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NONCLASSIFIABILITY OF UHF L^p-OPERATOR ALGEBRAS

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Abstract. For $p \in [1, \infty)$, we prove that simple, separable, monotracial UHF L^p-operator algebras are not classifiable up to (complete) isomorphism using countable structures, such as K-theoretic data, as invariants. The same assertion holds even if one only considers UHF L^p-operator algebras of tensor product type obtained from a diagonal system of similarities. For $p = 2$, it follows that separable nonselfadjoint UHF operator algebras are not classifiable by countable structures up to (complete) isomorphism. Our results, which answer a question of N. Christopher Phillips, rely on Borel complexity theory, and particularly Hjorth's theory of turbulence.

1. Introduction

Suppose that X is a standard Borel space and λ is a Borel probability measure on X. For $p \in [1, \infty)$, we denote by $L^p(\lambda)$ the Banach space of Borel-measurable complex-valued functions on X (modulo null sets), endowed with the L^p-norm. Let $B(L^p(\lambda))$ denote the Banach algebra of bounded linear operators on $L^p(\lambda)$ endowed with the operator norm. We will identify the Banach algebra $M_n(B(L^p(\lambda)))$ of $n \times n$ matrices with entries in $B(L^p(\lambda))$, with the algebra $B(L^p(\lambda)^{\oplus n})$ of bounded linear operators on the p-direct sum $L^p(\lambda)^{\oplus n}$ of n copies of $L^p(\lambda)$.

A (concrete) separable, unital *L^p-operator algebra*, is a separable, closed subalgebra of $B(L^p(\lambda))$ containing the identity operator. (Such a definition is consistent with Definition 1.1 in [P2], in view of Proposition 1.25 in [P2].) In the following, all L^p-operator algebras will be assumed to be separable and unital. Every unital L^p-operator algebra $A \subseteq B(L^p(\lambda))$ is in particular a p-operator space in the sense of Section 4 in [D], with matrix norms obtained by identifying $M_n(A)$ with a subalgebra of $M_n(B(L^p(\lambda)))$. Such algebras have been introduced and studied by N. Christopher Phillips in [P1, P3, P4]. Many important classes of C*-algebras have been shown to have L^p-analogs, including Cuntz algebras [P1], UHF algebras [P3], AF algebras [PV], and more generally groupoid C*-algebras [GL].

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If A is a unital complex algebra, then an *L^p-representation* of A on a standard Borel probability space (X, λ) is a unital algebra homomorphism $\rho\colon A \to B(L^p(\lambda))$. The closure inside $B(L^p(\lambda))$ of $\rho(A)$ is an L^p-operator algebra, called the *L^p-operator algebra associated with ρ*. It can be identified with the completion of A with respect to the operator seminorm structure $\|[a_{ij}]\|_\rho = \|[\rho(a_{ij})]\|_{M_n(B(L^p(\lambda)))}$ for $[a_{ij}] \in M_n(A)$; see [BLM, 1.2.16]. If A and B are L^p-operator algebras, a *unital homomorphism* $\varphi\colon A \to B$ is an algebra homomorphism such that $\varphi(1) = 1$. The n-th *amplification* $\varphi^{(n)}\colon M_n(A) \to M_n(B)$ is defined by $[a_{ij}] \mapsto [\varphi(a_{ij})]$. A unital homomorphism φ is *completely bounded* if every amplification $\varphi^{(n)}$ is bounded and its completely bounded norm

$$\|\varphi\|_{\mathrm{cb}} = \sup_{n\in\mathbb{N}} \left\|\varphi^{(n)}\right\|$$

is finite.

Definition 1.1. Let A and B be unital L^p-operator algebras.

(1) A and B are said to be *(completely) isomorphic* if there is a (completely) bounded unital isomorphism $\varphi\colon A \to B$ with (completely) bounded inverse $\varphi^{-1}\colon B \to A$.
(2) A and B are said to be *(completely) commensurable* if there are (completely) bounded unital homomorphisms $\varphi\colon A \to B$ and $\psi\colon B \to A$.

For $d \in \mathbb{N}$, we denote by M_d the unital algebra of $d \times d$ complex matrices, with matrix units $\{e_{i,j}\}_{1\le i,j\le d}$. Let $\boldsymbol{d} = (d_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{N}$, and let $\boldsymbol{\rho} = (\rho_n)_{n\in\mathbb{N}}$ be a sequence of representations $\rho_n\colon M_{d_n} \to B(L^p(X_n, \lambda_n))$. Define $M_{\boldsymbol{d}}$ to be the algebraic infinite tensor product $\bigotimes_{n\in\mathbb{N}} M_{d_n}$. Let $X = \prod_{n\in\mathbb{N}} X_n$ be the product Borel space and $\lambda = \bigotimes_{n\in\mathbb{N}} \lambda_n$ be the product measure. We naturally regard the algebraic tensor product $\bigotimes_{n\in\mathbb{N}} B(L^p(\lambda_n))$ as a subalgebra of $B(L^p(\lambda))$. The correspondence

$$M_{\boldsymbol{d}} \to \bigotimes_{n\in\mathbb{N}} B(L^p(\lambda_n)) \subseteq B(L^p(\lambda))$$
$$a_1 \otimes \cdots \otimes a_k \mapsto \rho_1(a_1) \otimes \cdots \otimes \rho_k(a_k),$$

extends to a unital homomorphism $\rho\colon M_{\boldsymbol{d}} \to B(L^p(\lambda))$.

