PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 144, Number 2, February 2016, Pages 487–502
http://dx.doi.org/10.1090/proc/12765
Article electronically published on June 10, 2015

CRITICAL VALUES OF GAUSSIAN $SU(2)$ RANDOM POLYNOMIALS

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(Communicated by Walter Van Assche)

ABSTRACT. In this article we will get the estimate of the expected distribution of critical values of Gaussian $SU(2)$ random polynomials as the degree is large enough. The result about the expected density is a direct application of the Kac-Rice formula. The critical values will accumulate at infinity, then we will study the rate of this convergence and its rescaling limit as $n \to \infty$.

1. Introduction

Random polynomials and random holomorphic functions are studied as ways to gain insight for problems arising in string theory and analytic number theory [5, 11, 16]. In [14] Kac studied and determined a formula for the expected distribution of zeros of some real Gaussian random polynomials. His work was generalized to complex random polynomials and random analytic functions throughout the years; we refer to [3, 4, 6, 13, 17] for more background and results.

1.1. $SU(2)$ polynomials.

When the random polynomial is defined invariant with respect to some group action, the problem can turn out to be particularly interesting; we refer §2.3 in [13] for examples. In this article we will study a special family: the Gaussian $SU(2)$ random polynomials. This is of particular interest in the physics literature as the zeros describe a random spin state for the Majorana representation (modulo phase) on the unit sphere [11].

Given a probability space Ω and $\{a_j\}_{j=0}^{\infty}$, a collection of i.i.d complex random variables with density $\frac{1}{\pi}e^{-|z|^2}$ on it, the family of $SU(2)$ random polynomials is defined as

$$p_n(z) = \sum_{j=0}^{n} a_j \sqrt{\binom{n}{j}} z^j. \tag{1}$$

Although this polynomial is defined on $\mathbb{C}$, we may also view it as an analytic function on $\mathbb{CP}^1 = \mathbb{C} \cup \infty$ with a pole at ∞.

Various properties of the zeros of random $SU(2)$ polynomials have been studied such as the distribution of zeros and the two points correlation function [3, 11]. First, zeros of these polynomials are uniformly distributed on $S^2 \cong \mathbb{CP}^1$ with respect to the Fubini-Study metric, i.e., the average distribution of zeros is invariant under

Received by the editors July 26, 2014 and, in revised form, January 12, 2015.
2010 *Mathematics Subject Classification.* Primary 12E10.

the $SU(2)$ action on $\mathbb{CP}^1$ [13]. To be more precise, let us denote

$$\mathcal{Z}_{p_n} = \sum_{z\in\mathbb{CP}^1:\ p_n(z)=0} \delta_z$$

as the empirical measure of zeros of Gaussian $SU(2)$ random polynomials and define the pairing

$$\langle \mathcal{Z}_{p_n}, \phi\rangle = \sum_{z\in\mathbb{CP}^1:\ p_n(z)=0} \phi(z), \quad \text{where } \phi \in C^\infty(\mathbb{CP}^1).$$

We define the expectation

$$\langle \mathbb{E}\mathcal{Z}_{p_n}, \phi\rangle := \mathbb{E}\langle \mathcal{Z}_{p_n}, \phi\rangle = \frac{1}{\pi^{n+1}} \int_{\mathbb{C}^{n+1}} \left(\sum_{z\in\mathbb{CP}^1:\ p_n(z)=0} \phi(z)\right) e^{-|a|^2} d\ell_{a_0}\cdots d\ell_{a_n},$$

where $d\ell_{a_j} = \frac{1}{2i} da_j \wedge d\bar{a}_j$ is the Lebesgue measure on $\mathbb{C}$.

Then the expected density of zeros is calculated in [3] as

$$\mathbb{E}\mathcal{Z}_{p_n} = n\omega_{FS},$$

in the sense that

$$\mathbb{E}\langle \mathcal{Z}_{p_n}, \phi\rangle = n\int_{\mathbb{CP}^1} \phi\omega_{FS}, \quad \text{where } \phi \in C^\infty(\mathbb{CP}^1),$$

where ω_{FS} is the Fubini-Study form on $\mathbb{CP}^1$ [10].

We can also study the two points correlation function of zeros of $SU(2)$ polynomials and its scaling property. We define the two points correlation function as [3]

$$K_n(z,w) := \mathbb{E}\left(Z_{p_n}(z)\otimes Z_{p_n}(w)\right),$$

such that for any smooth test function $\phi_1(z)\otimes\phi_2(w)$, we have the pairing

$$\langle K_n(z,w), \phi_1(z)\otimes\phi_2(w)\rangle = \mathbb{E}\left(\langle \mathcal{Z}_{p_n}, \phi_1\rangle\right)\left(\langle \mathcal{Z}_{p_n}, \phi_2\rangle\right).$$

If we scale the two points correlation function by a factor $\frac{1}{\sqrt{n}}$, then we have

$$K_n(\frac{z}{\sqrt{n}}, \frac{w}{\sqrt{n}}) = \frac{(\sinh t^2 + t^2)\cosh t - 2t\sinh t}{\sinh t^3} + O(\frac{1}{\sqrt{n}}),$$

where $t = \frac{|z-w|^2}{2}$ and $|z-w|$ is the geodesic distance of z and w on $\mathbb{CP}^1$. It is easy to see that

$$K_n(\frac{z}{\sqrt{n}}, \frac{w}{\sqrt{n}}) = t - \frac{2}{9}t^3 + O(t^5) \text{ as } t\to 0,$$

which implies zeros repel each other. We refer to [3, 11] for more details.

1.2. **Main results.** In this article we will study the expected distribution of nonvanishing critical values of $|p_n|$ as n tends to infinity.

Note that the modulus $|p_n|$ is a subharmonic function, thus there is no local maximum; local minimums are all zeros and thus nonvanishing critical values are obtained only at saddlepoints [7]. Hence, the expected density of nonvanishing critical values of $|p_n|$ that we study in this article are in fact the expected density of values of saddlepoints of $|p_n|$.

The nonvanishing critical values of $|p_n|$ are obtained at points

$$\{z\in\mathbb{C}:\ p_n' = 0 \text{ and } p_n \neq 0\}. \tag{2}$$

A random polynomial p_n has no repeated zeros almost surely, which implies that the set (2) is almost surely equivalent to

$$\{z \in \mathbb{C} : \ p_n' = 0\}; \tag{3}$$

i.e., (nonvanishing) $|p_n|$ and p_n have the same critical points almost surely.

