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PARRY'S TOPOLOGICAL TRANSITIVITY AND f-EXPANSIONS

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ABSTRACT. In his 1964 paper on f-expansions, Parry studied piecewise-continuous, piecewise-monotonic maps F of the interval $[0, 1]$, and introduced a notion of topological transitivity different from any of the modern definitions. This notion, which we call *Parry topological transitivity* (PTT), is that the *backward orbit* $O^-(x) = \{y : x = F^n y \text{ for some } n \geq 0\}$ of some $x \in [0, 1]$ is dense. We take *topological transitivity* (TT) to mean that some x has a dense *forward* orbit. Parry's application of PTT to f-expansions is that PTT implies the partition of $[0, 1]$ into the "fibers" of F is a generating partition (i.e., f-expansions are "valid"). We prove the same result for TT, and use this to show that for interval maps F, TT implies PTT. A separate proof is provided for continuous maps F of perfect Polish spaces. The converse is false.

1. INTRODUCTION

The concept of *topological transitivity* plays an important role in dynamical systems theory. A map $F : X \to X$, where X is a topological space, is said to be topologically transitive (TT) if the *forward orbit* of some $x \in X$, defined as

$$O^+(x) = \{F^n x : n \geq 0\},$$

is dense in X. Note that for $U, V \subseteq X$ and $F : X \to X$,

$$U \cap F^n V \neq \emptyset \Leftrightarrow F^{-n} U \cap V \neq \emptyset. \tag{1.1}$$

Another definition, sometimes called *regional topological transitivity* (RTT), is that for any $U, V \subseteq X$ nonempty and open, there exists $n > 0$ so (1.1) holds. When F is continuous, there are many situations where TT and RTT are equivalent (see, e.g., [19], [15], [3] and Proposition 1 below). However, one often wants to apply topological transitivity with less ideal hypotheses. The benefit of TT is that it makes sense even when F is not continuous.

In this paper, we will mostly be interested in piecewise-monotonic, piecewise-continuous maps F on the unit interval. In 1964, Parry [23] gave a different definition of topological transitivity in this situation. Parry's definition, which we will refer to here as *Parry topological transitivity* (PTT), says that for some $x \in X$, the *backward orbit*, defined as

$$O^-(x) := \{F^{-n}(x) : n \geq 0\} = \{y : x = F^n(y) \text{ for some } n \geq 0\},$$

is dense.

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Note that when F is invertible, the backward orbit of x is simply the forward orbit of x under F^{-1}. For invertible maps, topological transitivity is often defined as the existence of $x \in X$ with dense *two-sided orbit*, defined as $O(x) = \{F^n(x) : n \in \mathbb{Z}\} = O^+(x) \cup O^-(x)$. We will refer to this as *two-sided* topological transitivity (TTT). Again, it is often the case for homeomorphisms F (see Corollary 2 below) that TTT is equivalent to TT, and thus to PTT. But the situation becomes more complicated if F is not invertible, and even more complicated if F is not continuous.

A constant function F obviously satisfies PTT in a trivial way. As we will see, even for surjective maps F, PTT generally does not imply TT. But in many situations, TT does imply PTT. It is not hard to obtain such results under "nice" hypotheses, like for continuous maps of perfect Polish spaces (see Theorem 3). In this case, we show that TT implies $O^-(x)$ is dense for a dense G_δ set of $x \in X$. Recently, it was shown in [19] that continuous TT maps $F : \mathbb{R} \to \mathbb{R}$ satisfy PTT (in fact [19] proves more: if such an F is TT, then $O^-(x)$ is dense for all but possibly two points $x \in \mathbb{R}$).

Our main goal in this paper is to better understand the situation studied by Parry [23], namely, piecewise-continuous, piecewise-monotonic maps F of the interval. We call a map $F : [0,1] \to [0,1]$ a *piecewise interval map* if there is a finite or countable partition ξ of Lebesgue almost all of $[0,1]$ into disjoint intervals, indexed by a "digit set" $\mathcal{D}$, such that each $F|_{\Delta(d)}$ is continuous and strictly monotonic (see Section 3 for details). In his paper [23], Parry considered piecewise interval maps in the context of f-expansions, as defined by Rènyi [24], Bissinger [5] and Everett [11] in the 1940's and 1950's. Unknown to these authors, the same idea had previously been studied in 1924 by Kakeya [13].

The idea of f-expansions (the term is due to Rènyi [24]) is to use piecewise interval maps F to obtain what we call the F-representation $\mathbf{r}(x) \in \mathcal{D}^{\mathbb{N}}$ of $x \in [0,1]$ by recording the sequence $\mathbf{r}(x) = .d_1d_2d_3\ldots$ of ξ-intervals visited by the F-iterates of x (see Section 3). The goal is to find conditions on F so that distinct x have distinct F-representations ("valid" in Parry's terminology [23]). We also study the algorithm (see Section 5) to recover x from $\mathbf{r}(x)$. In particular, under appropriate conditions the "f-expansion" $f(d_1 + f(d_2 + f(d_3 + \ldots)))$ converges to x, where $f : \mathbb{R} \to [0,1]$ is a function satisfying $F(x) = f^{-1}(x) \bmod 1$.

There are, of course, two especially well know cases of valid F-representations and f-expansions. Binary representations/expansions[1] of real numbers correspond to the map $F(x) = 2x \bmod 1$ with $f(x) = x/2$. Continued fraction representations/expansions correspond to the map $F(x) = 1/x \bmod 1$ with $f(x) = 1/x$. In each of these cases, F satisfies both TT and PTT.

In [23] Parry proved that PTT implies F-representations are valid (and also, with some additional hypotheses, that valid F-expansions implies PTT). In this paper, we prove a slightly strengthened version of Parry's first result, as well as a "modern" version of Parry's result, which says that TT implies F-representations are valid. One benefit is that TT is often easy to verify. For example (see Proposition 5), any F that is ergodic for an invariant measure μ equivalent to Lebesgue measure will satisfy TT. In the end, we show (Theorem 23) that TT implies PTT for piecewise interval maps F.

[1]Decimal representations/expansions correspond to replacing the "base" 2 with base 10.

2. Topological transitivity

We begin by carefully stating two standard results mentioned in the introduction.

Proposition 1. *Suppose $F : X \to X$ is a continuous map on a Polish space (i.e., X is separable and completely metrizable). If F satisfies RTT, then the set $X_0 = \{x : O^+(x)$ is dense in $X\}$ contains a dense G_δ set. In particular, RTT implies TT. If, in addition, X is perfect (i.e., has no isolated points), then TT implies RTT.*

Proof. Assume $U \cap F^{-n}V \neq \emptyset$ for some $n \geq 0$. Then for any V open, $\bigcup_{n\geq 0} F^{-n}V$ is dense open, since it meets any open set U. Thus, whenever $\{V_k\}$ is a countable base for X, the Baire category theorem implies the G_δ set $X_0 = \bigcap_{k\geq 0}\bigcup_{n\geq 0} F^{-n}V_k$ is dense. Clearly $x \in X_0$ implies that for any k, there exists $n \geq 0$ so that $F^n(x) \in V_k$, so $O^+(x)$ is dense.