Definition 1.2 ([P1, Definition 3.9])**.** Let $\boldsymbol{d} = (d_n)_{n\in\mathbb{N}}$ be a sequence of positive integers, and let $\boldsymbol{\rho} = (\rho_n)_{n\in\mathbb{N}}$ be a sequence of unital homomorphisms $\rho_n\colon M_{d_n} \to B(L^p(\lambda_n))$. Let $\rho\colon M_{\boldsymbol{d}} \to B(L^p(\lambda))$ be the unital homomorphism described above. The UHF L^p-operator algebra $A(\boldsymbol{d}, \boldsymbol{\rho})$ associated with $\boldsymbol{\rho}$ is defined as the closure of $\rho(M_{\boldsymbol{d}})$ inside of $B(L^p(\lambda))$.

A Banach algebra A is said to be a *UHF L^p-operator algebra of tensor product type $\boldsymbol{d}$* if there exists a sequence $\boldsymbol{\rho}$ such that A is isometrically isomorphic to $A(\boldsymbol{d}, \boldsymbol{\rho})$.

A special class of UHF L^p-operator algebras of tensor product type has been introduced in [P3, Section 5]. For $d \in \mathbb{N}$, denote by c_d the normalized counting measure on $d = \{0, 1, 2, \ldots, d-1\}$, and set $\ell^p(d) = L^p(\{0, \ldots, d-1\}, c_d)$. The *(canonical) spatial representation* σ^d of M_d on $\ell^p(d)$ is defined by setting

$$\left(\sigma^d(a)\xi\right)(j) = \sum_{i=0}^{d-1} a_{ij}\xi(i)$$

for $a \in M_d$, for $\xi \in \ell^p(d)$, and $j = 0, \ldots, d-1$; see [P1, Definition 7.1]. Observe that the corresponding matrix norms on M_d are obtained by identifying M_d with the algebra of bounded linear operators on $\ell^p(d)$.

Fix a real number α in $[1, +\infty)$, and an enumeration $(w_{d,\alpha,k})_{k\in\mathbb{N}}$ of all diagonal $d \times d$ matrices with entries in $[1, \alpha] \cap \mathbb{Q}$. Let X be the disjoint union of countably many copies of $\{0, 1, \ldots, d-1\}$, and let λ_d be the Borel probability measure on X that agrees with $2^{-k}c_d$ on the k-th copy of $\{0, 1, \ldots, d-1\}$. We naturally identify the algebraic direct sum $\bigoplus_{n\in\mathbb{N}} B(\ell^p(d))$ with a subalgebra of $B(L^p(\lambda_d))$. The map

$$\begin{aligned} M_d &\to \bigoplus_{n\in\mathbb{N}} B(\ell^p(d)) \subseteq B(L^p(\lambda_d)) \\ x &\mapsto \left(\sigma^d\left(w_{d,\alpha,k}\, x\, w_{d,\alpha,k}^{-1}\right)\right)_{k\in\mathbb{N}} \end{aligned}$$

defines a representation $\rho^\alpha \colon M_d \to B(L^p(\lambda_d))$. We denote the corresponding Banach algebra structure on M_d by M_d^α.

For a sequence $\boldsymbol{\alpha} = (\alpha_n)_{n\in\mathbb{N}}$ in $[1, +\infty)$, we will denote by $\boldsymbol{\rho^\alpha}$ the sequence of representations $\rho^{\alpha_n} \colon M_{d_n} \to B(L^p(\lambda_{d_n}))$ described in the paragraph above. Following the terminology in [P3, Section 3 and Section 5], we say that the corresponding UHF L^p-operator algebras $A(\boldsymbol{d}, \boldsymbol{\rho^\alpha})$ are *obtained from a diagonal system of similarities* . When the type $\boldsymbol{d} = (d_n)_{n\in\mathbb{N}}$ is clear from the context, we will simply write $A^{\boldsymbol{\alpha}}$ for $A(\boldsymbol{d}, \boldsymbol{\rho^\alpha})$.

Definition 1.3. If A is a unital Banach algebra, a *normalized trace* on A is a continuous linear functional $\tau \colon A \to \mathbb{C}$ with $\tau(1) = 1$, satisfying $\tau(ab) = \tau(ba)$ for all $a, b \in A$. The algebra A is said to be *monotracial* if it has a unique normalized trace.

Recall that a Banach algebra is said to be *simple* if it has no nontrivial closed two-sided ideals.

Remark 1.4. It was shown in [P4, Theorem 3.19(3)] that UHF L^p-operator algebras obtained from a diagonal system of similarities are always simple and monotracial.

