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THE GRADED COUNT OF QUASI-TREES IS NOT A KNOT INVARIANT

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Abstract. In "A survey on the Turaev genus of knots", Champanerkar and Kofman propose several open questions. The first asks whether the polynomial whose coefficients count the number of quasi-trees of the all-A ribbon graph obtained from a diagram with minimal Turaev genus is an invariant of the knot. We answer negatively by showing a counterexample obtained from the two diagrams of 8_{21} on the Knot Atlas and KnotScape.

1. Introduction

The Tutte polynomial is a graph invariant with many beautiful properties, including duality and a deletion-contraction relation. Bollobás devotes an entire chapter to the subject in his text [Bol98, Chapter X]. Bollobás and Riordan generalize this polynomial invariant in [BR01, BR02] for the setting of ribbon graphs, generalizing many of its properties. We use the definition summing over all subsets:

$$C(\mathbb{G}; X, Y, Z) = \sum_{\mathbb{H} \subseteq \mathbb{G}} (X-1)^{k(\mathbb{H})-k(\mathbb{G})} Y^{n(\mathbb{H})} Z^{g(\mathbb{H})}$$

where: $v(\mathbb{G})$, $e(\mathbb{G})$, and $k(\mathbb{G})$ are the numbers of vertices, edges, and components of $\mathbb{G}$, respectively; $r(\mathbb{G}) = v(\mathbb{G}) - k(\mathbb{G})$ is the rank of $\mathbb{G}$; $n(\mathbb{G}) = e(\mathbb{G}) - r(\mathbb{G})$ is the nullity of $\mathbb{G}$; and $bc(\mathbb{G})$ is the number of connected components of the boundary of $\mathbb{G}$. Finally $2g(\mathbb{G}) = k(\mathbb{G}) - bc(\mathbb{G}) + n(\mathbb{G})$ is twice the genus of $\mathbb{G}$.

Dasbach, Futer, Kalfagianni, Lin, and Stoltzfus show in [DFK$^+$08] that the Jones polynomial is a specialization of the Bollobás-Riordan-Tutte polynomial $C(\mathbb{G}, X, Y, Z)$, where $\mathbb{G}$ is the all-A ribbon graph of a knot diagram. In [CKS11] Champanerkar, Kofman, and Stoltzfus defined a polynomial $q(\mathbb{G}; t, Y)$ for ribbon graphs, which is also a specialization of the Bollobás-Riordan-Tutte polynomial, whose coefficients count the number of quasi-trees of $\mathbb{G}$.

Proposition 1.1 ([CKS11, Proposition 2]). *Let* $q(\mathbb{G}; t, Y) = C(\mathbb{G}; 1, Y, tY^{-2})$. *Then* $q(\mathbb{G}; t, Y)$ *is a polynomial in* t *and* Y *such that*

$$q(\mathbb{G}; t) := q(\mathbb{G}; t, 0) = \sum_j a_j t^j \tag{1.1}$$

where a_j *is the number of quasi-trees of genus* j. *Consequently* $q(\mathbb{G}; 1)$ *equals the number of quasi-trees of* $\mathbb{G}$.

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Dasbach, Futer, Kalfagianni, Lin, and Stoltzfus in [DFK$^+$10, Theorem 3.2] show that the absolute value of the evaluation of this polynomial with $t = -1$ gives the determinant of the knot, an invariant defined to be the absolute value of the evaluation of the Jones polynomial with $t = -1$.

In "A survey on the Turaev genus of knots" [CK14], Champanerkar and Kofman ask whether the polynomial itself is an invariant when the Turaev genus g_T of the diagram is equal to that of the knot, that is, when it is minimal.

Question 1.2 ([CK14, Question 1])**.** Let $\mathbb{G}$ be the all-A ribbon graph for a diagram D of a knot K. If $g_T(D) = g_T(K)$, is $q(\mathbb{G}; t)$ an invariant of K?

We give a negative answer to this question by providing a counterexample.

Theorem 1.3. *The polynomial whose coefficients count the number of quasi-trees of the all-A ribbon graph obtained from a diagram with minimal Turaev genus is not an invariant of the knot.*

We prove this Theorem 1.3 by considering the two diagrams of 8_{21} obtained from the Knot Atlas [BNMea] and KnotScape [HT99]. We address these cases in Examples 3.1 and 3.2, respectively.

