ by

$$S_n^{II}(Z) = s_n^{II}(Z) \quad \text{for } Z \in ASM_{n,n}^{\mathbb{C}},$$

$$S_n^{III}(Z) = \det(I_n - ZZ^*) \quad \text{for } Z \in SM_{n,n}^{\mathbb{C}},$$

and

$$S_n^{IV}(Z) = 1 - 2ZZ^* + |ZZ^t|^2 \quad \text{for } Z \in \mathbb{C}^n,$$

where $S_{r,s}^I(Z)$, $S_n^{II}(Z)$, $S_n^{III}(Z)$, and $S_n^{IV}(Z)$ are called the *generic norm* of the corresponding domains (cf. [18]). $S_n^{II}(Z)$ is a polynomial of degree 2 in each $\text{Re } z_{ij}$, $\text{Im } z_{ij}$ for $1 \leq i < j \leq n$. $S_n^{III}(Z)$ is a polynomial of degree 4 in each $\text{Re } z_{ij}$, $\text{Im } z_{ij}$ for $1 \leq i < j \leq n$ and of degree 2 in each $\text{Re } z_{ii}$, $\text{Im } z_{ii}$ for $1 \leq i \leq n$.

Lemma 3.1.

(1) For $Z = (z_{ij}) \in M_{r,s}^{\mathbb{C}}$, the coefficient of $(\text{Re } z_{ij})^2$ in $S_{r,s}^I(Z)$ is

$$-\det(I_{r-1} - Z'Z'^*),$$

where Z' is the (i, j) minor of Z .

(2) For $Z = (z_{ij}) \in ASM_{n,n}^{\mathbb{C}}$, the coefficient of $(\text{Re } z_{ij})^4$, $1 \leq i < j \leq n$ in $S_n^{II}(Z)$ is

$$\det(I_{n-2} - Z''Z''^*),$$

where Z'' is $(n-2) \times (n-2)$ matrix obtained by removing i, j -th rows and i, j -th columns in Z .

(3) For $Z = (z_{ij}) \in SM_{n,n}^{\mathbb{C}}$, the coefficient of $(\text{Re } z_{ij})^4$, $1 \leq i < j \leq n$, and $(\text{Re } z_{ij})^2$ for $1 \leq i \leq n$ in $S_n^{III}(Z)$ are

$$\det(I_{n-2} - Z''Z''^*) \quad \text{and} \quad -\det(I_{n-1} - Z'Z'^*),$$

respectively, where Z'' is $(n-2) \times (n-2)$ matrix removing i, j -th rows and i, j -th columns in Z and Z' is the (i, i) minor of Z .

Proof. (1). For $Z \in M_{r,s}^{\mathbb{C}}$, denote $Z = \begin{pmatrix} X \\ Y \end{pmatrix}$ with $X = (X', X'') = (x_{ij}) \in M_{r-1,s}^{\mathbb{C}}$, where $X' \in M_{r-1,1}^{\mathbb{C}}$, $X'' \in M_{r-1,s-1}^{\mathbb{C}}$ and $Y = (y_1, \dots, y_s) \in M_{1,s}^{\mathbb{C}}$. We

only consider the coefficient of $(\operatorname{Re} y_1)^2$. The coefficient of $(\operatorname{Re} y_1)^2$ in $S_{r,s}^I(Z)$ is $\frac{\partial^2}{\partial \bar{y}_1 \partial y_1} S_{r,s}^I(Z)$. Since

$$(3.1) \quad \frac{\partial}{\partial y_1} \det(I_r - ZZ^*) = \frac{\partial}{\partial y_1} \det \begin{pmatrix} I_{r-1} - XX^* & -XY^* \\ -YX^* & 1 - YY^* \end{pmatrix}$$

and $YX^* = (y_1 \bar{x}_{11} + \cdots + y_s \bar{x}_{1s}, \dots, y_1 \bar{x}_{(r-1)1} + \cdots + y_s \bar{x}_{(r-1)s})$, we obtain

$$\frac{\partial}{\partial y_1} \det(I_r - ZZ^*) = \det \begin{pmatrix} I_{r-1} - XX^* & -XY^* \\ (-\bar{x}_{11}, -\bar{x}_{21}, \dots, -\bar{x}_{(r-1)1}) & -\bar{y}_1 \end{pmatrix}$$