Problem 5.15 of [P3] asks to provide invariants which classify, up to isomorphism, some reasonable class of UHF L^p-operator algebras, such as those constructed using diagonal similarities. The following is the main result of the present paper.

Theorem 1.5. *The simple, separable, monotracial UHF L^p-operator algebras are not classifiable by countable structures up to any of the following equivalence relations:*

(1) *complete isomorphism;*
(2) *isomorphism;*
(3) *complete commensurability;*
(4) *commensurability.*

The same conclusions hold even if one only considers UHF L^p-operator algebras of tensor product type $\boldsymbol{d}$ obtained from a diagonal system of similarities for a fixed sequence $\boldsymbol{d} = (d_n)_{n\in\mathbb{N}}$ of positive integers such that, for every distinct $n, m \in \mathbb{N}$, neither d_n divides d_m nor d_m divides d_n.

It follows from Theorem 1.5 that simple, separable, monotracial UHF L^p-operator algebras are not classifiable by K-theoretic data, even after adding to the K-theory a countable collection of invariants consisting of countable structures. When $p = 2$, Theorem 1.5 asserts that separable nonselfadjoint UHF operator algebras are not classifiable by countable structures up to isomorphism. This conclusion is in stark constrast with Glimm's classification of UHF C*-algebras by their corresponding supernatural number [G].

In fact, two UHF C*-algebras are $*$-isomorphic if and only if they are isomorphic as Banach algebras (with not necessarily contractive isomorphism). For the nontrivial implication, note that any Banach algebra isomorphism $\varphi\colon A \to B$ induces an (order) isomorphism $K_0(\varphi)\colon K_0(A) \to K_0(B)$, simply by functoriality of K_0. Now, by Glimm's classification, if two UHF C*-algebras have isomorphic K_0-groups, then there exists a $*$-isomorphism between them.

The argument above really shows that two UHF C*-algebras are $*$-isomorphic if and only if they are isomorphic as unital rings, because the K_0-group of a C*-algebra depends only on its ring structure. (Of course, this does not mean that any ring isomorphism is automatically a $*$-isomorphism.)

For the convenience of the reader, we recall the definition of the K_0-group of a unital ring R. Recall that an element $e \in R$ is said to be an *idempotent* if $e^2 = e$.

Definition 1.6. Let R be a unital ring. Two idempotents e and f in R are said to be *similar* if there exists an invertible element $s \in R$ satisfying $ses^{-1} = f$.

Define the *monoid of projections* $V(R)$ of R to be the set of equivalence classes of idempotents in $\bigcup_{n\in\mathbb{N}} M_n(R)$, where we identify e with $e \oplus 0$, with addition given by direct sum. The K_0-group $K_0(R)$ of R is the Grothendieck group of $V(R)$. The *positive cone* $K_0(R)_+$ of $K_0(R)$ is the image of $V(R)$ in $K_0(R)$ under the canonical Grothendieck map.

It is easy to see that K_0 is a functor from the category of unital rings, with unital homomorphisms, to the category of abelian groups with a positive cone, with positive group homomorphisms which preserve a distinguished element (the class of the unit).

As an easy example, the rank determines an isomorphism $K_0(M_n) \cong \mathbb{Z}$, with the class of the unit corresponding to $n \in \mathbb{Z}$.

2. Borel complexity theory

In order to obtain our main result, we will work in the framework of Borel complexity theory. In such a framework, a classification problem is regarded as an equivalence relation E on a standard Borel space X. If F is another equivalence relation on another standard Borel space Y, a *Borel reduction* from E to F is a Borel function $g\colon X \to Y$ with the property that

$$xEx' \quad \text{if and only if} \quad g(x)Fg(x').$$

The map g can be seen as a classifying map for the objects of X up to E. The requirement that g is Borel captures the fact that g is *explicit* and *constructible* (and not, for example, obtained by using the Axiom of Choice). The relation E is *Borel reducible* to F if there is a Borel reduction from E to F. This can be interpreted as asserting that is it possible to explicitly classify the elements of X up to E using F-classes as invariants.

The notion of Borel reducibility provides a way to compare the complexity of classification problems in mathematics. Some distinguished equivalence relations are then used as benchmarks of complexity. The first such benchmark is the relation $=_{\mathbb{R}}$ of equality of real numbers. (One can replace $\mathbb{R}$ with any other Polish space.) An equivalence relation is called *smooth* if it is Borel reducible to $=_{\mathbb{R}}$. Equivalently, an equivalence relation is smooth if its classes can be explicitly parametrized by the points of a Polish space. For instance, the above mentioned classification of UHF C*-algebras due to Glimm [G] shows that the classification problem of UHF C*-algebras is smooth. Smoothness is a very restrictive notion, and many natural classification problems transcend such a benchmark. For instance, the relation of isomorphism of rank 1 torsion-free abelian groups is not smooth; see [H3].