Now suppose $O^+(x)$ is dense and let U and V be open. There exist $n, m \geq 0$ with $F^n x \in U$ and $F^m x \in V$. Since X is perfect, we have for any nonempty open V that $V_n = V \setminus \{F^k(x) : x = 0, 1, \dots, n-1\}$ is nonempty and open. Thus $\{k \in \mathbb{N} : F^k(x) \in V\}$ is infinite, and we may assume $m > n$. It follows that $F^n x \in U \cap F^{-m+n}V \neq \emptyset$. □

Corollary 2. *If $F : X \to X$ is a homeomorphism of a perfect Polish space, then TT is equivalent to TTT. In fact, F and F^{-1} both satisfy TT, and there is a dense G_δ subset $X_0 \subseteq X$ so that for any $x \in X_0$, $O^-(x)$ and $O^+(x)$ are both dense.*

To see this, note that $O(x)$ dense implies either $O^+(x)$ or $O^-(x)$ is dense. In the latter case, $U \cap F^{-n}V \neq \emptyset$ for some $n \geq 0$.

2.1. The relation between TT and PTT.

The following example shows that even for continuous surjective maps of the closed interval, PTT does not imply TT.

Example 1. Define $F : [0, 1] \to [0, 1]$ by[2]

$$F(x) = \begin{cases} -2x + 1/2 & \text{if } x \in [0, 1/4), \\ \ \ 2x - 1/2 & \text{if } x \in [1/4, 3/4), \text{ and} \\ -2x + 5/2 & \text{if } x \in [3/4, 1]. \end{cases}$$

Note that $O^-(1/2)$ is dense, whereas $F^n(1/8, 3/8) \cap (5/8, 7/8) = \emptyset$ for all $n \geq 0$. Note also that just a single point has $O^-(x)$ dense, and not a dense G_δ set.

In the other direction, we have the following:

Theorem 3. *Suppose $F : X \to X$ is continuous and TT on a perfect Polish space X. Then F satisfies PTT, and moreover, $X_0 = \{x : O^-(x)$ is dense$\}$ contains a dense G_δ subset.*

Proof. It is well known that X Polish implies $X^{\mathbb{N}}$ is Polish: A complete separable metric on $X^{\mathbb{N}}$ is given by $\tilde{d}(\tilde{x}, \tilde{y}) = \sup_{n\geq 1}\{d(x_n, y_n)/2^n(1 + d(x_n, y_n))\}$, where d is a complete metric on X. The closed subspace $\widetilde{X} = \{\tilde{x} = (x_1, x_2, x_3, \dots) \in X^{\mathbb{N}_0} : x_n = F(x_{n+1})\} \subseteq X^{\mathbb{N}}$ is also Polish. Define $\widetilde{F} : \widetilde{X} \to \widetilde{X}$ by $\widetilde{F}(x_1, x_2, x_3, \dots) = (F(x_1), x_1, x_2, x_3, \dots)$, and note that $\widetilde{F}^{-1}(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4, \dots)$, so $\widetilde{F}$ is a homeomorphism. Call $\widetilde{F}$ the *natural extension* of F.

[2] My thanks to Ethan Akin for suggesting this example.

The map $\pi_k : \widetilde{X} \to X$, defined by $\pi_k(x_1, x_2, x_3, \dots) = x_k$, is surjective and open. It follows that $\widetilde{X}$ is perfect since X is perfect. Given a countable base $\mathcal{U}$ for X, let $\widetilde{\mathcal{U}}$ consist of all nonempty sets of the form

$$\widetilde{U} = \pi_1^{-1}(U_1) \cap \pi_2^{-1}(U_2) \cap \cdots \cap \pi_\ell^{-1}(U_\ell) \subseteq \widetilde{X}, \tag{2.1}$$

for some $\ell \geq 1$ and $U_1, U_2, \dots, U_\ell \in \mathcal{U}$. Then $\widetilde{\mathcal{U}}$ is a countable base for $\widetilde{X}$.

We **claim** that $O^+(\tilde{x})$ is dense in $\widetilde{X}$ for any $\tilde{x} \in \pi_1^{-1}(X_0)$, and thus $\widetilde{F}$ satisfies TT.

To prove the claim, fix $x \in X_0$ and let $\tilde{x} = (x, x_2, x_3, \dots) \in \pi_1^{-1}(X_0)$. Note that for any $k \geq 1$,

$$\tilde{F}^k(\tilde{x}) = (F^k(x), F^{k-1}(x), \dots, F(x), x, x_2, x_3, \dots).$$

Choose $\widetilde{U} \in \widetilde{\mathcal{U}}$, satisfying (2.1), so that

$$U = F^{-\ell+1}U_1 \cap F^{-\ell+2}U_2 \cap \cdots \cap U_\ell \subseteq X \tag{2.2}$$

is nonempty. Since F satisfies TT, the set

$$X_U := \bigcup_{n \geq 0} F^{-n}U = \{x \in X : O^+(x) \cap U \neq \emptyset\}$$

is dense open, so $X_0 = \bigcap_{U \in \mathcal{U}} X_U$ is dense G_δ.

Since $O^+(x)$ is dense and X is perfect, we can choose $n \geq \ell - 1$ so that $F^{n-\ell+1}x \in U$. Then

$$F^n(x) \in U_1,\ F^{n+1}(x) \in U_2,\ \dots,\ F^{n+\ell-1}(x) \in U_\ell,$$

and it follows from (2.1) that $\widetilde{F}^n(\tilde{x}) \in \widetilde{U}$. Since $\widetilde{U} \in \widetilde{\mathcal{U}}$ was arbitrary, $O^+(\tilde{x})$ is dense, proving the claim.

Now, since F satisfies TT, and $\widetilde{F}$ is a homeomorphism of a perfect metric space $\widetilde{X}$, it follows from Corollary 2 that $\widetilde{F}$ satisfies TTT. Thus $O^-(\tilde{x})$ is dense for $\tilde{x} \in \widetilde{X}_0 \subseteq \widetilde{X}$, where $\widetilde{X}_0$ contains a dense G_δ. Since π_1 is surjective and open, $X_0 = \pi_1(\widetilde{X}_0)$ contains a dense G_δ, and for $\tilde{x} \in \widetilde{X}_0$, $\pi_1(O^-(\widetilde{x}))$ is dense in X. But $\pi_1(O^-(\widetilde{x})) \subseteq O^-(\pi_1(\widetilde{x})) = O^-(x)$, so F satisfies PTT. □

For $F : X \to X$, we call a set $B^- \subseteq X$ *a backward orbit* of $x \in X$ if $x_1 = x$ and $B^- = \{x_1, x_2, x_3, \dots\}$ with $x_n = F(x_{n+1})$ for all $n \geq 1$. We say F satisfies *strong Parry topological transitivity* (SPTT) if there exists a dense backward orbit for some $x \in X$. Clearly SPTT implies PTT. Conversely, the proof of Theorem 3 shows that when F is a continuous map on a perfect Polish space, TT implies SPTT. Note that Example 1 does not satisfy SPTT although it does satisfy PTT.

2.2. **PTT for symbolic dynamical systems.** Here we consider the one-sided full shift

$$\mathcal{D}^{\mathbb{N}} = \{\mathbf{x} = .d_1 d_2 d_3 \cdots : d_j \in \mathcal{D}\},$$

where $2 \leq \#(\mathcal{D}) \leq \aleph_o$, with left shift map $S(.d_1 d_2 d_3 \dots) = .d_2 d_3 \dots$, and also the two-sided full shift $\mathcal{D}^{\mathbb{Z}}$ with left shift homeomorphism $\widetilde{S}(\dots d_{-1}.d_0 d_1 d_2 \dots) = \dots d_{-1} d_0 . d_1 d_2 \dots$, and with the product topology in each case. If $\#(\mathcal{D}) < \infty$, these are perfect compact metric spaces, homeomorphic to the Cantor set, but in any case, they are uncountable, totally disconnected, perfect Polish spaces (since $\mathcal{D}$ is Polish).

