# RESTRICTED COHOMOLOGY OF MODULAR WITT ALGEBRAS

#### TYLER J. EVANS, ALICE FIALOWSKI, AND MICHAEL PENKAVA

(Communicated by Kailash C. Misra)

ABSTRACT. In this paper we compute the restricted 0, 1-, and 2-dimensional cohomology groups with trivial coefficients of the modular Witt algebras W for any prime p > 3, and give explicit description of all one-dimensional restricted central extensions of W.

## 1. INTRODUCTION

Let p > 3 be a prime,  $\mathbb{K}$  be an algebraically closed field of characteristic p and  $A = \mathbb{K}[x]/(x^p-1)$ . The Lie algebra W = Der(A) is called the *modular Witt algebra*. It has a basis  $e_i = x^{i+1}\partial$  where  $i = -1, 0, \ldots, p-2$  and  $\partial = d/dx$ . The Lie bracket in W satisfies  $[e_i, e_j] = (j - i)e_{i+j}$  where i + j is computed modulo p.

In characteristic zero, the analogous Lie algebra of derivations of the algebra  $\mathbb{C}[t, t^{-1}]$  is also called the Witt algebra. Fuchs and Gelfand have shown that  $\operatorname{Der}(\mathbb{C}[t, t^{-1}])$  has, up to equivalence, exactly one non-trivial one-dimensional central extension (the Virasoro algebra) [5]. Block has shown in characteristic p > 5 that the modular Witt algebra also has, again up to equivalence, exactly one non-trivial one-dimensional central extension as an ordinary (i.e. non-restricted) Lie algebra [1].

Since W is a Lie algebra of derivations of an algebra over  $\mathbb{K}$ , it has the canonical structure of a restricted Lie algebra. It is therefore natural to ask how many onedimensional *restricted* central extensions W has. To answer this question, one needs to compute the restricted cohomology group  $H^2(W) = H^2(W, \mathbb{K})$ . In this paper we will use the (partial) complex given in [3] to carry out this computation. We will give a complete computation of the restricted cohomology groups  $H^q(W) = H^q(W, \mathbb{K})$  for q = 0, 1 and 2. We will show that if p > 3, W has p + 1 non-equivalent non-trivial one-dimensional restricted central extensions. Moreover, when considered only as ordinary Lie algebra extensions, p of these extensions are trivial, and one is equivalent to the central extension of W in [1].

The structure of the paper is as follows. In section 2, we review the definitions of restricted Lie algebras, the (partial) cochain complex for restricted Lie algebra cohomology for the case of trivial coefficients, and establish the notation used throughout the paper. In section 3, we review the correspondence between onedimensional central extensions and cohomology for both ordinary and restricted Lie algebras and state our main theorem on restricted central extensions of W. Section 4 gives explicit representative cocycles for the ordinary cohomology group

Received by the editors February 16, 2015 and, in revised form, May 18, 2015.

<sup>2010</sup> Mathematics Subject Classification. Primary 17B50, 17B56; Secondary 17B68, 17B66.

 $H^2_{\rm cl}(W)$ , and section 5 contains the computations of the restricted cohomology groups  $H^q(W)$  for  $q \leq 2$ . Section 5 concludes with the proof of the main theorem and explicit descriptions of all one-dimensional restricted central extensions of W.

The authors are grateful to the referee for helpful suggestions.

## 2. Definitions and notation

Restricted Lie algebras (also called Lie *p*-algebras) were first introduced by Jacobson [8,9]. Recall that a modular Lie algebra  $\mathfrak{g}$  is called *restricted* if it is endowed with an additional unary operation  $g \mapsto g^{[p]}$  that satisfies for all  $g, h \in \mathfrak{g}$  and all  $\lambda \in \mathbb{K}$ 

$$\begin{aligned} &(\lambda g)^{[p]} = \lambda^p g^{[p]};\\ &\mathrm{ad}(g^{[p]}) = (\mathrm{ad}\,g)^p;\\ &(g+h)^{[p]} = g^{[p]} + h^{[p]} + \sum_{i=1}^{p-1} s_i(g,h); \end{aligned}$$

where  $s_i(g,h)$  is the coefficient of  $\lambda^{i-1}$  in  $(\operatorname{ad}(\lambda g+h))^{p-1}(h)$ .