A more generous notion of classifiability is being *classifiable by countable structures.* Informally speaking, an equivalence relation E on a standard Borel space X is classifiable by countable structures if it is possible to explicitly assign to the elements of X complete invariants up to E that are countable structures, such as countable (ordered) groups, countable (ordered) rings, et cetera. To formulate this definition precisely, let $\mathcal{L}$ be a countable first-order language [M, Definition 1.1.1]. The class $\mathrm{Mod}(\mathcal{L})$ of $\mathcal{L}$-structures supported by the set $\mathbb{N}$ of natural numbers can be regarded as a Borel subset of $\prod_{n\in\mathbb{N}} 2^{\mathbb{N}^n}$. As such, $\mathrm{Mod}(\mathcal{L})$ inherits a Borel structure making it a standard Borel space. Let $\cong_{\mathcal{L}}$ be the relation of isomorphism of elements of $\mathrm{Mod}(\mathcal{L})$.

Definition 2.1. An equivalence relation E on a standard Borel space is said to be *classifiable by countable structures*, if there exists a countable first-order language $\mathcal{L}$ such that E is Borel reducible to $\cong_{\mathcal{L}}$.

The Elliott-Bratteli classification of AF C*-algebras [E, B] shows, in particular, that AF C*-algebras are classifiable by countable structures up to $*$-isomorphism. Any smooth equivalence relation is in particular classifiable by countable structures.

Many naturally occurring classification problems in mathematics, and particularly in functional analysis and operator algebras, have recently been shown to transcend countable structures. This has been obtained for the relation of unitary conjugacy of irreducible representations and automorphisms of non-type I C*-algebras [H1, KLP1, F, L], conjugacy of ergodic measure-preserving transformations of the Lebesgue measure space [FW], conjugacy of automorphisms of $\mathcal{Z}$-stable C*-algebras and McDuff II_1 factors [KLP2], unitary conjugacy of unitary and self-adjoint operators [KS], and isomorphism of von Neumann factors [ST1, ST2]. The main tool involved in these results is the theory of turbulence developed by Hjorth in [H2].

Suppose that $G \curvearrowright X$ is a continuous action of a Polish group G on a Polish space X. The corresponding orbit equivalence relation E_G^X is the relation on X obtained by setting $xE_G^X x'$ if and only if x and x' belong to the same orbit. Hjorth's theory of turbulence provides a dynamical condition, called *(generic) turbulence*, that ensures that a Polish group action $G \curvearrowright X$ yields an orbit equivalence relation E_G^X that is not classifiable by countable structures. This provides, directly or indirectly, useful criteria to prove that a given equivalence relation is not classifiable by countable structures. A prototypical example of turbulent group action is the action of ℓ^1 on $\mathbb{R}^{\mathbb{N}}$ by translation. A standard argument allows one to deduce the following nonclassification criterion from turbulence of the action $\ell^1 \curvearrowright \mathbb{R}^{\mathbb{N}}$ and Hjorth's turbulence theorem [H2, Theorem 3.18]; see, for example, [L, Lemma 3.2 and Criterion 3.3].

Recall that a subspace of a topological space is *meager* if it is contained in the union of countably many closed nowhere dense sets.

Criterion 2.2. Suppose that E is an equivalence relation on a standard Borel space X. If there is a Borel map $f\colon [0,+\infty)^{\mathbb{N}} \to X$ such that

(1) $f(\boldsymbol{t})Ef(\boldsymbol{t}')$ whenever $\boldsymbol{t},\boldsymbol{t}' \in [0,+\infty)^{\mathbb{N}}$ satisfy $\boldsymbol{t}-\boldsymbol{t}' \in \ell^1$, and
(2) the preimage under f of any E-class is meager,

then E is not classifiable by countable structures.

We will apply this criterion to establish our main result.

3. Nonclassification

Fix a sequence $\boldsymbol{d} = (d_n)_{n\in\mathbb{N}}$ of integers such that for every distinct $n,m \in \mathbb{N}$, neither d_n divides d_m nor d_m divides d_n. In particular, this holds if the numbers d_n are pairwise coprime. The same argument works if one only assumes that all but finitely many values of $\boldsymbol{d}$ satisfy such an assumption. We endow $[1,+\infty)^{\mathbb{N}}$ with the product topology, and regard it as the parametrizing space for UHF L^p-operator algebras of type $\boldsymbol{d}$ obtained from a diagonal system of similarities, as described in the previous section; see also [P3, Section 3 and Section 5]. We therefore regard (complete) isomorphism and (complete) commensurability of UHF L^p-operator algebras of type $\boldsymbol{d}$, obtained from a diagonal system of similarities, as equivalence relations on $[1,+\infty)^{\mathbb{N}}$.

For $\boldsymbol{\alpha} \in [1,+\infty)^{\mathbb{N}}$, we denote by $A^{\boldsymbol{\alpha}}$ the corresponding UHF L^p-operator algebra; see the comments before Definition 1.3. For $\alpha \in [1,+\infty)$, the corresponding matrix norms on M_d^α are denoted by $\|\cdot\|_\alpha$. In particular, when $\alpha = 1$, one obtains the matrix norms induced by the spatial representation σ^d of M_d. The algebra $A^{\boldsymbol{\alpha}}$ can be seen as the L^p-operator tensor product $\bigotimes^p_{n\in\mathbb{N}} M_{d_n}^{\alpha_n}$, as defined in [P4, Definition 1.9]. (Note that, unlike in [P4], we write the Hölder exponent p as a superscript in the notation for tensor products.)