Call a subset $X \subseteq \mathcal{D}^{\mathbb{N}}$ a *one-sided subshift* if it is closed and S-invariant: $S(X) \subseteq X$. Similarly, call a subset $Y \subseteq \mathcal{D}^{\mathbb{Z}}$ a *two-sided subshift* if it is closed and $\widetilde{S}$-invariant: $\widetilde{S}(Y) = Y$. The *language* $\mathcal{L}$ of X (or $\mathcal{L}$ of Y) is the set of all finite words $w = w_0 w_1 \dots w_{\ell-1}$ (we say $|w| = \ell$) so that there exist $\mathbf{x} = .d_1 d_2 d_2 \dots \in X$ (or $\mathbf{y} = \dots d_{-1} d_0 . d_1 d_2 \dots \in Y$) and $k \in \mathbb{N}$ (or $k \in \mathbb{Z}$) with $w_0 w_1 \dots w_{\ell-1} = d_k d_{k+1} \dots d_{k+\ell-1}$.

Given a one-sided subshift X, we define $\widetilde{X}$ to be the two-sided subshift with the same language. The two-sided shift $\widetilde{S}$ on $\widetilde{X}$ is conjugate to the natural extension of the one-sided shift S on X in the sense of (the proof of) Theorem 3. Indeed, for $\widetilde{X}_1 := \{\tilde{x} = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots) \in X^{\mathbb{N}_0} : \mathbf{x}_n = S(\mathbf{x}_{n+1})\} \subseteq X^{\mathbb{N}}$, one has $\mathbf{x}_1 = .d_1 d_2 d_3 \dots$, $\mathbf{x}_2 = .d_0 d_1 d_2 \dots$, $\mathbf{x}_3 = .d_{-1} d_0 d_1 \dots$, etc., so the map $\dots d_{-1}.d_0 d_1 d_2 \dots \mapsto \tilde{x} : \widetilde{X} \to \widetilde{X}_1$ provides the desired conjugacy.

Corollary 4. *Let S be a one-sided shift and let $\widetilde{S}$ be its two-sided natural extension. Then S satisfies TT if and only if $\widetilde{S}$ satisfies TT, if and only if $\widetilde{S}^{-1}$ satisfies TT, if and only if $\widetilde{S}^{-1}$ satisfies TTT, if and only if S satisfies SPTT.*

Remark 1. A sub-base for the topology on X (or on Y) is given by *cylinder sets* $[w] := \{\mathbf{x} \in X : \mathbf{x}|_{[1,2,\dots,|w|]} = w\}$, for $w \in \mathcal{L}$ (or $[w] := \{\mathbf{y} \in Y : \mathbf{y}|_{[-\ell,-\ell+1,\dots,\ell-1,\ell]} = w\}$, $w \in \mathcal{L}$ and $|w| = 2\ell + 1$). One can characterize TT for S (or TTT for $\widetilde{S}$) by the property that $\mathcal{L}$ satisfies

$$\forall\, v, w \in \mathcal{L}\ \exists\, c \in \mathcal{L} \text{ so that } vcw \in \mathcal{L}.$$

Example 2. Let $X \subseteq \{1, 2, \overline{1}, \overline{2}\}^{\mathbb{N}}$ be the subshift[3] defined by forbidding the words $\mathcal{F} = \{\overline{k}\ell : k, \ell \in \{1, 2\}\}$. Here we have two one-sided two-shifts: "unbarred" $\{1, 2\}^{\mathbb{N}}$ and "barred" $\{\overline{1}, \overline{2}\}^{\mathbb{N}}$, with the possibility of "barring" the tail of a point $\mathbf{x} \in \{1, 2\}^{\mathbb{N}}$. Any point $\mathbf{x} \in X$ has $S^n(\mathbf{x}) \in \{1, 2\}^{\mathbb{N}} \cup \{\overline{1}, \overline{2}\}^{\mathbb{N}}$ for n sufficiently large, so $O^+(\mathbf{x})$ is never dense. However, any $\overline{\mathbf{x}} \in \{\overline{1}, \overline{2}\}^{\mathbb{N}}$ has $O^-(\overline{\mathbf{x}})$ dense.

3. Piecewise interval maps

Let λ denote Lebesgue measure on $[0, 1]$. An *interval partition* is a finite or countable indexed collection $\xi = \{\Delta(d) \subseteq [0, 1] : d \in \mathcal{D}\}$ of $2 \le \#(\xi) \le \aleph_0$ disjoint intervals such that $D := \bigcup_{d \in \mathcal{D}} \Delta(d)$ satisfies $\lambda(D) = 1$. The intervals $\Delta(d)$, which have endpoints $a_d < b_d$, may be open, closed, half-open $(a_d, b_d]$ or half-closed $[a_d, b_d)$. Let $\Delta(d)^\circ = (a_d, b_d)$ and note that $\bigcup_{d \in \mathcal{D}} \Delta(d)^\circ$ is open and dense. We generally refer to elements of the index set $\mathcal{D}$ as *digits*.

A *piecewise interval map* (PIM) F on $[0, 1]$ is an interval partition ξ, together with a map $F : D \to [0, 1]$ (informally, $F : [0, 1] \to [0, 1]$) such that

(1) each $F|_{\Delta(d)}$ is continuous and strictly monotonic,
(2) $\lambda(B^c) = 0$, where $B := \{x : F^n x \in D \text{ for all } n \ge 0\} = \bigcap_{n \ge 0} F^{-n} D$,
(3) for all $d, d' \in \mathcal{D}$ (including $d = d'$) and $n \ge 0$, $\Delta(d) \cap F^n(\Delta(d'))$ is either empty or an interval (i.e., it does not consist of a single point, or equivalently, $\Delta(d)^\circ \cap F^n(\Delta(d')^\circ) \ne \emptyset$).

We often assume (but do not require) that F is surjective (or, at least "almost surjective": $\overline{F(D)} = [0, 1]$, which is implied by TT). We say $F|_\Delta$, $\Delta \in \xi$, is *type A* if it is increasing and *type B* if it is decreasing. We say F is type A (or type B) if every $F|_\Delta$ is type A (or type B). Otherwise, F is called *mixed type.* We say F

[3]It is a nonprimitive subshift of finite type.

is *full* on $\Delta \in \xi$ if $(0,1) \subseteq F(\Delta)$. Condition (3) can always be achieved by taking each $\Delta \in \xi$ to be an open interval. The process of removing some endpoints from ξ to make F satisfy (3) only changes D on a countable set. However, in certain examples, it is natural to keep the endpoints (see the examples below). We define TT and PTT for such F as in the introduction (in terms of a point having a dense orbit).

Since each $F|_\Delta$ is strictly monotonic, condition (2) is automatic if D^c is countable. In particular, (2) always holds if ξ is finite. Condition (2) also holds if $\lambda(\{x : F'(x) = 0\}) = 0$, since this is equivalent to F being *nonsingular* in the sense that $\lambda(F^{-1}(E)) = 0$ for each $E \subseteq [0,1]$ with $\lambda(E) = 0$.