and

$$(3.2) \quad \frac{\partial^2}{\partial \bar{y}_1 \partial y_1} \det(I_r - ZZ^*)$$

$$(3.3) \quad = \det \left(\begin{array}{c|c} I_{r-1} - XX^* & \begin{pmatrix} -x_{11} \\ \vdots \\ -x_{(r-1)1} \end{pmatrix} \\ \hline (-\bar{x}_{11}, -\bar{x}_{21}, \dots, -\bar{x}_{(r-1)1}) & -1 \end{array} \right)$$

$$= \det \left(\begin{array}{c|c} I_{r-1} - X''X''^* & 0 \\ \hline (-\bar{x}_{11}, -\bar{x}_{21}, \dots, -\bar{x}_{(r-1)1}) & -1 \end{array} \right)$$

$$(3.4) \quad = -\det(I_{r-1} - X''X''^*).$$

The second equation comes from subtracting j -th row of (3.3) by the product of x_{j1} and the r -th row of (3.3).

(2). We will only consider the coefficient of $(\operatorname{Re} z_{12})^4$. By Lemma 3.1(1), for $W = (w_{ij}) \in M_{n,n}^C$, $\det(I_n - WW^*) = a_2(\operatorname{Re} w_{12})^2 + a_1(\operatorname{Re} w_{12}) + a_0$, where a_i are polynomials in $\operatorname{Re} w_{ij}$, $\operatorname{Im} w_{ij}$ for $i \neq 1, j \neq 2$ and $\operatorname{Im} w_{12}$. Since we can obtain the coefficient of $(\operatorname{Re} z_{12})^4$ by substituting $w_{ij} = z_{ij}$ for $1 \leq i < j < n$, $w_{ji} = -z_{ij}$ for $1 \leq i < j \leq n$, and $w_{ii} = 0$ for $1 \leq i \leq n$ in the coefficient of $(\operatorname{Re} w_{12})^2(\operatorname{Re} w_{21})^2$, the coefficient of $(\operatorname{Re} z_{12})^4$ is $\det(I_{n-2} - Z''Z''^*)$, where $Z'' = (z_{ij})_{3 \leq i \leq n, 3 \leq j \leq n}$.

(3). We can obtain the result by similar method in (1) and (2). $\square$

Proposition 3.2. $S_{r,s}^I(Z)$, $S_{r,s}^{II}(Z)$, $S_{r,s}^{III}(Z)$, and $S_n^{IV}(Z)$ are irreducible.

Proof. In the case of $S_{r,s}^I$: At first we will prove that $S_{r,s}^I(Z)$ is irreducible. Note that $r \leq s$. We use induction with respect to k on $S_{k,s-r+k}^I$. For $Z \in M_{1,s-r+1}^C$, $1 - ZZ^*$ is irreducible. Suppose that $S_{r-1,s-1}^I(Z)$ is irreducible and $S_{r,s}^I(Z) = AB$ for some polynomial A and B . Denote $Z = \begin{pmatrix} X \\ Y \end{pmatrix}$ as in the proof of Lemma 3.1. The degree of $\operatorname{Re} y_1$ is 2.

Step 1. Suppose that there is a non-zero monomial of $(\operatorname{Re} y_1)^2$ in A . Then we can write $S_{r,s}^I(Z)$ as

$$S_{r,s}^I(Z) = (\mu(\operatorname{Re} y_1)^2 + \sigma \operatorname{Re} y_1 + \nu) B$$

for some suitable polynomials μ, σ, ν and B which do not contain $\operatorname{Re} y_1$ variable. Without loss of generality, we may assume that B is not a constant. Then by Lemma 3.1, μB is $\det(I_{r-1} - X''X''^*)$ which is irreducible by the induction hypothesis. Hence B is $\det(I_{r-1} - X''X''^*)$ (up to constants). This implies that it consists of the monomials of variable X'' . Then A has every variable of X' . Consider the coefficient of $(\operatorname{Re} x_{11})^2$ which is irreducible and does not contain $\operatorname{Re} x_{1j}$, $\operatorname{Im} x_{1j}$ variables.