Lemma 3.1. *Let $\boldsymbol{\alpha},\boldsymbol{\alpha}' \in [1,+\infty)^{\mathbb{N}}$ satisfy*

$$L := \prod_{n\in\mathbb{N}} \frac{\alpha_n}{\alpha'_n} < +\infty.$$

Then the identity map on the algebraic tensor product $M_{\boldsymbol{d}} = \bigotimes_{n\in\mathbb{N}} M_{d_n}$ extends to a completely bounded unital homomorphism $A^{\boldsymbol{\alpha}} \to A^{\boldsymbol{\alpha}'}$, with $\|\varphi\|_{cb} \le L$. In other words, the matrix norms $\|\cdot\|_{\boldsymbol{\alpha}}$ and $\|\cdot\|_{\boldsymbol{\alpha}'}$ on the algebraic tensor product $\bigotimes_{n\in\mathbb{N}} M_{d_n}$ satisfy

$$\|\cdot\|_{\boldsymbol{\alpha}'} \le L\|\cdot\|_{\boldsymbol{\alpha}}.$$

Proof. For $j\in\mathbb{N}$, set $L_j = \alpha_j/\alpha'_j$. Fix $\varepsilon > 0$. In order to prove our assertion, it is enough to show that if $k \in \mathbb{N}$ and x is an element of $M_k\left(\bigotimes_{j\in\mathbb{N}} M_{d_j}\right)$, then $\|x\|_{\boldsymbol{\alpha}'} \le (1+\varepsilon)L\|x\|_{\boldsymbol{\alpha}}$. Let $x \in M_k\left(\bigotimes_{j\in\mathbb{N}} M_{d_j}\right)$, and choose $n,m\in\mathbb{N}$ and $X_{i,j} \in M_k(M_{d_i})$

for $1 \leq i \leq n$ and $1 \leq j \leq m$, satisfying

$$x = \sum_{j=1}^{m} X_{1,j} \otimes \cdots \otimes X_{n,j}.$$

By definition of the matrix norms on $A_{\boldsymbol{\alpha}}$, for $1 \leq i \leq n$ there exists a diagonal matrix $w_i \in M_{d_i}$ with entries in $[1, \alpha_i]$ such that, if $W_i \in M_k(M_{d_i})$ is the diagonal matrix with entries in M_{d_i}, and nonzero entries equal to w_i (in other words, $W_i = 1_{M_k} \otimes w_i$), then

$$\|x\|_{\boldsymbol{\alpha}} \leq (1+\varepsilon) \left\| \sum_{j=1}^{m} W_1 X_{1,j} W_1^{-1} \otimes \cdots \otimes W_n X_{n,j} X_n^{-1} \right\|.$$

For $1 \leq i \leq n$, we denote the diagonal entries of $w_i \in M_{d_j}$ by $a_{i,\ell}$, for $\ell = 1, \ldots, d_i$. We will define two other diagonal matrices

$$w_i' = \operatorname{diag}(a_{i,1}', \ldots, a_{i,d_i}') \quad \text{and} \quad r_i = \operatorname{diag}(r_{i,1}, \ldots, r_{i,d_i})$$

in M_{d_i}, with entries in $[1, \alpha_i']$ and $[1, L_i]$, respectively, as follows. For $1 \leq \ell \leq d_i$, we set

$$a_{i,\ell}' = \begin{cases} a_{i,\ell}, & \text{if } a_{i,\ell} < \alpha_i'; \\ \alpha_i', & \text{if } a_{j,\ell} \geq \alpha_i'. \end{cases}$$

and

$$r_{i,\ell} = \begin{cases} 1, & \text{if } a_{i,\ell} < \alpha_i'; \\ \frac{1}{\alpha_i'} a_{i,\ell}, & \text{if } a_{i,\ell} \geq \alpha_i'. \end{cases}$$

Observe that $r_{i,\ell}$ belongs to $[1, L_i]$ (since $a_{i,\ell} \leq \alpha_i \leq L_i \alpha_i'$), and that $a_{i,\ell}'$ belongs to $[1, \alpha_i']$ for all $1 \leq i \leq n$ and $1 \leq \ell \leq d_i$.

Define w_i' and r_i to be the diagonal $d_i \times d_i$ matrices with diagonal entries $a_{i,\ell}'$ and $r_{i,\ell}$ for $1 \leq \ell \leq d_i$. Let $W_i', R_i \in M_k(M_{d_i})$ be the diagonal $k \times k$ matrices with entries in M_{d_i} having diagonal entries equal to, respectively, w_i' and r_i. (In other words, $W_i' = 1_{M_k} \otimes w_i'$ and $R_i = 1_{M_k} \otimes r_i$.)