Restricted Lie algebras naturally arise in positive characteristic as derivation algebras of any algebra, or as the Lie algebra of algebraic groups with the operations [a, b] = ab - ba and  $a^{[p]} = a^p$  [7,10]. In particular, in the Witt algebra W we have  $g^{[p]} = g^p$ . Whereas the operation  $g \mapsto g^{[p]}$  is not linear in general, it is completely determined by its values on a basis. In W we have  $e_0^{[p]} = e_0$ ,  $e_i^{[p]} = 0$  for  $i \neq 0$ , and moreover,  $g^{[p]} = \gamma(g)g$  where  $\gamma(g) \in \mathbb{K}$  is a constant ([2, Theorem 1(a)]). Since W is simple, this restricted structure is unique.

The classification of one-dimensional restricted central extensions of W below is carried out through the calculation of the restricted cohomology  $H^2(W)$  with coefficients in  $\mathbb{K}$  taken as a trivial W module. We limit our description of the restricted cochain spaces and coboundary operators to the case of trivial coefficients and refer the reader to [3] for the general description. For details on ordinary Lie algebra cohomology see [4]. Let  $C^q(W)$  denote restricted cochains of degree q $(0 \leq q \leq 3)$  and  $C^q_{cl}(W) = \operatorname{Hom}(\wedge^q W, \mathbb{K})$  the space of ordinary Lie algebra cochains. We use similar notation for the coboundary operators. We will denote multiple Lie bracket products as  $[g_1, g_2, g_3, \ldots, g_j] = [[\ldots [[g_1, g_2], g_3], \ldots, ]g_j]$ . In particular, we take these products from the right so that the equality  $\operatorname{ad}(g^{[p]})(h) = (\operatorname{ad} g)^p(h)$  is written

$$[h, g^{[p]}] = [h, \underbrace{g, \dots, g}_{p}].$$

Let us give an explicit description of the (partial) complex

$$C^0(W) \xrightarrow{\delta^0} C^1(W) \xrightarrow{\delta^1} C^2(W) \xrightarrow{\delta^2} C^3(W).$$

For  $q \leq 1$ , we set  $C^q(W) = C^q_{\rm cl}(W) = \operatorname{Hom}(\wedge^q W, \mathbb{K})$  and  $\delta^0 = \delta^0_{\rm cl}$ . If  $\varphi \in C^2_{\rm cl}(W)$ and  $\omega : W \to \mathbb{K}$ , then we say  $\omega$  has the \*-**property** with respect to  $\varphi$ , if for all  $g, h \in W$  and  $\lambda \in \mathbb{K}$ , we have  $\omega(\lambda g) = \lambda^p \omega(g)$  and

(1) 
$$\omega(g+h) = \omega(g) + \omega(h) + \sum_{\substack{g_i = g \text{ or } h \\ g_1 = g, g_2 = h}} \frac{1}{\#(g)} \varphi([g_1, g_2, \dots, g_{p-1}] \wedge g_p).$$

Here #(g) is the number of factors  $g_i$  equal to g. We remark that  $\omega$  has the \*-property with respect to the zero map precisely when  $\omega$  is a *p*-semilinear map on W, that is,  $\omega(q+h) = \omega(q) + \omega(h)$  and  $\omega(\lambda q) = \lambda^p \omega(q)$  for all  $q, h \in W$  and all  $\lambda \in \mathbb{K}$ . Moreover, given  $\varphi$ , we can assign the values of  $\omega$  arbitrarily on a basis for W and use (1) to define  $\omega: W \to \mathbb{K}$  that has the \*-property with respect to  $\varphi$ . Our space of 2-cochains is

 $C^{2}(W) = \{(\varphi, \omega) \mid \varphi : \Lambda^{2}W \to \mathbb{K}, \omega : W \to \mathbb{K} \text{ has the } * \text{-property w.r.t. } \varphi \},\$ 

and

$$\dim_{\mathbb{K}} C^2(W) = \frac{p(p+1)}{2}.$$

A linear map  $\psi: W \to \mathbb{K}$  induces a map  $\operatorname{ind}^1 \psi: W \to \mathbb{K}$  by the formula

$$\operatorname{ind}^1 \psi(g) = \psi(g^{[p]}),$$

and this map has the \*-property with respect to  $\delta_{cl}^1 \psi$  ([3, Lemma 4]). The coboundary operator  $\delta^1: C^1(W) \to C^2(W)$  is given by

$$\delta^1 \psi = (\delta^1_{\rm cl} \psi, {\rm ind}^1 \psi).$$

If  $\alpha : \Lambda^3 W \to \mathbb{K}$  is a skew-symmetric multilinear map on W and  $\beta : W \times W \to \mathbb{K}$ , we say that  $\beta$  has the \*\*-property with respect to  $\alpha$ , if the following conditions hold:

(i)  $\beta(q, h)$  is linear with respect to q; (ii)  $\beta(q, \lambda h) = \lambda^p \beta(q, h)$  for all  $\lambda \in \mathbb{K}$ ;

(iii)  
$$\beta(a, h_1 + h_2) = \beta(a, h_1) + \beta(a, h_2)$$

(2) 
$$(2) \qquad -\sum_{\substack{l_1,\ldots,l_p=1 \text{or} 2\\l_1=1,l_2=2}} \frac{1}{\#\{i_i=1\}} \alpha(g \wedge [h_{l_1},\cdots,h_{l_{p-1}}] \wedge h_{l_p}).$$

Again we remark that  $\beta$  has the \*\*-property with respect to the zero map precisely when  $\beta$  is linear in the first variable and *p*-semilinear in the second variable. We can use (2) to define  $\beta$  for a given  $\alpha$  and values of  $\beta$  on a basis for W. Our space of 3-cochains is

$$C^{3}(W) = \{(\alpha, \beta) \mid \alpha \in C^{3}_{cl}(W), \beta : W \times W \to \mathbb{K} \text{ has the } **\text{-property w.r.t. } \alpha\}$$

and

$$\dim_{\mathbb{K}} C^{3}(W) = \frac{p(p+1)(p+2)}{6}.$$

An element  $(\varphi, \omega) \in C^2(W)$  induces a map  $\operatorname{ind}^2(\varphi, \omega) : W \times W \to \mathbb{K}$  by the formula

$$\operatorname{ind}^{2}(\varphi,\omega)(g,h) = \varphi(g,h^{[p]}) - \varphi([g,\underbrace{h,\cdots,h}_{p-1}] \wedge h),$$

and this map has the \*\*-property with respect to  $\delta^2_{cl}\varphi$  ([3, Lemma 5]). The coboundary operator  $\delta^2 : C^2(W) \to C^3(W)$  is given by the formula

$$\delta^2(\varphi, \omega) = (\delta^2_{cl}\varphi, ind^2(\varphi, \omega)).$$

We write  $e^i = e^*_i$  (dual basis vector),  $e_{i,j} = e_i \wedge e_j$ ,  $e^{i,j} = e^*_{i,j}$  and  $e_{r,s,t} = e_r \wedge e_s \wedge e_t$  where  $-1 \leq i < j \leq p-2$  and  $-1 \leq r < s < t \leq p-2$ .

The ordinary cochain spaces  $C^1_{\rm cl}(W), C^2_{\rm cl}(W)$  and  $C^3_{\rm cl}(W)$  admit a natural grading:

$$\begin{aligned} & (C_{\rm cl}^1)_k(W) = {\rm span}\{e^k\}; \\ & (C_{\rm cl}^2)_k(W) = {\rm span}\{e^{i,j} \mid i+j=k \pmod{p}\}; \\ & (C_{\rm cl}^3)_k(W) = {\rm span}\{e^{r,s,t} \mid r+s+t=k \pmod{p}\}; \end{aligned}$$

where  $k = -1, \ldots, p - 2$ . Moreover, we have

$$\dim_{\mathbb{K}} (C_{\rm cl}^1)_k(W) = 1;$$
  
$$\dim_{\mathbb{K}} (C_{\rm cl}^2)_k(W) = \frac{p-1}{2};$$
  
$$\dim_{\mathbb{K}} (C_{\rm cl}^3)_k(W) = \frac{(p-1)(p-2)}{6}$$

The coboundary operators  $\delta_{cl}^1$  and  $\delta_{cl}^2$  preserve this grading, and we denote by  $(\delta_{cl}^1)_k$ and  $(\delta_{cl}^2)_k$  the restrictions of  $\delta_{cl}^1$  and  $\delta_{cl}^2$  to  $(C_{cl}^1)_k(W)$  and  $(C_{cl}^2)_k(W)$ , respectively.

## 3. Restricted central extensions

As stated in the Introduction, the main result of this paper is the classification of one-dimensional restricted central extensions of W. We now state the main theorem.

**Theorem 3.1.** If p > 3, then  $H^2(W)$  is (p + 1)-dimensional. Moreover, there is a p-dimensional subspace of  $H^2(W)$  for which each corresponding one-dimensional restricted central extension is trivial when considered as an ordinary Lie algebra extension.

Theorem 3.1 implies that exactly one of the p+1 non-trivial classes of restricted one-dimensional central extensions remains non-trivial when considered only as a Lie algebra extension. For p > 5, this is the central extension described first by Block in [1]. Our method here is different, and also gives the result for p = 5.