However the coefficient of $(\operatorname{Re} x_{11})^2$ contains B and this induces a contradiction. Therefore, there is no second-order term in each A and B .

Step 2. Suppose that $S_{r,s}^I(Z) = (\mu \operatorname{Re} y_1 + \sigma)(\nu \operatorname{Re} y_1 + \rho)$ with polynomials μ, σ, ν, ρ , which do not have the $\operatorname{Re} y_1$ variable. Then $\mu\nu$ is irreducible by Lemma 3.1 and hence μ or ν is a constant. If μ is a constant, $\nu = \det(I_{r-1} - X''X''^*)$ up to a constant. Note that ν contains a second-order term of variables in X'' . Since there is no $\operatorname{Re} y_1$ term in ρ , $\nu \operatorname{Re} y_1 + \rho$ has a second-order term. But by Step 1, $\nu \operatorname{Re} y_1 + \rho$ cannot have second-order term. Hence, $S_{r,s}^I(Z)$ is irreducible.

In the case of S_n^{II} , S_n^{III} , S_n^{IV} : Since for $n = 1$, $S_1^{III}(z) = 1 - |z|^2$ for $z \in \Delta$, which is irreducible, use the same method (induction) as in the proof of the case $S_{r,s}^I$ considering factorization with respect to the $\operatorname{Re} z_{11}$ variable. In the case of S_n^{IV} , we can easily show that it is irreducible. We omit the proof.

We only sketch the proof that $S_n^{II}(Z)$ is irreducible. For $n = 2$, let $Z = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$. Then $S_2^{II}(Z) = 1 - |a|^2$ which is irreducible. For $n = 3$, let $Z = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$. Then $S_3^{II}(Z) = 1 - |a|^2 - |b|^2 - |c|^2$ which is also irreducible.

Assume that $S_{2n}^{II}(Z)$ and $S_{2n-1}^{II}(Z)$ are irreducible. Since the even-dimensional case is similar to the odd-dimensional case, we will only consider the odd-dimensional case. Since the coefficient of $(\operatorname{Re} z_{ij})^4$ in $S_{2n+1}^{II}(Z)^2$ is $\det(I_{2n-1} - Z''Z''^*)$ as in Lemma 3.1 (2), the coefficient of $(\operatorname{Re} z_{ij})^2$ in $S_{2n+1}^{II}(Z)$ is irreducible. Hence, a similar proof of the case $S_{r,s}^I$ can be applied. $\square$

Let $f : \Omega_1 \rightarrow \Omega_2$ be a proper holomorphic polynomial map where Ω_1, Ω_2 are irreducible bounded symmetric domains of classical type. Let S_1, S_2 be the corresponding generic norms. Since f is proper, by Proposition 3.2,

$$S_2(f(Z)) = 0 \text{ whenever } S_1(Z) = 0.$$

(Note that if $Z \in \partial\Omega_n^{IV}$, $S_n^{IV}(Z) = 0$ since if $ZZ^* = 1$, $S_n^{IV}(Z) < 0$. Hence, we do not need to consider the $ZZ^* - 1$ term in the definition of Ω_n^{IV} .) Hence, there is a real analytic map F_f satisfying

$$(3.5) \quad S_2(f(Z)) = S_1(Z) F_f(Z).$$

We can polarize this equation by

$$(3.6) \quad S_2(f(Z), f(W)) = S_1(Z, W) F_f(Z, W).$$

Example 3.3. (1) Let $f : \Omega_{r,s} \rightarrow \Omega_{r',s'}$ be a proper holomorphic polynomial map. Then (3.6) is expressed by

$$\det(I_{r'} - f(Z)f(W)^*) = \det(I_r - ZW^*) F_f(Z, W).$$

(2) Let $f : \Omega_n^{IV} \rightarrow \Omega_N^{IV}$ be a proper holomorphic polynomial map. Then (3.6) is expressed by

$$\begin{aligned} 1 - 2f(Z)f(W)^* + (f(Z)f(Z)^t) \overline{(f(W)f(W)^t)} \\ = \left(1 - 2ZW^* + (ZZ^t) \overline{(WW^t)}\right) F_f(Z, W). \end{aligned}$$