Then $W_i = R_i W_i'$ for all $1 \leq i \leq n$. Additionally,

$$\|R_i\| \leq L_i \quad \text{and} \quad \left\|R_i^{-1}\right\| \leq 1.$$

Therefore,

$$\begin{aligned}
\|x\|_{\boldsymbol{\alpha}} &\leq (1+\varepsilon) \left\| \sum_{j=1}^{m} W_1 X_{1,j} W_1^{-1} \otimes \cdots \otimes W_n X_{n,j} W_n^{-1} \right\| \\
&= (1+\varepsilon) \left\| \sum_{j=1}^{m} R_1 W_1' X_{1,j} W_1^{'-1} R_1^{-1} \otimes \cdots \otimes R_n W_n' X_{n,j} W_n^{'-1} R_n^{-1} \right\| \\
&\leq (1+\varepsilon) \|R_1\| \, \|R_2\| \cdots \|R_n\| \left\| \sum_{j=1}^{m} W_1' X_{1,j} W_1'^{-1} \otimes \cdots \otimes W_n' X_{n,j} W_n'^{-1} \right\| \\
&\leq (1+\varepsilon) L_1 \cdots L_n \left\| \sum_{j=1}^{m} W_1' X_{1,j} W_1'^{-1} \otimes \cdots \otimes W_n' X_{n,j} W_n'^{-1} \right\| \\
&\leq (1+\varepsilon) L \, \|x\|_{\boldsymbol{\alpha}'} .
\end{aligned}$$

This concludes the proof. $\square$

Corollary 3.2. *If $\boldsymbol{\alpha}, \boldsymbol{\alpha}' \in [1, +\infty)^{\mathbb{N}}$ satisfy*

$$\prod_{n \in \mathbb{N}} \max\left\{\frac{\alpha_n}{\alpha'_n}, \frac{\alpha'_n}{\alpha_n}\right\} < +\infty,$$

then A^{α} and $A^{\alpha'}$ are completely isomorphic.

The following lemma can be proved in the same way as [P3, Lemma 5.11] with the extra ingredient of [P3, Lemma 5.8]. As before, we denote by $\otimes^p$ the L^p-operator tensor product; see [P4, Definition 1.9].

Lemma 3.3 (Phillips)**.** *Let $L > 0$ and let $d \in \mathbb{N}$. Then there is a constant $R(L, d) > 0$ such that the following holds. Whenever A is a unital L^p-operator algebra, whenever $\alpha, \alpha' \in [1, +\infty)$ satisfy*

$$\alpha' \geq R(L, d)\alpha,$$

and $\varphi \colon M_d^{\alpha} \to M_d^{\alpha'} \otimes^p A$ is a unital homomorphism with $\|\varphi\| \leq L$, there exists a unital homomorphism $\psi \colon M_d^{\alpha} \to A$ with $\|\psi\| \leq L + 1$.

Our assumption on the values of $\boldsymbol{d}$ will be used for the first time in Lemma 3.5, where it is shown that sufficiently different sequences yield noncommensurable UHF L^p-operator algebras.

The K_0-group of a Banach algebra A is defined as the K_0-group of its underlying ring structure; see Definition 1.6. The following easy lemma is standard, but we include its proof here for the convenience of the reader.

Lemma 3.4. *Let A be a unital ring and let $d \in \mathbb{N}$. If there exists a unital homomorphism $M_d \to A$, then the class of unit of A in $K_0(A)$ must be divisible by d.*

Proof. Let $\varphi \colon M_d \to A$ be a unital homomorphism. For $j = 1, \ldots, d$, denote by $e_j \in M_d$ the idempotent $\operatorname{diag}(0, \ldots, 1, \ldots, 0)$ (there is a 1 on the j-th entry). Choose $u_j \in M_n$ invertible satisfying $u_j e_j u_j^{-1} = e_1$ (one can choose u_j to be a suitable permutation matrix).

Since φ is unital, we have

$$1_A = \varphi(1_{M_d}) = \sum_{j=1}^{d} \varphi(e_j),$$

so by taking classes in K_0 we get $[1_A] = \sum_{j=1}^{d} [\varphi(e_j)]$. Since φ is unital and u_j is invertible, $\varphi(u_j)$ is also invertible, and clearly $\varphi(u_j)\varphi(e_j)\varphi(u_j)^{-1} = \varphi(e_1)$ for all $j = 1, \ldots, d$. We deduce that $[\varphi(e_j)] = [\varphi(e_1)]$, and thus $[1_A] = d[\varphi(e_1)]$, as desired. ☐

Recall the notation M_d^{α} and $A^{\boldsymbol{\alpha}}$ from before Definition 1.3.

Lemma 3.5. *Suppose that $\boldsymbol{\alpha}, \boldsymbol{\alpha}' \in [1, +\infty)^{\mathbb{N}}$ satisfy $\alpha'_n \geq R(n, d_n)\alpha_n$ for infinitely many $n \in \mathbb{N}$. Then there is no continuous unital homomorphism $\varphi \colon A^{\boldsymbol{\alpha}} \to A^{\alpha'}$.*

Proof. Assume by contradiction that $\varphi \colon A^{\boldsymbol{\alpha}} \to A^{\boldsymbol{\alpha}'}$ is a continuous unital homomorphism and set $L = \|\varphi\|$. Choose $n \in \mathbb{N}$ such that $n \geq L$ and $\alpha'_n \geq R(n, d_n)\alpha_n$.