Hence, we will first get the expected density of critical values of p_n in Theorem 1. As a direct consequence, we can apply the polar coordinate to get the expected density of nonvanishing critical values of $|p_n|$ in Theorem 2.

We denote the empirical measure of critical values of p_n as

$$\mathcal{C}_{p_n} = \sum_{z:\ p_n'(z)=0} \delta_{p_n(z)}. \tag{4}$$

We now define the pairing

$$\langle \mathcal{C}_{p_n}, \phi \rangle = \sum_{z:\ p_n'(z)=0} \phi(p_n(z)), \quad \forall \phi(x) \in C_c^\infty(\mathbb{R}^2), \tag{5}$$

where $C_c^\infty(\mathbb{R}^2)$ is the space of smooth functions on $\mathbb{R}^2$ with compact support.

We denote $\mathbb{D}_{p_n}(x)$ as the expected density of critical values of p_n in the sense that

$$\mathbb{E}\langle \mathcal{C}_{p_n}, \phi \rangle = \int_{\mathbb{C}} \phi(x) \mathbb{D}_{p_n}(x) d\ell_x, \quad \forall \phi(x) \in C_c^\infty(\mathbb{R}^2), \tag{6}$$

whereas $d\ell_x$ is the Lebesgue measure of $\mathbb{C}$.

Those definitions also apply to the empirical measure of the nonvanishing critical values of $|p_n|$ which is

$$\mathcal{C}_{|p_n|} = \sum_{z:\ p_n'=0} \delta_{|p_n|}, \tag{7}$$

which is a measure defined on the nonnegative real line $\mathbb{R}_+$.

We define its expectation as

$$\langle \mathbb{E}\mathcal{C}_{|p_n|}, \phi \rangle := \mathbb{E}\langle \mathcal{C}_{|p_n|}, \phi \rangle = \int_0^\infty \phi(x) \mathbb{D}_{|p_n|} dx, \ \forall \ \phi(x) \in C_c^\infty(\mathbb{R}_+), \tag{8}$$

where dx is the Lebesgue measure on $\mathbb{R}$.

In this article, we will first get the exact formula for the expected density $\mathbb{D}_{p_n}$ in Proposition 1 by the Kac-Rice formula (see section §2), then we study the asymptotic behavior of $\mathbb{D}_{p_n}$ as $n \to \infty$. Our main results follow.

Theorem 1. *The expected density $\mathbb{D}_{p_n}$ of the empirical measure $\mathcal{C}_{p_n}$ of the critical values of p_n satisfies the estimate*

$$\mathbb{D}_{p_n} = \frac{1 - e^{-|x|^2}}{\pi |x|^2} + \frac{1}{\pi} \int_0^1 e^{-(s - s\log s)|x|^2} ds + o(1) \ \text{ as } \ n \to \infty, \tag{9}$$

for any $x \in \mathbb{C}$.

As proved in Proposition 1, the density $\mathbb{D}_{p_n} d\ell_x$ only depends on $|x|$, i.e., the modulus of $|p_n|$, thus we can rewrite it as $\mathbb{D}_{p_n}(|x|)|x|d|x|d\theta$ under the polar coordinate. If we integrate on θ variable, then

$$\mathbb{D}_{|p_n|} = \int_0^{2\pi} \mathbb{D}_{p_n}(|x|)|x| d\theta = 2\pi |x| \mathbb{D}_{p_n}(|x|) \tag{10}$$

will be the density of critical values of $|p_n|$. Thus we get

Theorem 2. *The expected density* $\mathbb{D}_{|p_n|}$ *of the empirical measure* $\mathcal{C}_{|p_n|}$ *of the nonvanishing critical values of* $|p_n|$ *satisfies the estimate*

$$\mathbb{D}_{|p_n|}(x) = \frac{2(1-e^{-x^2})}{x} + 2x\int_0^1 e^{-(s-s\log s)x^2}ds + o(1) \quad as \ n\to\infty, \tag{11}$$

for any $x \in \mathbb{R}_+$.

The decay of $\mathbb{D}_{|p_n|}(x)$ is of order $1/x$ as x goes to infinity, thus the total mass on the interval $[a,\infty)$ is infinity for any $a>0$, i.e., the critical values will accumulate at infinity as $n\to\infty$. In order to study the rate of this accumulation, we consider the distribution function $F_n(x)$ of the probability density

$$\frac{\mathbb{D}_{|p_n|}}{n-1} = \frac{1}{n-1}\mathbb{E}(\sum_{p_n'=0}\delta_{|p_n|}).$$

Next, we will show that the critical values are spreading out exponentially.

Theorem 3. *For any fixed* $\epsilon>0$, $F_n(e^{n^{\frac{1-\epsilon}{2}}})\to 0$ *and* $F_n(e^{n^{\frac{1+\epsilon}{2}}})\to 1$ *as* $n\to\infty$.

Then the modulus of critical values of p_n will mainly concentrate in the interval $[e^{n^{\frac{1-\epsilon}{2}}}, e^{n^{\frac{1+\epsilon}{2}}}]$ as n large enough. Thus we need to consider the following rescaled probability density to get more information about this convergence

$$R_n(x) = (F_n(e^{\frac{nx}{2}}))'.$$

Then we prove that $R_n(x)$ satisfies the rescaled limit

Theorem 4.

$$\lim_{n\to\infty} R_n(x) = \begin{cases} e^{-x} & \text{if } x>0,\\ \lim_{n\to\infty}\mathbb{D}_{p_n}(1) & \text{if } x=0,\\ 0 & \text{if } x<0,\end{cases}$$

where $\lim_{n\to\infty}\mathbb{D}_{p_n}(1)$ *is the constant given by the leading term in* (9) *evaluated at* 1.

1.3. **Further remarks.** First note that our setting is different from the one in [5]. For example, in [5], critical points of $SU(2)$ polynomials are defined to be the points

$$\{z\in\mathbb{CP}^1:\ \nabla' p_n = 0\},$$

where $\nabla' = \frac{\partial}{\partial z} - \frac{n\bar{z}dz}{1+|z|^2}$ is the smooth Chern connection on the line bundle $\mathcal{O}(n)\to\mathbb{CP}^1$ with respect to the Fubini -Study metric and p_n is a global holomorphic section of the line bundle $\mathcal{O}(n)\to\mathbb{CP}^1$ [10]. By choosing such a smooth Chern connection, the expected distribution of critical points is also invariant under the $SU(2)$ action [5]. But in this article, the critical points are defined by the usual derivative

$$\{z\in\mathbb{C}:\ \frac{\partial p_n}{\partial z} = 0\}.$$

In fact, the derivative $\frac{\partial}{\partial z}$ is a meromorphic flat Chern connection on $\mathcal{O}(n)\to\mathbb{CP}^1$ with a pole at ∞. Under this setting, the expected density of critical points is not $SU(2)$ invariant; we refer to [12] for more details.