We rely on an algorithm given by Armond, Druivenga, and Kindred in [ADK14] to obtain alternating diagrams on a surface with minimal Turaev genus.

2. Definitions

A *ribbon graph* $\mathbb{G}$ is a graph embedded on a given surface Σ such that each component of $\Sigma \backslash \mathbb{G}$ is a disk; these components are called *faces.* This embedding provides a cycling orientation of the edges around each vertex. The *genus* $g(\mathbb{G})$ is the genus of the surface $g(\Sigma)$. Given a sub-ribbon graph $\mathbb{H}$ of $\mathbb{G}$, the surface on which $\mathbb{H}$ naturally embeds may not be the same as that of $\mathbb{G}$, although it is the case that $g(\mathbb{H}) \leq g(\mathbb{G})$. A *spanning* sub-ribbon graph is one that contains all of the vertices of $\mathbb{G}$. A *quasi-tree* $\mathbb{H}$ of $\mathbb{G}$ is a spanning sub-ribbon graph that has only one face.

The *all-A ribbon graph* is obtained by converting the all-A state into a ribbon graph. Given a knot diagram D, we can smooth a crossing in two ways as in Figure 1. A *state* of the diagram is a choice of smoothing for each crossing, resulting in a disjoint union of circles in the plane. We consider the state where the A-smoothing is chosen for each crossing. We convert it to a ribbon graph by taking each circle in the state to be a vertex, each crossing to be an edge, and the cyclic ordering of the edges around each vertex to be the cyclic ordering of the crossings connected to each circle. The orientation of a circle is taken to be counter-clockwise if the circle is nested in an even number of other circles and clockwise if it is nested in an odd number of other circles.

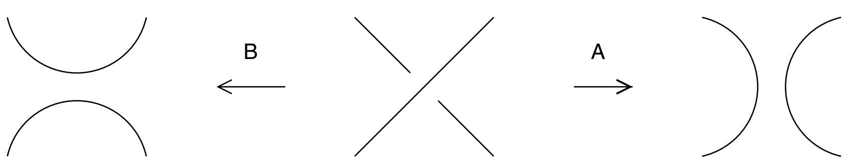

FIGURE 1. The A and B smoothings of a crossing in a knot diagram.

Example 2.1. The construction of the all-A ribbon graph may be seen in an example from [DFK$^+$10, Figure 3] for a digram for 8_{21} shown in Figure 2 below.

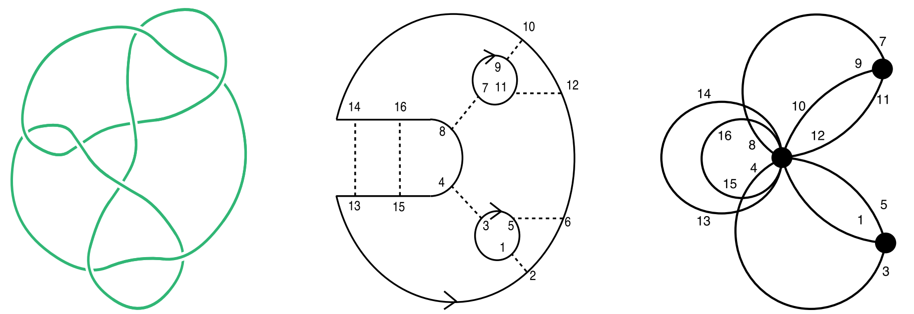

FIGURE 2. The construction of the all-A ribbon graph from a diagram for 8_{21} (as appearing in [DFK$^+$10, Figure 3]).

The genus of the all-A ribbon graph is called the *Turaev genus of the diagram* and is denoted $g_T(D)$. The *Turaev genus* $g_T(K)$ of a knot K is the minimum value of $g_T(D)$ where D can be any diagram for K.

The Turaev genus of a knot was first defined in [DFK$^+$08]. Champanerkar and Kofman offer a very complete "survey on the Turaev genus of knots" [CK14], and we refer the reader to this short survey for further details.