Lemma 3.4. Let $g : \Omega_1 \rightarrow \Omega_2$, and let $f : \Omega_2 \rightarrow \Omega_3$ be proper holomorphic polynomial maps. Then

$$F_{f \circ g}(Z, W) = F_g(Z, W) F_f(g(Z), g(W)).$$

Proof.

$$\begin{aligned} S_1(Z, W) F_{f \circ g}(Z, W) &= S_3(f \circ g(Z), f \circ g(W)) \\ &= S_2(g(Z), g(W)) F_f(g(Z), g(W)) \\ &= S_1(Z, W) F_g(Z, W) F_f(g(Z), g(W)). \end{aligned} \quad \square$$

Lemma 3.5. Let $U = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be an automorphism of a domain Ω with appropriate block matrices A, B, C, D . Then F_U is given as the following:

(1) If $\Omega = \Omega_{r,s}^I$ or Ω_n^{III} ,

$$F_U(Z, W) = \frac{1}{\det(A + ZC) \overline{\det(A + WC)}}.$$

(2) If $\Omega = \Omega_n^{II}$,

$$F_U(Z, W) = \frac{1}{f_U(z, w)} \quad \text{where} \quad f_U(Z, W)^2 = \det(A + ZC) \overline{\det(A + WC)}.$$

(3) If $\Omega = \Omega_n^{IV}$,

$$F_U(Z, W) = -\frac{1}{2} \frac{1}{\{(-2iZB + Z'D)(i, 1)^t\} \{(-2iWB + W'D)(i, 1)^t\}}.$$

Proof. (1 and 2) Since by (2.1),

$$\begin{aligned} U(Z)U(W)^* &= (A + ZC)^{-1}(B + ZD)\{(A + WC)^{-1}(B + WD)\}^* \\ &= (A + ZC)^{-1}((A + ZC)(A + WC)^* + ZZ^* - I_r)((A + WC)^{-1})^* \\ &= I_r - (A + ZC)^{-1}(I_r - ZZ^*)((A + WC)^{-1})^*, \end{aligned}$$

$$(3.7) \quad \det(I_r - U(Z)U(W)^*) = \frac{\det(I_r - ZW^*)}{\det(A + ZC) \overline{\det(A + WC)}}.$$

(3) Note that for $Z = (z_1, \dots, z_n)$, $S_n^{IV}(Z) = -\frac{1}{2}Q(\tilde{\xi}^{IV}(Z))$, where $Q(x_1, \dots, x_{n+2}) = |x_1|^2 + \dots + |x_n|^2 - |x_{n+1}|^2 - |x_{n+2}|^2$ and $\tilde{\xi}^{IV}(Z) = (-2iZ, 1 + ZZ^t, i(1 - ZZ^t))$. Then

$$\begin{aligned} S_n^{IV}(g_M(Z)) &= -\frac{1}{2}Q\left(\tilde{\xi}^{IV}(\xi^{IV^{-1}}(\xi^{IV}(Z)M))\right) \\ &= -\frac{1}{2}Q\left(\frac{-2iZA + Z'C}{(-2iZB + Z'D)(i, 1)^t}, \frac{-2iZB + Z'D}{(-2iZB + Z'D)(i, 1)^t}\right) \\ &= -\frac{1}{2} \frac{S_n^{IV}(Z)}{|(-2iZB + Z'D)(i, 1)^t|^2}. \end{aligned} \quad \square$$

Remark 3.6. In the case of Ω_n^{II} , since for $Z \in ASM_{n,n}^C$, $C^t A + C^t Z C$ is anti-symmetric, this implies that

$$\det(A + ZC) = \frac{\det(C^t A + C^t Z C)}{\det(C)} = \frac{(Pf(C^t A + C^t Z C))^2}{\det(C)},$$

where $Pf(Y)$ is the Pfaffian polynomial of a matrix Y , and hence F_U is a rational function.