Our second remark is as following. In [8] and [9] the authors studied the expected density of nonvanishing critical values of the pointwise norm of Gaussian random holomorphic sections of the positive holomorphic line bundle over compact Kähler manifolds. Now let us briefly explain the main result in [8] and [9] and compare it with Theorem 2. Take Gaussian $SU(2)$ random polynomials (sections) p_n for example. We equip the line bundle $\mathcal{O}(n) \to \mathbb{CP}^1$ with a Hermitian metric $h^n = e^{-n\phi}$, where $\phi = \log(1+|z|^2)$ is the Kähler potential of the Fubini-Study metric. Then the pointwise h-norm of the holomorphic section $|p_n|_{h^n} = |p_n|e^{-\frac{n\phi}{2}}$ is globally defined on $\mathbb{CP}^1$ [10], and hence the critical points of $|p_n|_{h^n}$ are defined as

$$\Sigma_n = \{z \in \mathbb{CP}^1 : \ \frac{\partial |p_n|_{h^n}}{\partial z} = 0\}.$$

We define the (normalized) empirical measure of critical values of $|p_n|_{h^n}$ as

$$\mathcal{C}_{|p_n|_{h^n}} := \frac{1}{n}\left(\sum_{z\in\Sigma_n} \delta_{|p_n|_{h^n}}\right),$$

which is also a measure defined on $\mathbb{R}_+$.

Then the expectation of $\mathcal{C}_{|p_n|_{h^n}}$ satisfies the estimate

$$\mathbb{E}\mathcal{C}_{|p_n|_{h^n}} = x\left(2x^2 - 4 + 8e^{-\frac{x^2}{2}}\right)e^{-x^2} + O(\frac{1}{n}), \quad x \in \mathbb{R}_+, \tag{12}$$

as n large enough. In fact, this estimate is universal: it holds on any Riemannian surfaces [8], [9].

Thus, the (normalized) density $\mathbb{E}\mathcal{C}_{|p_n|_{h^n}}$ is decaying exponentially as x is large enough, which is quite different from the behavior of (nonnormalized) density $\mathbb{E}\mathcal{C}_{|p_n|}$ in Theorem 2. This is mainly because of the connection we choose: the usual derivative $\frac{\partial}{dz}$ in this article is a meromorphic flat connection on $\mathbb{CP}^1$ with a pole at ∞, while in [8] and [9] the proof of (12) relies on a choice of smooth Chern connection $\nabla' = \frac{\partial}{\partial z} - \frac{n\bar{z}dz}{1+|z|^2}$.

2. Kac-Rice formula

In this section, we first review the Kac-Rice formula for a stochastic process, referring to [1, 14, 15] for more details. Then we generalize the formula to the expected distribution of critical values of p_n.

The Kac-Rice formula is as follows: let $f(z)$ be a real valued stochastic process indexed by a compact interval $I \subset \mathbb{R}$. Then the Kac-Rice formula for the expected number of zeros is

$$\mathbb{E}\#\{z \in I : \ f(z) = 0\} = \int_I \int_{\mathbb{R}} |y| p_z(0, y) dy dz,$$

where $p_z(0, y)$ is the joint density $p_z(x, y)$ of (f, f') evaluated at $(0, y)$. If f is a Gaussian process, then the joint density $p_z(x, y)$ is determined by the covariance matrix of (f, f') [1].

The proof of this formula is explained in more detail in [1]. The idea of the proof is based on the observation that

$$\#\{z \in I : \ f(z) = 0\} = \int_I \delta_0(f(z))|f'(z)| dz.$$

We take expectation on both sides to get

$$\begin{aligned}\mathbb{E}\#\{z\in I:\ f(z)=0\} &= \int_I\int_{\mathbb{R}_y}\int_{\mathbb{R}_x}\delta_0(x)p_z(x,y)|y|dxdydz\\ &= \int_I\int_{\mathbb{R}}|y|p_z(0,y)dydz.\end{aligned}$$

Thus the expected density of zeros of f is given by

$$\mathbb{E}\left(\sum_{z\in I:\ f(z)=0}\delta_z\right)=\left(\int_{\mathbb{R}}|y|p_z(0,y)dy\right)dz. \tag{13}$$

If $f(z)$ is a complex stochastic process indexed by a compact complex domain, the above formula reads

$$\mathbb{E}\left(\sum_{z\in I:\ f(z)=0}\delta_z\right)=\left(\int_{\mathbb{C}}|y|^2p_z(0,y)d\ell_y\right)d\ell_z, \tag{14}$$

where $d\ell_y$ and $d\ell_z$ are Lebesgue measures on $\mathbb{C}$. Compared with (13), we get $|y|^2$ since a one-dimensional complex random process is a two-dimensional real random process. In fact, this formula is based on the definition of the delta function and the identity

$$\#\{z\in I:\ f(z)=0\}=\int_I\delta_0(f(z))\frac{1}{2i}df\wedge d\bar{f}=\int_I\delta_0(f(z))|f'|^2d\ell_z.$$

The formula arises when we take expectation on both sides.

2.1. **Kac-Rice formula: Revisited.** In this subsection, let us get the formula for the expected density of critical values of a (real or complex) stochastic process f by the method of Kac and Rice.

For simplicity, let us first consider a smooth real Gaussian process $f\in C^\infty(I)$, where I is a compact subset in $\mathbb{R}$.

Let $\Theta\subset\mathbb{R}$ be a compact subset. Let us denote the set of critical values in Θ as

$$\mathcal{C}_\Theta=\{z\in I:\ f(z)\in\Theta,\ \ f'(z)=0\}.$$

Let us denote the measure $\mu(x)dx$ on Θ as

$$\mu(x)dx=\mathbb{E}\left(\sum_{z\in\mathcal{C}_\Theta}\delta_{f(z)}\right),$$

in the sense that

$$\mathbb{E}\left(\langle\sum_{z\in\mathcal{C}_\Theta}\delta_{f(z)},\phi\rangle\right)=\int_\Theta\phi\mu(x)dx,$$

where ϕ is any smooth test function defined on Θ.