In many cases (see, e.g., [8], [7]) a stronger version of (2) holds. A Borel probability measure μ on $[0,1]$ is invariant if $\mu(D^c) = 0$, and $\mu(F^{-1}J) = \mu(J)$. Call an invariant measure μ *full* (FIM) if $\mu(J) > 0$ for any open interval J. An FIM is *absolutely continuous* (ACIM) if there exists $\rho \in L^1([0,1], \lambda)$, $\rho(x) > 0$, with $\mu(J) = \int_J \rho(x)\, d\lambda(x)$. The existence of an ACIM implies that F is nonsingular. Often, one can also show that an FIM (or ACIM) μ is *ergodic*: this means $F(E) = E$, for $E \subseteq [0,1]$ Borel, implies $\mu(E)\mu(E^c) = 0$.

Proposition 5. *If a PIM F has an ergodic FIM (or ACIM), then F satisfies TT.*

To prove this, let $\mathcal{J}$ be the collection of open intervals with rational endpoints. The Birkhoff ergodic theorem implies that for μ-a.e. $x \in [0,1]$ and any interval $J \in \mathcal{J}$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \chi_J(F^k x) = \mu(J) > 0,$$

so $O^+(x) \cap J \neq \emptyset$, and $O^+(x)$ is dense.

3.1. ***F*-representations.** Recall that

$$D = \bigcup_{d\in\mathcal{D}} \Delta(d) \subseteq [0,1] \quad \text{and} \quad B = \bigcap_{n\geq 0} F^{-n}(D).$$

By abuse of notation, we also denote by ξ the map $\xi : D \to \mathcal{D}$ with $\xi(x) = d$ for $x \in \Delta(d)$. We call ξ, defined for λ-a.e. $x \in [0,1]$, the *digit function*. Given a PIM F, we define the F-representation of $x \in B$ to be the sequence

$$\mathbf{r}(x) = .d_1 d_2 d_3 d_4 \cdots \in \mathcal{D}^{\mathbb{N}},$$

where $d_n = \xi(F^{n-1}x)$ for $n \in \mathbb{N}$. In ergodic theory, $\mathbf{r}(x)$ is called the (F,ξ)-name of x. We say that F-representations are *valid* if the map $\mathbf{r}$ is injective for λ-a.e. $x \in B$.

Parry observed in his paper [23] that the conditions for validity known at the time (i.e., in [5], [11], [24]) were all sufficient conditions, and were all "metric" in nature. Probably the nicest result of this type is Kakeya's Theorem [13], which essentially says that F-representations are valid for PIMs F of type A or B, provided $|F'(x)| > 1$ almost everywhere. Parry observed that one ought to expect the necessary and sufficient conditions for validity to be *dynamical* in nature. He went on to prove that F-representations are valid if F satisfies what we have called Parry topological transitivity.

3.2. **Examples.**

Example 3 (β-representations)**.** A β-transformation is a type A map $F : [0,1] \to [0,1]$ defined by $F(x) = \beta x \bmod 1$, for $\beta > 1$. Here $\xi(x) = \lfloor \beta x \rfloor$ with $\mathcal{D} = \{0, 1, \ldots, \beta - 1\}$ for $\beta \in \mathbb{N}$, and $\mathcal{D} = \{0, 1, \ldots, \lfloor \beta \rfloor\}$ for $\beta \notin \mathbb{N}$. The β-representations were introduced in [24] by Rényi, who showed that every β-transformation F has an ergodic ACIM (so satisfies TT). An explicit formula for the density $\rho(x)$ was given by Parry [22]). Parry (see [23]) studied the more general α-β-transformation, $F(x) = \alpha + \beta x \bmod 1$, which he showed are not necessarily ergodic (or TT).

Example 4 (Generalized Gauss transformations)**.** For real numbers $r \geq 1$, define the type 2 map $F(x) = r/x \bmod 1$ with $\xi(x) = \lfloor r/x \rfloor$. The case $r = 1$, known as the *Gauss transformation*, has an ergodic ACIM with $\rho(x) = (\log(2)(x+1))^{-1}$. The existence of an ergodic ACIM for $r > 1$ is discussed in [17] (an explicit formula for each $r \in \mathbb{N}$ is given in [9]). Thus each such F satisfies TT. The corresponding F-representations are (generalized) continued fractions.

Example 5 (Quadratic maps)**.** For $s \approx 0.8$, $s < r \leq 1$, consider $F : [0,1] \to [0,1]$ by

$$F(x) = -4r\left((1 - r - 4r^2 + 4r^3) - (1 - 8r^2 + 8r^3)x + r(1-2r)^2x^2\right),$$

with $\xi(x) = 0$ if $x < (1+2r-4r^2)/(2r-4r^2)$ and $\xi(x) = 1$ otherwise (this is the map $q(x) = 4rx(1-x)$, restricted to the interval $[q(r), r]$, then renormalized). These maps are commonly studied in chaos theory (see [10]). It is known that there is a set of values for r of positive Lebesgue measure so that F has an ergodic ACIM, and hence is TT. Closely related to both the quadratic maps and β-transformations are the *tent maps* defined for $1 < \tau \leq 2$ by $P(x) = \tau x \operatorname{wod} 1$, where we define $y \operatorname{wod} 1 = y \bmod 1$ if $\lfloor y \rfloor$ is even, and $1 - (y \bmod 1)$ if $\lfloor y \rfloor$ is odd. For all τ sufficiently large, F has an ergodic ACIM and hence is TT (see [12]).

Example 6 (The Cantor map)**.** This map F is defined to be linear, increasing, and full on each interval ξ in the complement K^c of the Cantor set $K \subseteq [0,1]$. The intervals in ξ are naturally indexed by $\mathcal{D} = \mathbb{Z}[1/2] \cap (0,1)$, the dyadic rationals in $(0,1)$. Note that in this example $D = K^c$ is measure zero but uncountable. More generally, ξ can be replaced by any interval partition. Then F is defined to be full on each $\Delta \in \xi$. Such maps F are called *generalized Lüroth transformations* in [8]. All such maps are TT and ergodic for Lebesgue measure.

Example 7 (Generalized Egyptian fractions)**.** Define $F(x) = x - 1/\lceil 1/x \rceil$ and $\xi(x) = \lceil 1/x \rceil$. Note that $O^-(0)$ is dense, so F satisfies PTT, whereas $F^n(x) \searrow 0$ for all x, so F does not satisfy TT. Note also that $O^-(x)$ is dense only for $x = 0$, and not for a dense G_δ set of x. Here, B is the set of irrationals, and $x = 1/d_1 + 1/d_2 + 1/d_3 + \ldots$ is the infinite *greedy Egyptian fraction* expansion of an irrational x. More generally, for a strictly increasing sequence $\mathbf{a} = (a_1, a_2, a_3, \ldots)$ of positive integers, $a_1 > 1$, such that $1 \leq \sum 1/a_n \leq \infty$ (e.g. the primes). Let $\lceil y \rceil_{\mathbf{a}} = a_n$, where $a_{n-1} < y \leq a_n$, and $F(x) = x - 1/\lceil 1/x \rceil_{\mathbf{a}}$. The case $a_n = 2^n$ gives binary expansions.