Proof of Theorem 1.1. Suppose that $g \circ U = V \circ f$ for some $V = \begin{pmatrix} V_1 & V_2 \\ V_3 & V_4 \end{pmatrix} \in \text{Aut}(\Omega_2)$ and $U = \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix} \in \text{Aut}(\Omega_1)$. Then since $F_{g \circ U} = F_{V \circ f}$, by Lemma 3.4,

$$F_g(U(Z), U(W))F_U(Z, W) = F_V(f(Z), f(W))F_f(Z, W).$$

By multiplying $S^1(Z, W)$ to both sides, we obtain

$$(3.8) \quad S^2(g \circ U(Z), g \circ U(W)) = F_V(f(Z), f(W))S^1(f(Z), f(W)).$$

For simplicity, we only consider when Ω_1 and Ω_2 are bounded symmetric domains of type I. Equation (3.8) is

$$(3.9) \quad \det(I_{r'} - f(Z)f(W)^*) = \det(I_{r'} - g \circ U(Z)(g \circ U(W))^*) \frac{\det(V_1 + f(Z)V_3)}{\det(V_1 + f(W)V_3)}.$$

Put $W = 0$ in (3.9). Then

$$(3.10) \quad 1 = \det(I_{r'} - g \circ U(Z)(g \circ U(0))^*) \det(V_1 + f(Z)V_3) \overline{\det(V_1)}.$$

If $U_3 = 0$, then by $g(U_1^{-1}ZU_4) = V \circ f(Z)$, we obtain $0 = g(0) = V \circ f(0) = V_1^{-1}V_2$, and hence $V_2 = 0$. So assume $U_3 \neq 0$. Note that in this case, $\det(U_1 + ZU_3)$ is not a constant. Suppose that $g \circ U(0) \neq 0$. Then $\det(I_{r'} - g \circ U(Z)(g \circ U(0))^*)$ is not a constant, and hence it is p/q , where p and q are non-constant polynomials without common factors and $q = \det(U_1 + ZU_3)^l$ for some positive integer l . But since the product of p/q and polynomial cannot be a constant, (3.10) induces a contradiction. Hence $g \circ U(0)$ should be zero. This implies that $0 = g(U_1^{-1}U_2) = V \circ f(0) = V(0) = V_1^{-1}V_2$. Hence $V_2 = 0$ (and also $V_3 = 0$) and hence $1 = \det(V_1 + f(Z)V_3)$. Put this into (3.9). Then we obtain

$$(3.11) \quad \det(I_{r'} - f(Z)f(W)^*) = \det(I_{r'} - g \circ U(Z)(g \circ U(W))^*).$$

Since the right-hand side of (3.11) is singular on $\{Z \in \Omega_{r,s} : \det(U_1 + ZU_3) = 0\}$, U_3 should be zero. $\square$

4. APPLICATION

In this section we suggest examples which are one-parameter families of inequivalent proper holomorphic maps between bounded symmetric domains of classical type. We use Theorem 1.1 to prove that proper holomorphic maps $f_t : \Omega_1 \rightarrow \Omega_2$ are inequivalent for each $0 \leq t \leq 1$. Every example in this section is obtained in [21]. As in [6], we define the following equivalence relation:

Let Ω_1, Ω_2 be domains. Consider a continuous map $H : \Omega_1 \times [0, 1] \rightarrow \Omega_2$. Denote $H_t(z) = H(z, t)$. Suppose that $H_t : \Omega_1 \rightarrow \Omega_2$ is holomorphic for each $t \in [0, 1]$. Then we will say that H_t is a *continuous family* of holomorphic maps from Ω_1 to Ω_2 .

Definition 4.1. Let $f : \Omega_1 \rightarrow \Omega_2$ and $g : \Omega_1 \rightarrow \Omega_3$ be proper holomorphic maps. Then f and g are *homotopic in the target domain* Ω if for each $t \in [0, 1]$, there is a proper holomorphic maps $H_t : \Omega_1 \rightarrow \Omega$ such that

- there are totally geodesic embeddings $e_k : \Omega_k \rightarrow \Omega$ for $k = 2, 3$, with respect to their Bergman metrics,
- $H_0 = e_2 \circ f$ and $H_1 = e_3 \circ g$,
- H_t is a continuous family of holomorphic maps from Ω_1 to Ω .