Then we have the following lemma.

Lemma 1. *Let us denote $p_z(x,y,\xi)$ as the joint density of (f,f',f'') at z. Then*

$$\mu(x)dx=\left(\int_I\int_{\mathbb{R}}|\xi|p_z(x,0,\xi)d\xi dz\right)dx,$$

where dx, $d\xi$, and dz are Lebesgue measures on $\mathbb{R}$.

Proof. We will first do a formal calculation:

$$\langle \sum_{f\in\Theta,\ f'=0} \delta_{f(z)}, \phi(x)\rangle = \sum_{f\in\Theta,\ f'=0} \phi(f(z)) = \int_I \chi_{\{f\in\Theta\}}\phi(f(z))\delta(f')df'. \tag{15}$$

By taking expectation on both sides and considering $df' = f''dz$, we have

$$\mathbb{E}\left\langle \sum_{f\in\Theta,\ f'=0} \delta_{f(z)}, \phi(x)\right\rangle = \int_{\mathbb{R}_x}\int_I\int_{\mathbb{R}_\xi}\int_{\mathbb{R}_y} p_z(x,y,\xi)\chi_{\{x\in\Theta\}}\phi(x)\delta(y)|\xi|dyd\xi dzdx \tag{16}$$

$$= \int_\Theta \left(\int_I\int_{\mathbb{R}_\xi} p_z(x,0,\xi)|\xi|d\xi dz\right)\phi(x)dx \tag{17}$$

$$= \int_\Theta \phi(x)\mu(x)dx. \tag{18}$$

This calculation requires justification in (15), (16), and (17). The way to rigorously do that is to approximate the δ function by a sequence of simple functions and do a verbatim repetition of the proof in [1, Theorem 11.2.3,Corollary 11.2.4].

From (18) to the conclusion, one needs to prove that the density on both left and right hand sides are continuous. To prove the continuity of μ, it is again repeating the argument in [1, Theorem 11.2.3, Corollary 11.2.4]; see [1, Section 11.4] and [2] for details. $\square$

In the proof of Lemma 1 we have assumed I and Θ are compact subsets in $\mathbb{R}$. But the proof of Lemma 1 can be generalized to the $SU(2)$ random polynomials p_n, which are a collection of complex Gaussian stochastic processes indexed by $\mathbb{C}$.

The generalization of Θ to be $\mathbb{C}$ only requires picking up a sequence of discs centered at the origin with radius $m \in \{1, 2, \ldots\}$ and taking limit in weak sense. And the generalization from I to $\mathbb{C}$ is the same.

However, we do need to modify the pairing by choosing the test functions $\phi(z)$ in the smooth compact supported space $\mathcal{C}_c^\infty(\mathbb{R}^2)$ in order to change the order of the integration on $\mathbb{C}$. Following the proof of Lemma 1, we have

Lemma 2. *The expected density of critical values of p_n is*

$$\mathbb{D}_{p_n} d\ell_x = \left(\int_{\mathbb{C}}\int_{\mathbb{C}} |\xi|^2 p_z(x,0,\xi)d\ell_\xi d\ell_z\right)d\ell_x, \tag{19}$$

where $d\ell_x, d\ell_\xi$, and $d\ell_z$ are Lebesgue measures on $\mathbb{C}$ and

$$p_z(x,0,\xi) = \frac{1}{\pi^3 \det \Delta_z} \exp\left\{-\left\langle \begin{pmatrix} x \\ 0 \\ \xi\end{pmatrix}, \Delta_z^{-1}\begin{pmatrix} \bar{x} \\ 0 \\ \bar{\xi}\end{pmatrix}\right\rangle\right\} \tag{20}$$

is the joint density of (p_n, p_n', p_n''), where Δ_z is the covariance matrix of (p_n, p_n', p_n'').

Proof. The proof of this formula is the same as the one in Lemma 1.

We start with a disk U in place of I, take $\Theta \subset \mathbb{C}$ compact, and write $\mathcal{C}_\Theta = \{z \in U : \ f(z) \in \Theta,\ \ f'(z) = 0\}$ again. Then we have

$$\langle \sum_{f\in\Theta,\ f'=0} \delta_{f(z)}, \phi(x)\rangle = \sum_{f\in\Theta,\ f'=0} \phi(f(z)) = \int_U \chi_{\{f\in\Theta\}}\phi(f(z))\delta(f')(\frac{1}{2i}df' \wedge d\bar{f}'). \tag{21}$$

Taking expectation on both left and right hand sides and noting $df' \wedge d\bar{f}' = |f''|^2 dz \wedge d\bar{z} = |f''| d\ell_z$, we have

$$\begin{aligned}\mathbb{E}\left\langle \sum_{f\in\Theta,\ f'=0} \delta_{f(z)}, \phi(x)\right\rangle &= \int_{\mathbb{C}_x}\int_U\int_{\mathbb{C}_\xi}\int_{\mathbb{C}_y} p_z(x,y,\xi)\chi_{\{x\in\Theta\}}\phi(x)\delta(y)|\xi|^2 d\ell_y d\ell_\xi d\ell_z d\ell_x \\ &= \int_\Theta \left(\int_U\int_{\mathbb{C}_\xi} p_z(x,0,\xi)|\xi|^2 d\ell_\xi d\ell_z\right)\phi(x)d\ell_x \\ &= \int_\Theta \phi(x)\mu(x)d\ell_x.\end{aligned}$$

The justification process here is the same as in Lemma 1.

The lemma follows if we replace $f(z)$ by $p_n(z)$. Since p_n is a Gaussian process, the joint density $p_z(x,y,\xi)$ is uniquely determined by the covariance matrix of (p_n, p_n', p_n''). □

3. Proof of main theorems

3.1. The density $\mathbb{D}_{p_n}$.

In this subsection, we will derive the exact formula for $\mathbb{D}_{p_n}$ based on Lemma 2. We prove

Proposition 1. *The expected density of the empirical measure of $\mathcal{C}_{p_n}$ is given by the formula*

$$\mathbb{D}_{p_n} = \frac{n-1}{\pi}\int_1^\infty \frac{n(r-1)+1}{r^{n+2}} e^{-\frac{n(r-1)+1}{r^n}|x|^2} dr. \tag{22}$$

Thus $\mathbb{D}_{p_n}$ is a function only depending on $|x|$.