Example 8 (Interval exchange transformations)**.** Let ξ be an interval partition, and let ξ' be a "permutation" of ξ. Suppose there is a bijection $\varphi : \xi \to \xi'$ such that for each $\Delta \in \xi$ there is $r(\Delta) \in (-1,1)$ so that $\varphi(\Delta) = \Delta + r(\Delta)$. Define $F(x) = x + r(\Delta)$ for $x \in \Delta$ (see [14], [21]). Interval exchanges preserve

Lebesgue measure. Various conditions for ergodicity and TT are known (see [14], [27], [16]). Included here are the circle rotations F, which can be realized as 2-interval exchanges: $\xi = \{[0,\alpha),[\alpha,1)\}$ (labeled 0 and 1), with TT and ergodicity if and only if $\alpha \notin \mathbb{Q}$. The resulting F-representations are Sturmian sequences. Similarly, the von Neumann adding machine transformation F is an exchange of the partition ξ into intervals of lengths $1/2^n$, in order of decreasing length, ξ' the partition into the same intervals, but in order of increasing lengths. See [25] for applications of interval exchange transformations to F-representations. This is TT and ergodic. Up to metric isomorphism, any ergodic measure-preserving transformation F can be realized as a (usually infinite) interval exchange (see [4]). It should be noted that interval exchange transformations F differ from the other examples discussed here because they are invertible. Orientation reversing interval exchange transformations were studied in [20], but they rarely satisfy TT.

4. Parry's theorem

In this section we first state and then prove our main results about topological transitivity and valid F-expansions for piecewise interval maps F. The first result is essentially Parry's theorem [23]. Our contribution is to extend the proof to the mixed type case.

Theorem 6. *Suppose F is a PIM* (*type A, type B or mixed type*). *If F satisfies PTT, then F-representations are valid.*

Parry also proved the following partial converse, which we prove below for completeness.

Proposition 7 (Parry [23])**.** *Let F be a PIM such that $F^{-1}(0)$ includes all the endpoints of ξ except possibly* 0 *or* 1*. If F-representations are valid, then F satisfies PTT.*

Next, we state our "modern" version of Parry's theorem.

Theorem 8. *Suppose F is a PIM* (*type A, type B or mixed type*). *If F satisfies TT, then F-representations are valid.*

4.1. **Some preliminaries.**

Let F be a PIM. An open interval $I \subseteq [0,1]$ is called a *homterval* if $F^n|_I$ is continuous and strictly monotonic for each $n \geq 1$. In particular, for each $n \geq 0$, F^n is a homeomorphism between I and $F^n(I)$. There are two special kinds of homtervals. A homterval I is called a *wandering interval* if $F^n(I) \cap F^m(I) = \emptyset$ for all $m > n \geq 0$. A homterval I is called a *period-p absorbing interval* if $F^p(I) \subseteq I$, where $p \geq 1$ is as small as possible. Each $F^k(I)$, $k \geq 0$, is also a period-p absorbing interval. More generally, for $n \geq 1$ and $p \geq 1$, call a homterval I *n-pre period-p absorbing* if there exists a period-p absorbing interval J so that $I, F(I), \ldots, F^{n-1}(I)$ are pairwise disjoint, and $F^n(I) \subseteq J$, where n is as small as possible. If I is period-p absorbing, we regard it as 0-pre period-p absorbing.

Lemma 9. *If I is a homterval, then either I is a wandering interval or I is n-pre period-p absorbing, for some $n \geq 0$ and $p \geq 1$.*

Proof. Suppose I is a homterval that is not a wandering interval. Then there exist a smallest $n \geq 0$ so that $F^n(I) \cap F^{n+p}(I) \neq \emptyset$ for some $p \geq 1$, and then assume p is also as small as possible. Since I is open, $F^n(I) \cap F^{n+p}(I)$ is an interval, so $F^n(I) \cup F^{n+p}(I)$ is a homterval. Repeatedly applying F^p gives open intervals

$F^{n+\ell p}(I) \cap F^{n+(\ell+1)p}(I)$ for each $\ell \geq 0$. It follows that $J = \bigcup_{\ell=0}^{\infty} F^{n+\ell p}(I)$ is a homterval with $F^p(J) \subseteq J$. By assumption $I, F(I), \dots, F^{n-1}(I)$ are disjoint and $F^n(I) \subseteq J$. $\square$

Lemma 10. *If a PIM F satisfies TT, then there can be no homtervals.*

Proof. Suppose to the contrary that $O^+(x)$ is dense and there is a homterval I. If I is a wandering interval, then $O^+(x)$ can meet I at most once. This contradicts the density of $O^+(x)$. By Lemma 9, the only other possibility is that I is an n-pre period-p absorbing interval. Since $O^+(x)$ is dense, $F^k(x) \in O^+(x) \cap I$ for some $k \geq 0$, and then $F^{k+n}(x) \in J$, where J is period-p absorbing. We may assume without loss of generality that $n + k = 0$ so $x \in J$. We **claim** this implies that $O^+(x) \cap J$ is not dense, which is a contradiction.

To prove the claim, we first assume $p = 1$ and show $O^+(x)$ is not dense in J. Note F maps J homeomorphically onto $F(J) \subseteq J$. If $F(x) = x$, $O^+(x) = \{x\}$ is not dense, so assume $F(x) \neq x$. Either $F|_J$ is strictly increasing or decreasing. In the increasing case, for example, assume $F(x) > x$. Then the sequence $F^n(x)$ is bounded, increasing, has a limit point, so $O^+(x)$ is not. There are three more identical cases for $p = 1$. If $p > 1$, the same argument shows that the orbit of any $F^k(x)$, $k = 0, 1, \dots, p-1$, under F^p, has at most one limit point. Thus $O^+(x)$ has at most p limit points and is not dense. $\square$

Note that the limit points in the proof are p-periodic. In [26] this is described as F having a period-p *periodic attractor*. The next observation is essentially due to Parry [23].

Lemma 11. *If a PIM F satisfies PTT, then there can be no pre absorbing interval.*

Proof. It suffices to show there is no absorbing interval, so assume to the contrary that I is an absorbing interval of period p. As in the proof of Lemma 10, assume $p = 1$, so $F|_I : I \to F(I) \subseteq I$ is a homeomorphism. If $x \notin F(I)$, then $F^{-1}(x) \cap I = \emptyset$, so $O^-(x)$ cannot be dense. Thus we assume $x \in F(I)$, and show that $O^-(x)$ is not dense in $F(I)$.

Consider the homeomorphism $(F|_I)^{-1} : F(I) \to I$. We assume without loss of generality that $(F|_I)^{-1}$ is increasing (otherwise replace $F|_I$ with $(F|_I)^2$). If there is an $n > 0$ so that $(F|_I)^{-n}(x) \notin F(I)$, then $O^-(x) \cap F(I)$ is finite. Thus we assume $(F|_I)^{-n}(x) \in F(I)$ for all $n \geq 0$. One possibility is that $(F|_I)(x) = x$, but this implies $O^-(x)$ is not dense. Thus assume the case $(F|_I)(x) > x$. This implies that $(F|_I)^{-n}(x)$ is a bounded increasing sequence, has a limit, and $(F|_I)^{-n}(x)$ is not dense. $\square$

For $d_1 d_2 \dots d_n \in \mathcal{D}^n$, let $\Delta(d_1 d_2 \dots d_n) = \{x : \mathbf{r}(x)|_{[1,\dots,n]} = d_1 d_2 \dots d_n\}$. Equivalently,

$$
\begin{aligned}
\Delta(d_1 d_2 \dots d_n) &= \Delta(d_1) \cap F^{-1}\Delta(d_2) \cap \dots \cap F^{-n+1}\Delta(d_n) \\
&= \Delta(d_1) \cap F^{-1}\Delta(d_2 d_3 \dots d_n) \qquad (4.1)\\
&= \Delta(d_1 d_2 \dots d_{n-1}) \cap F^{-n+1}\Delta(d_n).
\end{aligned}
$$

By assumption (3), $\Delta(d_1 d_2 \dots d_n)$ is either empty or an interval, in which case we call it a *fundamental interval* of order n (or a *cylinder*). Let $\xi^{(n)}$ be the partition into fundamental intervals of order n, and define $||\xi^{(n)}|| = \sup\{|\Delta| : \Delta \in \xi^{(n)}\}$. It is clear that $\mathbf{r}$ is injective if and only if $||\xi^{(n)}|| \to 0$. In ergodic theory, one writes

$$\xi^{(n)} = \bigvee_{k=1}^{n} F^{-k+1}\xi,$$

and calls ξ a *generating partition* for F if $||\xi^{(n)}|| \to 0$.