Set

$$A = \bigotimes_{m\in\mathbb{N}, m\neq n}^{p} M_{d_m}^{\alpha'_m}.$$

Apply Lemma 3.3 to the unital homomorphism $\varphi\colon M_{d_n}^{\alpha_n} \to M_{d_n}^{\alpha'_n} \otimes^p A$, to get a unital homomorphism $\psi\colon M_{d_n}^{\alpha_n} \to A$ with $\|\psi\| \leq L+1$.

Using Lemma 3.4, we conclude that the class of the unit of A in $K_0(A)$ is divisible by d_n. On the other hand, the K-theory of A is easy to compute using that K-theory for Banach algebras commutes with direct limits (with contractive maps). We get

$$K_0(A) = \mathbb{Z}\left[\frac{1}{b} : b \neq 0 \text{ divides } d_m \text{ for some } m \neq n\right],$$

with the unit of A corresponding to $1 \in K_0(A) \subseteq \mathbb{Q}$.

Since there is a prime appearing in the factorization of d_n that does not divide any d_m, for $m \neq n$, we deduce that the class of the unit of A in $K_0(A)$ cannot be divisible by d_n. This contradiction shows that there is no continuous unital homomorphism $\varphi\colon A^{\boldsymbol{\alpha}} \to A^{\boldsymbol{\alpha}'}$ □

We say that a set is *comeager* if its complement is meager. Observe that, by definition, a nonmeager set intersects every comeager set. Recall that we regard $[1,+\infty)^{\mathbb{N}}$ as the parametrizing space of the UHF L^p-operator algebras of tensor product type $\boldsymbol{d}$ obtained from a diagonal system of similarities. Consistently, we regard (complete) isomorphism and commensurability of such algebras as equivalence relations on $[1,+\infty)^{\mathbb{N}}$.

Proof of Theorem 1.5. By [P4, Theorem 3.19(3)], every UHF L^p-operator algebra of tensor product type $\boldsymbol{d}$ obtained from a diagonal system of similarities is simple and monotracial. Therefore, it is enough to prove the second assertion of Theorem 1.5. For $\boldsymbol{t} \in [0,+\infty)^{\mathbb{N}}$, define $\exp(\boldsymbol{t})$ to be the sequence $(\exp(t_n))_{n\in\mathbb{N}}$ of real numbers in $[1,\infty)$. By Corollary 3.2, if $\boldsymbol{t}, \boldsymbol{t}' \in [0,+\infty)^{\mathbb{N}}$ satisfy $\boldsymbol{t}-\boldsymbol{t}' \in \ell^1$, then $A^{\exp(\boldsymbol{t})}$ and $A^{\exp(\boldsymbol{t}')}$ are completely isomorphic. We claim that for any nonmeager subset C of $[0,+\infty)^{\mathbb{N}}$, one can find $\boldsymbol{t}, \boldsymbol{t}' \in C$ such that $A^{\exp(\boldsymbol{t})}$ and $A^{\exp(\boldsymbol{t}')}$ are not commensurable. This fact together with Corollary 3.2 will show that the Borel function

$$\begin{aligned} [0,+\infty)^{\mathbb{N}} &\to [1,+\infty)^{\mathbb{N}} \\ \boldsymbol{t} &\mapsto \exp(\boldsymbol{t}) \end{aligned}$$

satisfies the hypotheses of Criterion 2.2 for any of the equivalence relations E in the statement of Theorem 1.5, yielding the desired conclusion.

Then let C be a nonmeager subset of $[0,+\infty)^{\mathbb{N}}$, and fix $\boldsymbol{t} \in C$. We want to find $\boldsymbol{t}' \in C$ such that $A^{\exp(\boldsymbol{t})}$ and $A^{\exp(\boldsymbol{t}')}$ are not commensurable. The set

$$\begin{aligned} &\left\{\boldsymbol{t}' \in [0,+\infty)^{\mathbb{N}} \colon \text{ for all but finitely many } n \in \mathbb{N},\ \exp(t'_n) \leq R(n,d_n)\exp(t_n)\right\} \\ &\qquad = \bigcup_{k\in\mathbb{N}} \left\{\boldsymbol{t}' \in [0,+\infty)^{\mathbb{N}} \colon \forall n \geq k,\ \exp(t'_n) \leq R(n,d_n)\exp(t_n)\right\} \end{aligned}$$

is a countable union of closed nowhere dense sets, hence meager. Therefore, its complement

$$\left\{\boldsymbol{t}' \in [0,+\infty)^{\mathbb{N}} \colon \text{ for infinitely many } n \in \mathbb{N},\ \exp(t'_n) > R(n,d_n)\exp(t_n)\right\}$$

is comeager. In particular, since C is nonmeager, there is $\boldsymbol{t}' \in C$ such that $\exp(t'_n) \geq R(n, d_n)\exp(t_n)$ for infinitely many $n \in \mathbb{N}$. By Lemma 3.3, there is no continuous unital homomorphism from $A^{\exp(\boldsymbol{t})}$ to $A^{\exp(\boldsymbol{t}')}$. Therefore $A^{\exp(\boldsymbol{t})}$ and $A^{\exp(\boldsymbol{t}')}$ are not commensurable. This concludes the proof of the above claim. □