Proof. By Lemma 2 in order to compute the expected density of critical values of p_n, we first need to compute the covariance matrix of (p_n, p_n', p_n'').

By definition, the covariance matrix of the Gaussian process (p_n, p_n', p_n'') is given by [1, 5]

$$\Delta = \begin{pmatrix} \mathbb{E}(p_n\overline{p_n}) & \mathbb{E}(p_n'\overline{p_n}) & \mathbb{E}(p_n''\overline{p_n}), \\ \mathbb{E}(p_n\overline{p_n'}) & \mathbb{E}(p_n'\overline{p_n'}) & \mathbb{E}(p_n''\overline{p_n'}), \\ \mathbb{E}(p_n\overline{p_n''}) & \mathbb{E}(p_n'\overline{p_n''}) & \mathbb{E}(p_n''\overline{p_n''}) \end{pmatrix}.$$

The covariance kernel for the Gaussian process p_n is

$$\mathbb{E}(p_n(z)\overline{p_n}(w)) := \Pi_n(z,w) = (1+z\bar{w})^n.$$

Then we can express each entry in the covariance matrix as follows:

$$
\begin{aligned}
\mathbb{E}(p_n\overline{p_n}) &= \Pi_n(z,z) = (1+|z|^2)^n,\\
\mathbb{E}(p_n'\overline{p_n}) &= \frac{\partial \Pi_n(z,w)}{\partial z}|_{z=w} = n\bar{w}(1+z\bar{w})^{n-1}|_{z=w} = n\bar{z}(1+|z|^2)^{n-1},\\
\mathbb{E}(p_n''\overline{p_n}) &= \frac{\partial^2 \Pi_n(z,w)}{\partial^2 z}|_{z=w} = n(n-1)\bar{w}^2(1+z\bar{w})^{n-2}_{z=w} = n(n-1)\bar{z}^2(1+|z|^2)^{n-2},\\
\mathbb{E}(p_n'\overline{p_n'}) &= \frac{\partial^2 \Pi_n(z,w)}{\partial z\partial \bar{w}}|_{z=w} = n(1+z\bar{w})^{n-2}((n-1)z\bar{w}+1+z\bar{w})|_{z=w}\\
&= n(n|z|^2+1)(1+|z|^2)^{n-2},\\
\mathbb{E}(p_n''\overline{p_n'}) &= \frac{\partial^3 \Pi_n(z,w)}{\partial^2 z\partial \bar{w}}|_{z=w} = n(n-1)(1+z\bar{w})^{n-3}\bar{w}(nz\bar{w}+2)|_{z=w}\\
&= n(n-1)(1+|z|^2)^{n-3}\bar{z}(n|z|^2+2),\\
\mathbb{E}(p_n''\overline{p_n''}) &= \frac{\partial^4 \Pi_n(z,w)}{\partial^2 z\partial^2 \bar{w}}|_{z=w}\\
&= n(n-1)((n-2)(n-3)z^2\bar{w}^2(1+z\bar{w})^{n-4}\\
&\quad + 4(n-2)z\bar{w}(1+z\bar{w})^{n-3} + 2(1+z\bar{w})^{n-2})|_{z=w}\\
&= n(n-1)(1+|z|^2)^{n-4}(n(n-1)|z|^4+4(n-1)|z|^2+2).
\end{aligned}
$$

These show the covariance matrix is

$$
\Delta_z = (1+|z|^2)^n \times \begin{pmatrix} 1 & \frac{n\bar{z}}{1+|z|^2} & \frac{n(n-1)\bar{z}^2}{(1+|z|^2)^2} \\ \frac{nz}{1+|z|^2} & \frac{n+n^2|z|^2}{(1+|z|^2)^2} & \frac{2n(n-1)\bar{z}+(n-1)n^2\bar{z}|z|^2}{(1+|z|^2)^3} \\ \frac{n(n-1)z^2}{(1+|z|^2)^2} & \frac{2n(n-1)z+(n-1)n^2z|z|^2}{(1+|z|^2)^3} & \frac{2n(n-1)+4n(n-1)^2|z|^2+n^2(n-1)^2|z|^4}{(1+|z|^2)^4} \end{pmatrix}.
$$

Hence

$$
\det \Delta_z = (1+|z|^2)^{3n}\frac{2n^3-2n^2}{(1+|z|^2)^6}, \tag{23}
$$

which never degenerates when $n>1$. We denote

$$
Q_z(x,\xi) =: \left\langle \begin{pmatrix} x\\ 0\\ \xi\end{pmatrix}, \Delta_z^{-1}\begin{pmatrix} \bar{x}\\ 0\\ \bar{\xi}\end{pmatrix}\right\rangle.
$$

Then by direct computations, we rewrite.

$$
Q_z(x,\xi) = \frac{(1+|z|^2)^{2n}}{\det\Delta_z} \times \left\langle \begin{pmatrix} x\\ \xi\end{pmatrix}, \begin{pmatrix} \frac{n^5|z|^4+2n^4(|z|^2-|z|^4)+n^3(|z|^4-2|z|^2+2)-2n^2}{(1+|z|^2)^6} & \frac{(n^3-n^2)\bar{z}^2}{(1+|z|^2)^4} \\ \frac{(n^3-n^2)z^2}{(1+|z|^2)^4} & \frac{n}{(1+|z|^2)^2}\end{pmatrix}\begin{pmatrix} \bar{x}\\ \bar{\xi}\end{pmatrix}\right\rangle
$$

(note we only need to calculate the four corner entries of the inverse matrix). We expand this expression and further rewrite $Q_z(x,\xi)$ as

$$
\frac{1}{2(1+|z|^2)^n}\left(\left|\sqrt{n^2-n}\bar{z}^2x + \frac{1}{\sqrt{n^2-n}}\xi(1+|z|^2)^2\right|^2 + 2(n|z|^2+1)|x|^2\right). \tag{24}
$$

By Lemma 2, the expected density of critical values of p_n is given by the formula

$$\mathbb{D}_{p_n}(x) = \frac{1}{\pi^3}\int_{\mathbb{C}}\int_{\mathbb{C}} \frac{e^{-Q_z(x,\xi)}}{\det \Delta_z}|\xi|^2 d\ell_\xi d\ell_z. \tag{25}$$