Proof of Proposition 7. Denote the endpoints of ξ by $|\xi|$. By the hypotheses $|\xi| = F^{-1}(0) \cup \{0,1\}$, and similarly $|\xi^{(n)}| = \bigcup_{k=0}^{n-1} F^{-k}(0) \cup \{0,1\}$. Since F-representations are valid, $||\xi^{(n)}|| \to 0$, which implies $O^-(0) \cup \{0,1\} = \bigcup_{n \geq 1} |\xi^{(n)}|$ is dense. It follows that F is PTT. □

For $x \in B$, let $\Delta^n(x)$ be the interval in $\xi^{(n)}$ that contains x. Thus, $||\xi^{(n)}|| \not\to 0$ if and only if there exists an x so that $|\Delta^n(x)| \not\to 0$. Note that $\Delta^{n+1}(x) \subseteq \Delta^n(x)$. Define

$$\Delta(x) = \bigcap_{n \in \mathbb{N}} \Delta^n(x).$$

Either $\Delta(x)$ is a (nontrivial) interval or $\Delta(x) = \{x\}$, with the former if and only if $|\Delta^n(x)| \not\to 0$ (i.e., if and only if F-representations are not valid).

All $y \in \Delta(x)$ satisfy $\mathbf{r}(y) = \mathbf{r}(x)$ and $\Delta(y) = \Delta(x)$. When $\Delta(x)$ is a nontrivial interval, each map $(F^n)|_{\Delta(x)}$, for $n \in \mathbb{N}$, is continuous and strictly monotonic (i.e., a homeomorphism onto its range). In particular, $\Delta(x)^\circ \subseteq \Delta(x)$ is a homterval. We summarize.

Lemma 12. *If F-representations are not valid, then there exists $x \in B$ so that $\Delta(x)^\circ$ is a homterval.*

Proof of Theorem 8. Suppose F-representations are not valid. By Lemma 12 there is a homterval $\Delta(x)^\circ$, and by Lemma 10, F cannot be TT. □

4.2. **Flip lexicographic order.** Let $\mathcal{A} = \{d \in \mathcal{D} : F|_{\Delta(d)}$ is increasing$\}$ and $\mathcal{B} = \{d \in \mathcal{D} : F|_{\Delta(d)}$ is decreasing$\}$, so that $\mathcal{D} = \mathcal{A} \cup \mathcal{B}$ is a disjoint union. Note that $\mathcal{D} = \mathcal{A}$ if F is type A, and $\mathcal{D} = \mathcal{B}$ if F is type B. For two intervals $\Delta, \Delta' \in \xi$ say $\Delta < \Delta'$ if $x < x'$ for all $x \in \Delta$, $x' \in \Delta'$. This induces an order on $\mathcal{D}$ by $d < d'$ if $\Delta(d) < \Delta(d')$. This order, in turn, leads to the following order on $\mathcal{D}^{\mathbb{N}}$, called *flip lexicographic order.*

Definition 13. Suppose $\mathcal{D} \subseteq \mathbb{Z}$. Given $\mathbf{d} = .d_1d_2d_3\cdots \in \mathcal{D}^{\mathbb{N}}$, $\mathbf{e} = .e_1e_2e_3\cdots \in \mathcal{D}^{\mathbb{N}}$, with $\mathbf{d} \neq \mathbf{e}$, let $n = \min\{j \geq 1 : d_j \neq e_j\}$. Let $p = 0$ if $n = 1$ and otherwise $p = \#\{j = 1, \ldots, n-1 : d_j = e_j \in \mathcal{B}\}$. Define $\mathbf{d} \prec \mathbf{e}$ if $d_n < e_n$ and p is even, or if $d_n > e_n$ and p is odd. Otherwise, define $\mathbf{e} \prec \mathbf{d}$. We will write $\mathbf{d} \preceq \mathbf{d}$ if $\mathbf{d} \prec \mathbf{e}$ or $\mathbf{d} = \mathbf{e}$.

If F is type A, this is lexicographic order, and if F is type B, it is alternating lexicographic order. Parry's proof [23] of Theorem 6 assumes one of these two cases. Flip lexicographic order appears in [18].

Lemma 14. *If $x < y$, then $\mathbf{r}(x) \preceq \mathbf{r}(y)$. Conversely, if $\mathbf{r}(x) \prec \mathbf{r}(y)$, then $x < y$. In particular, if $\mathbf{r}(x) \neq \mathbf{r}(y)$, then $x \neq y$.*

Proof. Let $x < y$ and $\mathbf{d} = \mathbf{r}(x)$ and $\mathbf{e} = \mathbf{r}(y)$. One possibility is that $y \in \Delta(x)$, so $\Delta(x) = \Delta(y)$, in which case $\mathbf{d} = \mathbf{e}$. Otherwise there is a smallest $n \geq 1$ so that $\Delta^n(x) \neq \Delta^n(y)$. If $n = 1$, then $\Delta^1(x) = \Delta(d_1) < \Delta(e_1) = \Delta^1(y)$, so $d_1 < e_1$. Since $p = 0$, this implies $\mathbf{r}(x) \prec \mathbf{r}(y)$. If $n > 1$, then $x, y \in \Delta(d_1 d_2 \dots d_{n-1})$ and $p \leq n-1$. If p is even, $F^{n-1}|_{\Delta(d_1 d_2 \dots d_{n-1})}$ is increasing, and since $x < y$, $F^{n-1}(x) < F^{n-1}(y)$. We then have $\Delta(d_n) < \Delta(e_n)$ so that $d_n < e_n$. This implies $\mathbf{d} \prec \mathbf{e}$ since p is even. If, on the other hand, p is odd, then $F^{n-1}|_{\Delta(d_0 d_1 \dots d_{n-1})}$ is decreasing, and $x < y$ implies $F^{n-}(y) < F^{n-1}(x)$, which implies $\Delta(e_n) < \Delta(d_n)$ and $e_n < d_n$. Since p is odd, this still implies $\mathbf{d} \prec \mathbf{e}$.

Conversely, suppose $\mathbf{r}(x) \prec \mathbf{r}(y)$. If $d_1 < e_1$, then $\Delta(d_1) < \Delta(e_1)$ and $x < y$. Now suppose $x, y \in \Delta(d_1 d_2 \dots d_{n-1})$, but $d_n \neq e_n$. Since $\mathbf{x} \prec \mathbf{y}$, we have $d_n < e_n$ if p is even and $e_n < d_n$ if p is odd. In the first case we have $F^n(x) < F^n(y)$ and in the second, $F^n(y) < F^n(x)$ (because $F^n(x) \in \Delta(x_n)$, and likewise for y). Note that $F^n|_{\Delta(x_0, x_1, \dots, x_{n-1})}$ is continuous, and either increasing or decreasing, depending on whether p is even or odd. In both cases, this implies $x < y$. $\square$

Lemma 15. *Let F satisfy PTT, and let x be such that $O^-(x)$ is dense in $[0,1]$. Then $\Delta(x) = \{x\}$.*

Proof. If $\Delta(x) \neq \{x\}$, then by Lemma 12, $\Delta(x)^\circ$ is a homterval. Since F satisfies PTT, Lemma 11 implies $\Delta(x)^\circ$ cannot be an absorbing interval, so by Lemma 9, $\Delta(x)^\circ$ must be a wandering interval. We show this is impossible.