3. A counterexample: Diagrams from the Knot Atlas and KnotScape

In Examples 3.1 and 3.2 below, we count the quasi-trees of the all-A ribbon graph obtained from diagrams coming from the Knot Atlas [BNMea] and KnotScape [HT99], respectively, as given on the left- and right-hand sides, respectively, in Figure 3. We will actually consider the mirror image of the diagram on the right. We show that the polynomial $q(\mathbb{G}, t)$ is not invariant on the knot.

FIGURE 3. The knot 8_{21} presented in diagrams given by the Knot Atlas [BNMea] and KnotScape [HT99], respectively. We will actually consider the mirror image of the diagram on the right.

Example 3.1. Consider first the knot diagram of 8_{21} given by the Knot Atlas [BNMea], as shown in Figure 3. This diagram has Turaev genus 2. We perform a Reidemeister III move on the upper central three crossings to obtain a diagram of Turaev genus 1.

Armond, Druivenga, and Kindred [ADK14] give an algorithm to obtain an alternating diagram on a surface. We apply this to obtain a Heegaard diagram, where the dashed and dotted lines represent α and β curves, respectively, as given on the left-hand side in Figure 4.

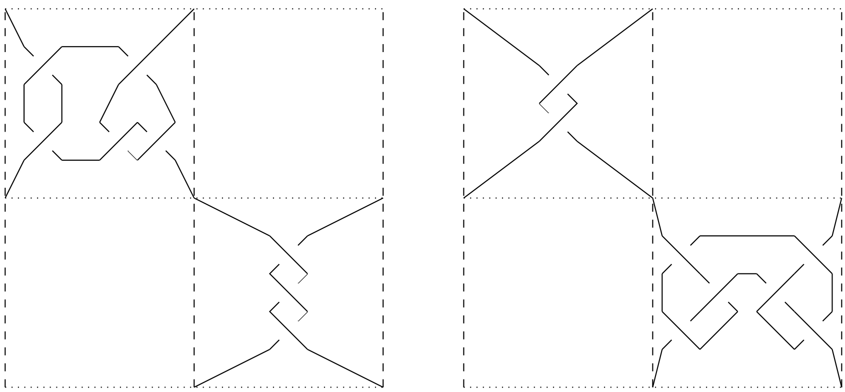

FIGURE 4. Alternating diagrams on the torus for 8_{21} coming from the Knot Atlas and KnotScape, respectively, after applying the algorithm of [ADK14]. We will actually consider the mirror image of the diagram on the right.

We checkerboard color this diagram on the torus to obtain the all-A ribbon graph given on the left-hand side in Figure 5. We proceed to count the number of quasi-trees.

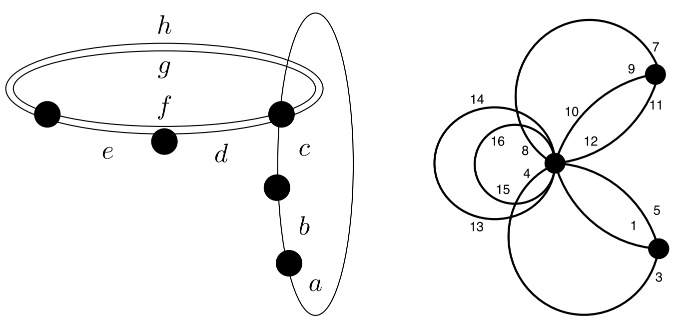

FIGURE 5. The all-A ribbon graphs for diagrams of 8_{21} coming from the Knot Atlas after a Reidemeister move III and from KnotScape (as appearing in [DFK$^+$10, Figure 3]), respectively. We will actually take the dual of the ribbon graph on the right.

First of all, any spanning tree of $\mathbb{G}$ must contain exactly two edges from the loop consisting of edges a, b, and c. From here the spanning trees fall into two classes: those with one of the two edges g and h and those with neither g nor h. Any spanning tree in the first class must contain one of the two edges d and e giving a total of $3 \times 2 \times 2 = 12$ spanning trees in the first class. Any spanning tree in the second class must contain two of the three edges d, e, and f giving a total of $3 \times 3 = 9$ spanning trees in the second class. Thus for this ribbon graph we get $a_0 = 21$.