Let us integrate ξ variable first. Plugging (24) into (25), we can rewrite (25) as

$$\mathbb{D}_{p_n}(x) = \frac{1}{\pi^3}\int_{\mathbb{C}} K_z \frac{e^{-\frac{n|z|^2+1}{(1+|z|^2)^n}|x|^2}}{\det \Delta_z} d\ell_z, \tag{26}$$

where K_z is the following integral in ξ variable:

$$K_z = \int_{\mathbb{C}} \exp\left\{-\frac{1}{2(1+|z|^2)^n}\left|\sqrt{n^2-n}\bar{z}^2 x + \frac{1}{\sqrt{n^2-n}}\xi(1+|z|^2)^2\right|^2\right\}|\xi|^2 d\ell_\xi.$$

We will first make the exponent into a perfect square. We change variables $\xi \to \frac{1}{\sqrt{n^2-n}}\xi(1+|z|^2)^2$ to get

$$K_z = \frac{(n^2-n)^2}{(1+|z|^2)^8}\int_{\mathbb{C}} \exp\left\{-\frac{1}{2(1+|z|^2)^n}\left|\sqrt{n^2-n}x\bar{z}^2+\xi\right|^2\right\}|\xi|^2 d\ell_\xi.$$

Further, we changing variable $\xi \to \sqrt{n^2-n}x\bar{z}^2+\xi$ to get

$$K_z = \frac{(n^2-n)^2}{(1+|z|^2)^8}\int_{\mathbb{C}} \exp\left\{-\frac{|\xi|^2}{2(1+|z|^2)^n}\right\}\left|\xi - \sqrt{n^2-n}x\bar{z}^2\right|^2 d\ell_\xi.$$

This turns into a Gaussian integral. Noting that the first moment terms are equal to zero after expanding the norm square, we have

$$K_z = \pi\frac{(n^2-n)^2}{(1+|z|^2)^8}\left[2(1+|z|^2)^n(n^2-n)|x|^2|z|^4 + 4(1+|z|^2)^{2n}\right].$$

If we change variable $r = 1+|z|^2$, we can rewrite

$$K_z = \pi\frac{(n^2-n)^2}{r^8}\left[2r^n(n^2-n)|x|^2(r-1)^2 + 4r^{2n}\right]$$

and

$$\det \Delta_z = r^{3n-6}(2n^3-2n^2), \quad e^{-\frac{n|z|^2+1}{(1+|z|^2)^n}|x|^2} = e^{-\frac{n(r-1)+1}{r^n}|x|^2}.$$

Now we plug these two lines back into formula (26) and use the polar coordinate $d\ell_z = \frac{1}{2}drd\theta$, integrate on θ variable, we can rewrite $\mathbb{D}_{p_n}$ as

$$\mathbb{D}_{p_n} = \frac{n-1}{\pi}\int_1^\infty \frac{(n^2-n)r^n(r-1)^2|x|^2 + 2r^{2n}}{r^{3n+2}} e^{-\frac{n(r-1)+1}{r^n}|x|^2} dr. \tag{27}$$

There are two parts in the numerator. We integrate by part to simplify the first term in the numerator. Note that

$$de^{-\frac{n(r-1)+1}{r^n}|x|^2} = e^{-\frac{n(r-1)+1}{r^n}|x|^2}[r^{-n-1}(n^2-n)(r-1)|x|^2]dr,$$

then the first part is equal to

$$\begin{aligned}
&\frac{n-1}{\pi}\int_1^\infty \frac{(n^2-n)r^n(r-1)^2|x|^2}{r^{3n+2}} e^{-\frac{n(r-1)+1}{r^n}|x|^2} dr \\
=&\frac{n-1}{\pi}\int_1^\infty \frac{(r-1)}{r^{n+1}} de^{-\frac{n(r-1)+1}{r^n}|x|^2} \\
=&\frac{n-1}{\pi}\int_1^\infty [\frac{n}{r^{n+1}} - \frac{n+1}{r^{n+2}}]e^{-\frac{n(r-1)+1}{r^n}|x|^2} dr.
\end{aligned}$$

Hence, the density (27) is further simplified to be

$$\mathbb{D}_{p_n}(x) = \frac{n-1}{\pi} \int_1^\infty \frac{n(r-1)+1}{r^{n+2}} e^{-\frac{n(r-1)+1}{r^n}|x|^2} dr,$$

which completes the proof. □

3.2. **Proof of Theorem 1.** Now we turn to the proof of our main Theorem 1.

We denote $t = \frac{1}{r}$ and

$$y_n(t) = \frac{n(r-1)+1}{r^n} = nt^{n-1} - (n-1)t^n,$$

then we have $t \in [0,1]$ and $y_n(t) \in [0,1]$ with $y_n(0) = 0$ and $y_n(1) = 1$.

Substituting $\frac{n(r-1)+1}{r^n}$ by $y_n(t)$, we rewrite $\mathbb{D}_{p_n}$ in Proposition 1 as

$$\begin{aligned} \mathbb{D}_{p_n} &= \frac{n-1}{\pi} \int_1^\infty \frac{y_n(t)}{r^2} e^{-y_n(t)|x|^2} dr \\ &= \frac{n-1}{\pi} \int_0^1 y_n(t) e^{-y_n(t)|x|^2} dt, \end{aligned} \tag{28}$$

where in the last step we change variable $t \to \frac{1}{r}$.

The trick to estimate $\mathbb{D}_{p_n}$ is to calculate

$$g_n(|x|^2) := \int_0^1 e^{-y_n(t)|x|^2} dt.$$

Integrating by part, we have

$$\begin{aligned} g_n(|x|^2) = &\int_0^1 t' e^{-y_n(t)|x|^2} dt \\ &= e^{-|x|^2} + \int_0^1 t y_n'(t)|x|^2 e^{-y_n(t)|x|^2} dt \\ &= e^{-|x|^2} + n(n-1)|x|^2 \int_0^1 (t^{n-1} - t^n) e^{-y_n(t)|x|^2} dt \\ &= e^{-|x|^2} + n|x|^2 \int_0^1 (nt^{n-1} - (n-1)t^n) e^{-y_n(t)|x|^2} dt \\ &\quad - n|x|^2 \int_0^1 t^{n-1} e^{-y_n(t)|x|^2} dt \\ &= e^{-|x|^2} + n|x|^2 \int_0^1 y_n(t) e^{-y_n(t)|x|^2} dt - |x|^2 h_n(|x|^2) \\ &= e^{-|x|^2} + \frac{\pi n|x|^2}{n-1} \mathbb{D}_{p_n} - |x|^2 h_n(|x|^2), \end{aligned}$$

where we denote

$$h_n(|x|^2) := n \int_0^1 t^{n-1} e^{-y_n(t)|x|^2} dt. \tag{29}$$

Thus,

$$\mathbb{D}_{p_n} = \frac{n-1}{n\pi} \left(\frac{g_n(|x|^2) - e^{-|x|^2}}{|x|^2} + h_n(|x|^2) \right). \tag{30}$$