4.1. One-parameter family of proper holomorphic maps among $\Omega_{r,s}^I$. Consider proper holomorphic maps $f, g : \Omega_{2,2}^I \rightarrow \Omega_{3,3}^I$, which are defined by

$$f \left(\begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \right) = \begin{pmatrix} z_1^2 & z_1 z_2 & z_2 \\ z_1 z_3 & z_2 z_3 & z_4 \\ z_3 & z_4 & 0 \end{pmatrix}, \text{ for } \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \in \Omega_{2,2}^I,$$

$$g \left(\begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \right) = \begin{pmatrix} z_1^2 & \sqrt{2} z_1 z_2 & z_2^2 \\ \sqrt{2} z_1 z_3 & z_1 z_4 + z_2 z_3 & \sqrt{2} z_2 z_4 \\ z_3^2 & \sqrt{2} z_3 z_4 & z_4^2 \end{pmatrix}.$$

Let $f_t : \Omega_{2,2}^I \rightarrow \Omega_{4,4}^I$ be proper holomorphic non-standard maps for $0 \leq t \leq 1$ defined by

$$(4.1) \quad \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \mapsto \begin{pmatrix} z_1^2 & \sqrt{2-t} z_1 z_2 & \sqrt{1-t} z_2^2 & \sqrt{t} z_2 \\ \sqrt{2-t} z_1 z_3 & \frac{2(1-t)}{2-t} z_1 z_4 + z_2 z_3 & 2\sqrt{\frac{1-t}{2-t}} z_2 z_4 & \sqrt{\frac{t}{2-t}} z_4 \\ \sqrt{1-t} z_3^2 & 2\sqrt{\frac{1-t}{2-t}} z_3 z_4 & z_4^2 & 0 \\ \sqrt{t} z_3 & \sqrt{\frac{t}{2-t}} z_4 & 0 & 0 \end{pmatrix}.$$

Then it is easily observed that f and g are homotopic in the target domain $\Omega_{4,4}^I$ through f_t .

Corollary 4.2. f_t are inequivalent for different t , $0 \leq t \leq 1$.

Proof. Suppose that $f_t \circ A = B \circ f_s$ for some $A \in U(2, 2)$ and $B \in U(4, 4)$. Without loss of generality, we may assume that $t \neq 0$. Then by Theorem 1.1, $f_t(UZV) = Lf_s(Z)M$ for some $U = \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix} \in U(2)$, $V = \begin{pmatrix} V_1 & V_2 \\ V_3 & V_4 \end{pmatrix} \in U(2)$, and $L = (L_{ij})$, $M = (M_{ij}) \in U(4)$. Denote $f_t = \sum f_{t,j}$ where $f_{t,j}$ is a homogeneous polynomial of degree j . Then $f_{t,j}(UZV) = Lf_{s,j}(Z)M$ for each j . Consider the linear part

$$f_{t,1} = \begin{pmatrix} 0 & 0 & 0 & \sqrt{t} z_2 \\ 0 & 0 & 0 & \sqrt{\frac{t}{2-t}} z_4 \\ 0 & 0 & 0 & 0 \\ \sqrt{t} z_3 & \sqrt{\frac{t}{2-t}} z_4 & 0 & 0 \end{pmatrix}.$$

At first, at $Z_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$,

$$f_{t,1}(UZ_1V) = f_{t,1} \begin{pmatrix} U_1V_1 & U_1V_2 \\ U_3V_1 & U_3V_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \sqrt{t}U_1V_2 \\ 0 & 0 & 0 & \sqrt{\frac{t}{2-t}}U_3V_2 \\ 0 & 0 & 0 & 0 \\ \sqrt{t}U_3V_1 & \sqrt{\frac{t}{2-t}}U_3V_2 & 0 & 0 \end{pmatrix}$$

and $Lf_{s,1}(Z_1)M = 0$. This implies that $U_1V_2 = U_3V_2 = U_3V_1 = U_3V_2 = 0$. Suppose that $U_3 \neq 0$. Then $V_1 = V_2 = 0$. This is a contradiction to $V \in U(2)$. Hence $U_3 = 0$. This implies that $U_2 = 0$ and $U_1 \neq 0$, hence $V_2 = V_3 = 0$.