A quasi-tree of $\mathbb{G}$ with genus 1 must contain all of the edges a, b, and c as well as one of the two edges g and h and again these quasi-trees fall into two classes: those that contain the edges d and e but not f and those that contain the edge f and exactly one of the edges d and e. This gives us $a_1 = 4 + 2 = 6$ for this ribbon graph.

Thus, we obtain $q(\mathbb{G}, t) = 6t + 21$.

Example 3.2. Now consider the knot diagram of 8_{21} given by KnotScape [HT99] having Turaev genus 1 already and appearing on the right-hand side of Figure 3. We will actually consider the mirror image.

Begin with the original diagram before taking the mirror image.

We apply the algorithm of [ADK14] to obtain an alternating diagram on a surface, which again is a Heegaard diagram, where the dashed and dotted lines represent α and β curves, respectively, as given on the right-hand side in Figure 4.

We checkerboard color this diagram on the torus to obtain the all-A ribbon graph, given in Figure 3 from [DFK$^+$10], which we include on the right-hand side in our Figure 5.

As observed in [DFK$^+$10], this ribbon graph contains 9 spanning trees and 24 genus-1 quasi-trees, yielding $q(\mathbb{G}, t) = 24t + 9$.

Now we take the mirror image, which exchanges the all-A ribbon graph and the similarly defined *all-B ribbon graph*, which is obtained from the *all-B state*; these ribbon graphs are dual to each other. Thus we obtain $q(\mathbb{G}, t) = 9t + 24$.

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References

[ADK14] Cody Armond, Nathan Druivenga, and Thomas Kindred, *Heegaard diagrams corresponding to Turaev surfaces*, J. Knot Theory Ramifications **24** (2015), no. 4, 1550026 (14 pages), DOI 10.1142/S0218216515500261. MR3346925

[BNMea] Dror Bar-Natan, Scott Morrison, and et al., *The Knot Atlas*, http://katlas.org.

[Bol98] Béla Bollobás, *Modern graph theory*, Graduate Texts in Mathematics, vol. 184, Springer-Verlag, New York, 1998. MR1633290 (99h:05001)

[BR01] Béla Bollobás and Oliver Riordan, *A polynomial invariant of graphs on orientable surfaces*, Proc. London Math. Soc. (3) **83** (2001), no. 3, 513–531, DOI 10.1112/plms/83.3.513. MR1851080 (2002f:05056)

[BR02] Béla Bollobás and Oliver Riordan, *A polynomial of graphs on surfaces*, Math. Ann. **323** (2002), no. 1, 81–96, DOI 10.1007/s002080100297. MR1906909 (2003b:05052)

[CK14] Abhijit Champanerkar and Ilya Kofman, *A survey on the Turaev genus of knots*, Acta Math. Vietnam. **39** (2014), no. 4, 497–514, DOI 10.1007/s40306-014-0083-y. MR3292579

[CKS11] Abhijit Champanerkar, Ilya Kofman, and Neal Stoltzfus, *Quasi-tree expansion for the Bollobás-Riordan-Tutte polynomial*, Bull. Lond. Math. Soc. **43** (2011), no. 5, 972–984, DOI 10.1112/blms/bdr034. MR2854567

[DFK$^+$08] Oliver T. Dasbach, David Futer, Efstratia Kalfagianni, Xiao-Song Lin, and Neal W. Stoltzfus, *The Jones polynomial and graphs on surfaces*, J. Combin. Theory Ser. B **98** (2008), no. 2, 384–399, DOI 10.1016/j.jctb.2007.08.003. MR2389605 (2009d:57020)

[DFK+10] Oliver T. Dasbach, David Futer, Efstratia Kalfagianni, Xiao-Song Lin, and Neal W. Stoltzfus, *Alternating sum formulae for the determinant and other link invariants*, J. Knot Theory Ramifications **19** (2010), no. 6, 765–782, DOI 10.1142/S021821651000811X. MR2665767 (2011d:57010)

[HT99] Jim Hoste and Morwen Thistlethwaite, *"KnotScape," a knot polynomial calculation program*, available at http://www.math.utk.edu/∼morwen/knotscape.html, 1999.

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