We claim

$$\lim_{n\to\infty} g_n(|x|^2) = 1.$$

This is quite straightforward. As $\forall \epsilon \in (0,1)$, we rewrite

$$g_n(|x|^2) = \int_0^1 e^{-y_n(t)|x|^2} dt = \int_0^{1-\epsilon} + \int_{1-\epsilon}^1 .$$

Since $y_n(t) \to 0$ uniformly on $[0, 1-\epsilon]$ as $n \to \infty$, thus

$$\lim_{n\to\infty} \int_0^{1-\epsilon} e^{-y_n(t)|x|^2} dt = \int_0^{1-\epsilon} \lim_{n\to\infty} e^{-y_n(t)|x|^2} dt = 1 - \epsilon.$$

For the second integration, since $y_n(t) \geq 0$ on $[0,1]$, we have $\int_{1-\epsilon}^1 e^{-y_n(t)|x|^2} dt \leq \epsilon$. Hence we get

$$1 - \epsilon \leq \varliminf_{n\to\infty} \int_0^1 e^{-y_n(t)|x|^2} dt \leq \varlimsup_{n\to\infty} \int_0^1 e^{-y_n(t)|x|^2} dt \leq 1.$$

As ϵ is chosen arbitrarily, letting $\epsilon \to 0^+$ yields the claim.

Now we estimate (30) to be

$$\begin{aligned} \mathbb{D}_{p_n} &= \frac{n-1}{n\pi} \left(\frac{1 - e^{-|x|^2}}{|x|^2} + h_n(|x|^2) + o(1) \right) \\ &= \frac{1 - e^{-|x|^2}}{\pi |x|^2} + \frac{1}{\pi} h_n(|x|^2) + o(1) \end{aligned} \tag{31}$$

as $n \to \infty$.

We now turn to estimate $h_n(|x|^2)$. Change variable $s = t^n$, h_n will be rewritten as

$$\int_0^1 e^{z_n(s)|x|^2} ds,$$

where

$$z_n(s) = -n s^{\frac{n-1}{n}} + (n-1)s.$$

It is easy to check that

$$z_n(s) \leq z_{n+1}(s)$$

for any fixed $s \in [0,1]$. Thus we have $z_n(s)$ monotone increasing to $-(s - s\log s)$ as $n \to \infty$; hence, $h_n(|x|^2)$ will satisfy

$$\lim_{n\to\infty} h_n(|x|^2) = \int_0^1 e^{-(s - s\log s)|x|^2} ds.$$

This will give us the estimate

$$h_n(|x|^2) = \int_0^1 e^{-(s - s\log s)|x|^2} ds + o(1)$$

as $n \to \infty$. Hence, we further estimate (31) to be

$$\mathbb{D}_{p_n} = \frac{1 - e^{-|x|^2}}{\pi |x|^2} + \frac{1}{\pi} \int_0^1 e^{-(s - s\log s)|x|^2} ds + o(1) \text{ as } n \to \infty,$$

which completes the proof of Theorem 1.

4. Growth of critical values and rescaling limit

By Theorem 2, the expected density of the modulus of critical values is $1/x$ decay as n large enough, which implies that the integration of the density over the interval $[a, \infty)$ is infinity for any $a > 0$ large enough, i.e., critical values accumulate at infinity as n tends to ∞. In this section, we will consider the rate of growth of the critical values and its rescaling limit.

4.1. **Rate of growth.** We consider the distribution function $F_n(x)$ of the probability density

$$\frac{\mathbb{D}_{|p_n|}}{n-1} = \frac{1}{n-1}\mathbb{E}(\sum_{p_n'=0} \delta_{|p_n|}).$$

We write the distribution function as

$$F_n(x) = \int_0^x \frac{\mathbb{D}_{|p_n|}}{n-1} dy = \frac{2\pi}{n-1}\int_0^x y\mathbb{D}_{p_n}(y)dy \tag{32}$$

by relation (10).

Using identity (28) and integrating by part, we will have

$$F_n(x) = 1 - g_n(|x|^2).$$

Now we turn to prove Theorem 3.

Proof. We only need to prove $g_n(e^{n^{1+\epsilon}}) \to 0$ and $g_n(e^{n^{1-\epsilon}}) \to 1$. Let us apply the dominated convergence theorem to $g_n(n^{1\pm\epsilon})$. We have

$$\lim_{n\to\infty} g_n(n^{1\pm\epsilon}) = \int_0^1 e^{-\lim_{n\to\infty} y_n(t)e^{n^{1\pm\epsilon}}} dt.$$

For the "+" part, we know that for any fixed $0 < t < 1$,

$$\lim_{n\to\infty} y_n(t)e^{n^{1+\epsilon}} = \lim_{n\to\infty} (nt^{n-1} - (n-1)t^n)e^{n^{1+\epsilon}} \geq \lim_{n\to\infty} t^{n-1}e^{n^{1+\epsilon}} = \infty.$$

For the "−" part, we know that $\forall t > 0$,

$$\lim_{n\to\infty} y_n(t)e^{n^{1+\epsilon}} \leq \lim_{n\to\infty} nt^{n-1}e^{n^{1-\epsilon}} = 0,$$

which implies the conclusions. □

4.2. **Rescaling limit.** As illustrated by Theorem 3, we need to consider the rescaled distribution

$$\tilde{F}_n(x) = F_n(e^{\frac{nx}{2}})$$

and the corresponding rescaled probability density

$$R_n(x) = (\tilde{F}_n(x))' = ne^{nx}\int_0^1 y_n(t)e^{-y_n(t)e^{nx}} dt$$

by relations (28) and (32), where $y_n(t) = nt^{n-1} - (n-1)t^n \in [0, 1]$.

Now we prove Theorem 4.