References

[B] O. Bratteli, *Inductive limits of finite dimensional C^*-algebras*, Trans. Amer. Math. Soc. **171** (1972), 195–234. MR0312282 (47 #844)

[BLM] D. P. Blecher and C. Le Merdy, *Operator algebras and their modules—an operator space approach*, London Mathematical Society Monographs. New Series, vol. 30, Oxford University Press, Oxford, 2004.

[D] M. Daws, *p-operator spaces and Figà-Talamanca–Herz algebras*, J. Operator Theory **63** (2010), no. 1, 47–83. MR2606882 (2011d:46118)

[E] G. A. Elliott, *On the classification of inductive limits of sequences of semisimple finite-dimensional algebras*, J. Algebra **38** (1976), no. 1, 29–44. MR0397420 (53 #1279)

[F] I. Farah, *A dichotomy for the Mackey Borel structure*, Proceedings of the 11th Asian Logic Conference, World Sci. Publ., Hackensack, NJ, 2012, pp. 86–93, DOI 10.1142/9789814360548_0005. MR2868507

[FW] M. Foreman and B. Weiss, *An anti-classification theorem for ergodic measure preserving transformations*, J. Eur. Math. Soc. (JEMS) **6** (2004), no. 3, 277–292. MR2060477 (2005c:37006)

[G] J. G. Glimm, *On a certain class of operator algebras*, Trans. Amer. Math. Soc. **95** (1960), 318–340. MR0112057 (22 #2915)

[GL] E. Gardella and M. Lupini, *Representations of étale groupoids on L^p-spaces*, arXiv:1408.3752 (2014).

[H1] G. Hjorth, *Non-smooth infinite-dimensional group representations*, 1997. Preprint.

[H2] G. Hjorth, *Classification and orbit equivalence relations*, Mathematical Surveys and Monographs, vol. 75, American Mathematical Society, Providence, RI, 2000. MR1725642 (2000k:03097)

[H3] G. Hjorth, *Measuring the classification difficulty of countable torsion-free abelian groups*, Proceedings of the Second Honolulu Conference on Abelian Groups and Modules (Honolulu, HI, 2001), Rocky Mountain J. Math. **32** (2002), no. 4, 1267–1280, DOI 10.1216/rmjm/1181070022. MR1987607 (2004j:20112)

[KLP1] D. Kerr, H. Li, and M. Pichot, *Turbulence, representations, and trace-preserving actions*, Proc. Lond. Math. Soc. (3) **100** (2010), no. 2, 459–484, DOI 10.1112/plms/pdp036. MR2595746 (2012e:37018)

[KLP2] D. Kerr, M. Lupini, and N. C. Phillips, *Borel complexity and automorphisms of C*-algebras*, Journal of Functional Analysis. To appear.

[KS] A. S. Kechris and N. E. Sofronidis, *A strong generic ergodicity property of unitary and self-adjoint operators*, Ergodic Theory Dynam. Systems **21** (2001), no. 5, 1459–1479, DOI 10.1017/S0143385701001705. MR1855842 (2002g:47077)

[L] M. Lupini, *Unitary equivalence of automorphisms of separable* C**-algebras*, Adv. Math. **262** (2014), 1002–1034, DOI 10.1016/j.aim.2014.05.022. MR3228448

[M] D. Marker, *Model theory*, Graduate Texts in Mathematics, vol. 217, Springer-Verlag, New York, 2002. An introduction. MR1924282 (2003e:03060)

[P1] N. C. Phillips, *Analogs of Cuntz algebras on L^p spaces*, arXiv:1201.4196 (2012).

[P2] N. C. Phillips, *Crossed products of L^p operator algebras and the K-theory of Cuntz algebras on L^p spaces*, arXiv:1309.6406 (2013).

[P3] N. C. Phillips, *Isomorphism, nonisomorphism, and amenability of L^p UHF algebras*, arXiv:1309.3694 (2013).

[P4] N. C. Phillips, *Simplicity of UHF and Cuntz algebras on L^p-spaces*, arXiv:1309.0115 (2013).

[PV] N. C. Phillips and M. G. Viola, *L^p analogs of AF algebras*. In preparation.

[ST1] R. Sasyk and A. Törnquist, *The classification problem for von Neumann factors*, Journal of Functional Analysis **256** (April 2009), no. 8, 2710–2724.

[ST2] R. Sasyk and A. Törnquist, *Turbulence and Araki-Woods factors*, Journal of Functional Analysis **259** (2010), no. 9, 2238–2252.

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