Second, at $Z_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$,

$$f_{t,1}(UZ_2V) = f_{t,1} \begin{pmatrix} 0 & U_1V_4 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \sqrt{t}U_1V_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$Lf_{s,1}(Z_2)M = \sqrt{s} \begin{pmatrix} L_{11}(M_{41}, M_{42}, M_{43}, M_{44}) \\ L_{21}(M_{41}, M_{42}, M_{43}, M_{44}) \\ L_{31}(M_{41}, M_{42}, M_{43}, M_{44}) \\ L_{41}(M_{41}, M_{42}, M_{43}, M_{44}) \end{pmatrix}.$$

This implies that $L_{21} = L_{31} = L_{41} = 0$, $L_{11} \neq 0$, $M_{41} = M_{42} = M_{43} = 0$, and $\sqrt{t}U_1V_4 = \sqrt{s}L_{11}M_{44}$. Since L is unitary, $L_{12} = L_{13} = L_{14} = 0$. Hence by taking square norm on $\sqrt{t}U_1V_4 = \sqrt{s}L_{11}M_{44}$, we obtain $t = s$. $\square$

Remark 4.3. In [21], there are generalizations of f_0, f_1 to $\tilde{f}_0, \tilde{f}_1$ as proper holomorphic maps $\tilde{f}_0 : \Omega_{r,s}^I \rightarrow \Omega_{2r-1,2s-1}^I$ and $\tilde{f}_1 : \Omega_{r,s}^I \rightarrow \Omega_{\frac{1}{2}r(r+1), \frac{1}{2}s(s+1)}^I$. Similar to (4.1), we can make homotopy $\tilde{f}_t$ of $\tilde{f}_0$ and $\tilde{f}_1$ which are inequivalent for all t , $0 \leq t \leq 1$. That is, $\tilde{f}_0$ and $\tilde{f}_1$ are homotopic in the target domain $\Omega_{r',s'}^I$ with $r' = \frac{1}{2}(r^2 + r + 2s - 2)$ and $s' = \frac{1}{2}(s^2 + s + 2r - 2)$.

Proof of Theorem 1.2. The same method of the proof in Corollary 4.2 can be applied. We omit the proof. $\square$

4.2. One-parameter family of proper holomorphic maps among Ω_n^{III} . Consider the proper holomorphic maps $h_0 : \Omega_2^{III} \rightarrow \Omega_3^{III}$ and $h_1 : \Omega_2^{III} \rightarrow \Omega_3^{III}$ defined by

$$h_0 \left(\begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} \right) = \begin{pmatrix} z_1^2 & \sqrt{2}z_1z_2 & z_2^2 \\ \sqrt{2}z_1z_2 & z_1z_2 + z_2^2 & \sqrt{2}z_2z_3 \\ z_2^2 & \sqrt{2}z_2z_3 & z_3^2 \end{pmatrix},$$

$$h_1 \left(\begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} \right) = \begin{pmatrix} z_1^2 & z_1z_2 & z_2 \\ z_1z_2 & z_2^2 & z_3 \\ z_2 & z_3 & 0 \end{pmatrix}, \text{ for } \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} \in \Omega_{2,2},$$

which are the restrictions of f_0 and f_1 to Ω_2^{III} . These are homotopically equivalent in the target domain Ω_4^{III} by $h_t : \Omega_2^{III} \rightarrow \Omega_4^{III}$ which are proper holomorphic maps

given by $h_t(Z) = f_t|_{\Omega_2^{III}}(Z)$. Explicitly,
(4.2)

$$h_t : \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} \mapsto \begin{pmatrix} z_1^2 & \sqrt{2-t}z_1z_2 & \sqrt{1-t}z_2^2 & \sqrt{t}z_2 \\ \sqrt{2-t}z_1z_2 & \frac{2(1-t)}{2-t}z_1z_3 + z_2^2 & 2\sqrt{\frac{1-t}{2-t}}z_2z_3 & \sqrt{\frac{t}{2-t}}z_3 \\ \sqrt{1-t}z_2^2 & 2\sqrt{\frac{1-t}{2-t}}z_2z_3 & z_3^2 & 0 \\ \sqrt{t}z_2 & \sqrt{\frac{t}{2-t}}z_3 & 0 & 0 \end{pmatrix}.$$

By applying a similar method to h_t and a higher version of h_t , we obtain the following:

Corollary 4.4. *There are uncountably many inequivalent proper holomorphic maps from Ω_n^{III} to $\Omega_{n'}^{III}$ with $n' = \frac{1}{2}(n^2 + 3n - 2)$.*

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