Proof. For $x = 0$, we have

$$\begin{aligned}\lim_{n\to\infty} R_n(0) &= \lim_{n\to\infty}\int_0^1 ny_n(t)e^{-y_n(t)}dt \\ &= \lim_{n\to\infty}\frac{\pi n}{n-1}\mathbb{D}_{p_n}(1) = \pi\lim_{n\to\infty}\mathbb{D}_{p_n}(1),\end{aligned}$$

where $\lim_{n\to\infty}\mathbb{D}_{p_n}(1)$ is a constant given by the leading term in (9).

For $x < 0$, we have

$$\begin{aligned}0 &\le \lim_{n\to\infty}\int_0^1 ny_n(t)e^{nx}e^{-y_n(t)e^{nx}}dt \le \lim_{n\to\infty}\int_0^1 ny_n(t)e^{nx}dt\\ &\le \lim_{n\to\infty}\int_0^1 ne^{nx}dt \le \lim_{n\to\infty} ne^{nx} = 0,\end{aligned}$$

which implies

$$\lim_{n\to\infty} R_n(x) = 0 \ \text{ for } x < 0. \tag{33}$$

Now we consider the case for $x > 0$. We integrate by part

$$\begin{aligned}\int_0^1 e^{-y_n(t)e^{nx}}dt =&\ te^{-y_n(t)e^{nx}}|_0^1 + \int_0^1 y_n'(t)e^{nx}te^{-y_n(t)e^{nx}}dt\\ =&\ e^{-e^{nx}} + \int_0^1 n(n-1)(t^{n-1}-t^n)e^{nx}e^{-y_n(t)e^{nx}}dt\\ =&\ e^{-e^{nx}} + R_n(x) - \int_0^1 nt^{n-1}e^{nx}e^{-y_n(t)e^{nx}}dt\\ =&:\ e^{-e^{nx}} + R_n(x) - Q_n(x).\end{aligned}$$

Thus we have

$$R_n(x) = \int_0^1 e^{-y_n(t)e^{nx}}dt + Q_n(x) - e^{-e^{nx}}. \tag{34}$$

For $x > 0$, the third term $e^{-e^{nx}} \to 0$ as $n \to \infty$.

Now we claim that

$$\int_0^1 e^{-y_n(t)e^{nx}}dt \to e^{-x}, \ \ Q_n(x) \to 0 \tag{35}$$

as $n \to \infty$.

If the claim holds, we will get

$$\lim_{n\to\infty} R_n(x) = e^{-x} \ \text{ for } x > 0, \tag{36}$$

which completes the proof of Theorem 4.

We now prove the first claim: For any $t < e^{-x}$,

$$\lim_{n\to\infty} y_n(t)e^{nx} \le \lim_{n\to\infty} nt^{n-1}e^{xn} = 0;$$

for any $t > e^{-x}$,

$$\lim_{n\to\infty} y_n(t)e^{nx} \ge \lim_{n\to\infty} t^{n-1}e^{xn} = \infty.$$

Therefore, by the dominated convergence theorem, we have

$$\lim_{n\to\infty}\int_0^1 e^{-y_n(t)e^{nx}}dt = \int_0^{e^{-x}} 1dt = e^{-x}.$$

Now we prove $Q_n(x) \to 0$ as $n \to \infty$. By changing variables, we write Q_n as

$$\begin{aligned}Q_n(x) &= \int_0^1 n(te^x)^{n-1}e^{-n(te^x)^{n-1}e^x + (n-1)(te^x)^n}d(te^x)\\ &= \int_0^{e^x} nr^{n-1}e^{-nr^{n-1}e^x + (n-1)r^n}dr.\end{aligned} \tag{37}$$

We separate the integral $\int_0^{e^x}$ in Q_n to be

$$Q_n(x) = \int_0^1 + \int_1^{e^x} := I_{1,n} + I_{2,n}.$$

First note $I_{1,n} \geq 0$, $I_{2,n} \geq 0$. We rewrite

$$\begin{aligned} I_{1,n} &= \int_0^1 nr^{n-1} e^{-nr^{n-1}e^x + (n-1)r^n} dr \\ &= \int_0^1 e^{-nu^{\frac{n-1}{n}} e^x + (n-1)u} du, \end{aligned} \tag{38}$$

where we change variable $u = r^n$.

Since $e^x > 1$ strictly, we must have $-nu^{\frac{n-1}{n}} e^x + (n-1)u \to -\infty$ as $n \to \infty$. By the dominated convergent theorem, we have $I_{1,n} \to 0$ as $n \to \infty$.

For the second integration, we further separate

$$I_{2,n} = \int_1^{1+\epsilon} + \int_{1+\epsilon}^{e^x} =: I_{3,n} + I_{4,n}, \tag{39}$$

where we choose $1 > \epsilon > 0$ such that $1 + 3\epsilon < e^x$.

For the first part, since $e^x > 1 + 3\epsilon > 1 + \epsilon$, we have

$$r^{n-1}(-ne^x + (n-1)r) \leq r^{n-1}(-nr(1+\epsilon) + (n-1)r) = -r^n(1 + n\epsilon),$$

thus

$$I_3 \leq \int_1^{1+\epsilon} e^{-r^n(1+n\epsilon) + \log n + (n-1)\log r} dr.$$

But $-r^n + (n-1)\log r \leq 0$ for $r \geq 1$; and $-r^n n\epsilon + \log n \leq -n\epsilon + \log n \to -\infty$ as $n \to \infty$. Thus, $I_{3,n} \leq n\epsilon e^{-n\epsilon} \to 0$ as $n \to \infty$.

For the second part in (39), we have

$$\begin{aligned} \int_{1+\epsilon}^{e^x} nr^{n-1} e^{-nr^{n-1}e^x + (n-1)r^n} dr &\leq \int_{1+\epsilon}^{e^x} nr^{n-1} e^{-r^{n-1}e^x} dr \\ &= \int_{1+\epsilon}^{e^x} e^{(n-1)\log r + \log n - r^{n-1}e^x} dr. \end{aligned}$$

But for $r \in [1+\epsilon, e^x]$, we have

$$-r^{n-1}e^x + (n-1)\log r + \log n \leq -(1+\epsilon)^{n-1}e^x + (n-1)x + \log n \to -\infty$$

as $n \to \infty$. Hence, the $I_{4,n}$ will tend to 0 as $n \to \infty$. Therefore $I_{2,n}$ tends to 0 as $n \to \infty$.

Now we must have $\lim_{n\to\infty} Q_n(x) \to 0$, which completes the claim. □

Acknowledgments

We would like to thank S. Zelditch for suggesting this problem and many helpful discussions. We also want to thank the referee for many valuable comments in the original version.

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