By (1.1), $F^n(\Delta(x)) \cap F^m(\Delta(x)) = \emptyset$ for all $m > n \geq 0$ implies $F^{-m}(\Delta(x)) \cap F^{-n}(\Delta(x)) = \emptyset$ for all $n > m \geq 0$. Now $F^{-n}(x) \subseteq F^{-n}(\Delta(x))$. It follows that $O^-(x) = \bigcup_{n \geq 0} F^{-n}(x)$ cannot be dense in $[0,1]$. Thus $\Delta(x) = \{x\}$. $\square$

Proof of Theorem 6. First note that $\Delta(z) = \{z\}$ whenever $Fz = y$ and $\Delta(y) = \{y\}$. Thus for any x with $O^-(x)$ dense, $z \in O^-(x)$ implies $\Delta(z) = \{z\}$.

Let $u < v$ and take $y, z \in O^-(x)$ so that $u < y < z < v$. By Lemma 14, $\mathbf{r}(u) \preceq \mathbf{r}(y) \prec \mathbf{r}(z) \preceq \mathbf{r}(v)$, so that $\mathbf{r}(u) \prec \mathbf{r}(v)$. Then by Lemma 14 again, $\mathbf{r}(u) \neq \mathbf{r}(v)$. $\square$

5. f-EXPANSIONS AND A GENERALIZATION

Given a PIM F, define the F-*shift*

$$X = \overline{\{\mathbf{r}(x) : x \in B\}} \subseteq \mathcal{D}^{\mathbb{N}},$$

with the left shift map S. Indeed, this is a one-sided shift since $S(\mathbf{r}(x)) = \mathbf{r}(F(x))$. Let $\widetilde{X}$, with $\widetilde{T}$, be the two-sided natural extension of X, and let $\mathcal{L}$ be the common language.

Lemma 16. *A word $d_1 d_2 \dots d_n \in \mathcal{L}$ if and only if $\Delta(d_1 d_2 \dots d_n)$ is an interval, or equivalently, $\Delta(d_1 d_2 \dots d_n)^\circ \neq \emptyset$.*

Proof. Note that $w \in \mathcal{L}$ if and only if $w = \mathbf{r}(x)_{[1,2,\dots,n]} = .d_1 d_2 \dots d_n$ for some $x \in B$. Then by (3), $\Delta(d_1 d_2 \dots d_n)$ is an interval. Conversely, suppose $\Delta(d_1 d_2 \dots d_n)$ is an interval. Let $x \in B \cap \Delta(d_1 d_2 \dots d_n)$. Then $.d_1 d_2 \dots d_n = \mathbf{r}(x)|_{[1,2,\dots,n]} \in \mathcal{L}$ since $\mathbf{r}(x) \in X$. $\square$

For $w = d_1 d_2 \dots d_n \in \mathcal{L}$, let $\overline{\Delta}(d_1 d_2 \dots d_n) = [a_n, b_n]$, so $\Delta(d_1 d_2 \dots d_n)^\circ = (a_n, b_n)$. Note that $\overline{\Delta}(d_1 d_2 \dots d_n) \subseteq \overline{\Delta}(d_1 d_2 \dots d_{n-1})$. Thus if F-representations are valid, $|\overline{\Delta}(d_1d_2\dots d_n)|\to 0$ as $n\to\infty$ for any $\mathbf{d}=.d_1d_2d_3\dots\in X$. Then

$$\{x\}=\bigcap_n\overline{\Delta}(d_1d_2\dots d_n)$$

and we define $E(\mathbf{d})=x$. If $\mathbf{d}=\mathbf{r}(x)$ for $x\in B$, then $x\in\Delta(d_1d_2\dots d_n)$ for all n, so in this case, $E(\mathbf{r}(x))=x$. We summarize.

Proposition 17. *Suppose F-representations are valid. Then there exists $E:X\to[0,1]$ so that for $\mathbf{d}=.d_1d_2d_3\dots\in X$, $\{E(\mathbf{d})\}=\bigcap_n\overline{\Delta}(d_1d_2\dots d_n)$. In particular, then $E(\mathbf{d})=\lim_n a_n=\lim_n b_n$. If $x\in B$ and $\mathbf{d}=\mathbf{r}(x)$, then $E(\mathbf{d})=x$.*

Lemma 18. *If $O^+(x)$ is dense and $\Delta(d_1d_2\dots d_n)^\circ\neq\emptyset$, then $\{N:F^N(x)\in\Delta(d_1d_2\dots d_n)^\circ\}$ is infinite.*

Proof. Since $O^+(x)$ is dense, and $\Delta(d_1d_2\dots d_n)^\circ$ is nonempty and open, there exists smallest $k_1\geq 0$ so that $F^{k_1}(x)\in\Delta(d_1d_2\dots d_n)^\circ$. We show there exists $k_2>k_1$ so that $F^{k_2}(x)\in\Delta(d_1d_2\dots d_n)^\circ$.

We know that $\mathbf{r}(F^{k_1}(x))|_{[1,2,\dots,n]}=.d_1d_2\dots d_n$ and $\Delta(d_1d_2\dots d_nd_{n+1}\dots d_m)^\circ\subseteq\Delta^m(F^{k_1}(x))$ for all $m>n$. Since $O^+(x)$ is dense, F satisfies TT, and thus Theorem 8 implies F-representations are valid. This implies that $|\Delta^m(F^{k_1}(x))|\to 0$ as $m\to\infty$. It follows that for some $m>n$, which we choose as small as possible, the inclusion $\Delta(d_1d_2\dots d_m)^\circ\subseteq\Delta(d_1d_2\dots d_n)^\circ$ is proper, and $F^{k_1}(x)\in\Delta(d_1d_2\dots d_m)^\circ$. Then there exists $e_m\neq d_m$ so that $\Delta(d_1d_2\dots d_{m-1}e_m)^\circ\neq\emptyset$, $\Delta(d_1d_2\dots d_{m-1}e_m)^\circ\subseteq\Delta(d_1d_2\dots d_m)^\circ$ and $F^\ell(x)\notin\Delta(d_1d_2\dots d_{m-1}e_m)^\circ$ for any $\ell=0,1,\dots,k_1$. Then there is a $k_2>k_1$ so that $F^{k_2}(x)\in\Delta(d_1d_2\dots d_{m-1}e_m)^\circ\subseteq\Delta(d_1d_2\dots d_n)^\circ$. □

Proposition 19. *If F satisfies TT, then so does the corresponding F-shift X, and its natural extension $\widetilde{X}$ satisfies TTT.*

Proof. For $w_1=d_1d_2\dots d_m, w_2=e_1e_2\dots e_k\in\mathcal{L}$, one has $\Delta(w_1)^\circ,\Delta(w_2)^\circ\neq\emptyset$. Choose $x\in B$ so that $O^+(x)$ is dense. By Lemma 18 there exist $k_2>k_1+m_1$ so that $F^{k_1}(x)\in\Delta(w_1)^\circ$ and $F^{k_2}(x)\in\Delta(w_2)^\circ$ so that $F^{k_2-k_1}(\Delta(w_1)^\circ)\cap\Delta(w_2)^\circ\neq\emptyset$. Equivalently, $w_1uw_2\in\mathcal{L}$ for some $u\in\mathcal{L}$. □

Fixing $d\in\mathcal{D}$, let $\overline{\Delta}(d)=[a_d,b_d]$, $\alpha_d=\lim_{x\to a_d^+}F(x)$ and $\beta_d=\lim_{x\to b_d^-}F(x)$. Define $f_d:[0,1]\to[0,1]$ by

$$f_d(x)=\begin{cases}a_d & \text{if } 0\leq x<F(\alpha_d),\\ (F|_{\Delta(d)})^{-1}(x) & \text{if } F(\alpha_d)\leq x<F(\beta_d),\\ \beta_d & \text{if } F(\beta)\leq x<1.\end{cases}\tag{5.1}$$

Each f_d is continuous because $F|_{\Delta(d)}:\Delta(d)\to[0,1]$ is continuous and strictly monotonic.

Lemma 20. *If $d_1d_2\dots d_n\in\mathcal{L}$, then*

$$\overline{\Delta}(d_1d_2\dots d_n)=f_{d_1}(f_{d_2}(\dots f_{d_n}([0,1])\dots)).$$

Proof. For $n = 1$ we have $f_{d_1}([0,1]) = [a_1, b_1] = \overline{\Delta}(d_1)$. Suppose

$$f_{d_2}(f_{d_3}(\dots f_{d_n}([0,1])\dots)) = \overline{\Delta}(d_2 d_3 \dots d_n) = [a', b'],$$

where $b' > a'$. Note that a' and b' are

$$f_{d_2}(f_{d_3}(\dots f_{d_n}(0)\dots)) \quad \text{and} \quad f_{d_2}(f_{d_3}(\dots f_{d_n}(1)\dots))$$

(in one order or the other). Then

$$f_{d_1}(f_{d_2}(\dots f_{d_n}([0,1])\dots)) = f_{d_1}(\overline{\Delta}(d_2 d_3 \dots d_n)) = f_{d_1}([a', b']).$$

Now for any interval $[a', b']$ and any $d \in \mathcal{D}$, (5.1) implies that $f_d([a', b']) = F^{-1}([a', b']) \cap \overline{\Delta}(d)$. The result now follows by (4.1). $\square$

Theorem 21. *Let F be a PIM such that F-representations are valid. Then for Lebesgue almost every $x \in [0,1]$ (i.e., for $x \in B$)*

$$x = E(\mathbf{x}) = \lim_{n\to\infty} f_{d_0}(f_{d_1}(\dots f_{d_n}(0)\dots)) = \lim_{n\to\infty} f_{d_0}(f_{d_1}(\dots f_{d_n}(1)\dots)), \tag{5.2}$$

where $\mathbf{x} = .d_0 d_1 d_2 \cdots = \mathbf{r}(x)$.

For $.d_1 d_2 d_3 \cdots \in \mathcal{D}^{\mathbb{N}}$ we call the limits (5.2) *generalized f-expansions.* Theorem 21 can be interpreted as saying that if F-representations are valid, then a.e. f-expansion converges to "what it should". This occurs whenever F satisfies either TT or PTT.

Traditionally, additional assumptions on F allow (5.2) to be expressed in a simpler form. We say F (i.e., the digit set $\mathcal{D}$, possibly relabeled) is *well ordered* if $\mathcal{D} \subseteq \mathbb{Z}$ and $\Delta(d) < \Delta(e)$ if and only if $d < e$. An example that is not well ordered is the Cantor transformation F in Example 6. If F is well ordered, we define $f : \mathbb{R} \to [0,1]$ by $f(x) = f_d(x - d)$ if $x \in [d, d+1)$ for each $d \in \mathcal{D}$. We extend f to a function $f : \mathbb{R} \to [0,1]$ by defining $f(x) = f(a)$ for all $x < a$, where $\Delta(d) = [a, b)$ is the left-most fundamental interval, and $f(x) = f(b)$ if $[a, b)$ is the first fundamental interval smaller than x. This is most natural if F is either type A or type B, in which case f is continuous, and respectively either (not necessarily strictly) increasing or decreasing.

If we restrict the function f, as defined above, to the intervals in $\mathbb{R}$ on which it is strictly monotonic, then f^{-1} exists, and we have

$$F(x) = f^{-1}(x) \bmod 1.$$

This is a traditional starting point for the theory (see [13], [23]). Equivalently, we can view f as the inverse of the function $F(x) + \xi(x)$ (where $\xi : D \to \mathcal{D} \subseteq \mathbb{Z}$ is the digit function).

Given $.d_1 d_2 d_3 \cdots \in \mathcal{D}^{\mathbb{N}}$ we define the (classical) f-expansion by

$$f(d_1 + f(d_2 + f(d_3 + \dots))).$$

In particular, we understand this expression to be the limit

$$\lim_{n\to\infty} f(d_1 + f(d_2 + f(d_3 + \dots f(d_n)\dots))).$$

Theorem 22. *Suppose F is a well ordered PIM such that F-representations are valid (i.e., if F satisfies either TT or PTT). Then f-expansions are* valid *in the sense that for λ-a.e $x \in [0,1]$ (i.e., for $x \in B$), $\mathbf{r}(x) = .d_1 d_2 d_3 \cdots \in \mathcal{D}^{\mathbb{N}}$ and*

$$x = f(d_1 + f(d_2 + f(d_3 + \dots))).$$

We also have $x = \lim_{n\to\infty} f(d_1 + f(d_2 + f(d_3 + \dots f(d_n + 1)\dots)))$.

6. Topological transitivity implies Parry topological transitivity

We can now prove our main result.

Theorem 23. *If F is a piecewise interval map (PIM) that satisfies TT, then it satisfies PTT.*

Proof. Since F satisfies TT, Proposition 19 implies that the two-sided F-shift $\widetilde{X}$ satisfies TTT. Let $\mathbf{y} \in \widetilde{X}$ be such that $O^-(\mathbf{y})$ is dense. For each $n \geq 0$, let $\mathbf{y}_n = \widetilde{S}^{-n}(\mathbf{y})$ and $\mathbf{x}_n = \pi_+(\mathbf{y}_n)$. Here $\pi_+ : \widetilde{X} \to X$ is the factor map $\pi_+(\dots d_{-1}d_0.d_1d_2\dots) = .d_1d_2\dots$. Note that $\pi_+(\widetilde{S}(\mathbf{y})) = S(\pi_+(\mathbf{y}))$, so $S^n(\mathbf{x}_n) = S^n(\pi_+(\mathbf{y}_n)) = \pi_+(\widetilde{S}^n(\mathbf{y}_n)) = \pi_+(\mathbf{y}) = \mathbf{x}$. Let $x_n := E(\mathbf{x}_n)$, which exists by Theorems 8 and 21. If $x := x_0$, then $F^n(x_n) = x$, so $B = \{x_0, x_1, x_2, \dots\}$ is a backward orbit for x. If $(\widetilde{S}^{-n}(\mathbf{y}))|_{[1,2,\dots,m]} = d_1d_2\dots d_m$, then $x_n \in \overline{\Delta}(d_1d_2\dots d_m)$. Since $O^-(\mathbf{y})$ is dense, B is dense too, and so F satisfies PTT. □

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