*Remark.* If p = 3, the algebra W is isomorphic to the Lie algebra  $\mathfrak{sl}_2(\mathbb{K})$ . In this case  $H^2(W)$  is three-dimensional so that W has just three non-equivalent one-dimensional restricted central extensions. Of course each of these extensions is trivial when considered as an ordinary Lie algebra extension.

Given a one-dimensional restricted central extension E of W, construct an element  $(\varphi, \omega) \in C^2(W)$  by choosing a  $\mathbb{K}$ -linear splitting map  $\sigma : W \to E$  and defining for all  $g, h \in W$ 

;

(3) 
$$\varphi(g,h) = [\sigma(g),\sigma(h)] = \sigma([g,h])$$
$$\omega(g) = \sigma(g)^{[p]} - \sigma(g^{[p]}).$$

The element  $(\varphi, \omega) \in C^2(W)$  is a cocycle, and the cohomology class of  $(\varphi, \omega)$  does not depend on the choice of the splitting map, but only on the equivalence class of the extension ([3, Corollary 4]).

Conversely, given a cocycle  $(\varphi, \omega) \in C^2(W)$ , define a restricted Lie algebra structure on  $E = W \oplus \mathbb{K}c$  by declaring for all  $g, h \in W$  and all  $\alpha, \beta \in \mathbb{K}$ 

(4) 
$$[g + \alpha c, h + \beta c] = [g, h] + \varphi(g, h)c;$$
$$(g + \alpha c)^{[p]} = g^{[p]} + \omega(g)c.$$

1880

The equivalence class of the resulting one-dimensional restricted central extension depends only on the cohomology class of  $(\varphi, \omega)$  ([3, Corollary 4]).

The map  $(\varphi, \omega) \mapsto \varphi$  induces a well-defined map  $H^2(W) \to H^2_{cl}(W)$  so that every one-dimensional restricted central extension of W is also a one-dimensional central extension as an ordinary Lie algebra, and equivalent one-dimensional restricted central extensions give equivalent one-dimensional ordinary central extensions.

Remark. It is known that modular Lie algebras do not always admit a Levi decomposition (a decomposition into the semidirect product of the radical and a semisimple algebra) [6,11]. Our computations below give examples of modular Lie algebras and restricted Lie algebras that are not Levi decomposable. If E is a Levi decomposable one-dimensional restricted central extension of W, then  $\mathbb{K}$  is the radical of E and there is a restricted Lie algebra homomorphism splitting map  $\sigma: W \to E$ . In this case  $\varphi$  and  $\omega$  in (3) are identically zero. If E is Levi decomposable only as an ordinary Lie algebra, then  $\sigma$  is only a Lie algebra homomorphism. In this case  $\varphi = 0$  but  $\omega \neq 0$ . Theorem 3.1 implies that there is a p-dimensional subspace of  $H^2(W)$  that classifies the one-dimensional restricted central extensions of W that are Levi decomposable as ordinary Lie algebras, but not as restricted Lie algebras. Elements in the complement to this subspace correspond to a onedimensional central extension that is not Levi decomposable as an ordinary nor restricted Lie algebra.

## 4. Ordinary Lie Algebra cohomology of W

We sketch a new method for computing  $H^2_{\rm cl}(W)$  by giving explicit descriptions of the cocycles. There are more results on the cohomology of W, for example [13, 14], but we will not use them in this paper. In the case of trivial coefficients, we have  $H^0_{\rm cl}(W) = \mathbb{K}$  and  $H^1_{\rm cl}(W) = 0$ .

Since the cochain spaces are graded and the coboundary maps are graded maps, we can compute  $H^2_{\rm cl}(W)$  by computing the cohomology in each graded component. An element  $\varphi \in (C^2_{\rm cl})_k(W)$  has the form

$$\varphi = \sum_{i+j=k \pmod{p}} a_{i,j} e^{i,j}$$

where  $a_{i,j} \in \mathbb{K}$ . The proof of the following lemma is a routine computation.

**Lemma 4.1.** For  $-1 \le k \le p - 2$ ,

$$\delta^1_{\rm cl}(e^k) = \sum_{\substack{-1 \le i < j \le p-2\\i+j=k \pmod{p}}} (j-i)e^{i,j}.$$

**Lemma 4.2.** If  $-1 \le k \le p-2$  and  $k \ne 0$ , then  $\dim_{\mathbb{K}}(\ker(\delta_{cl}^2)_k) = 1$ .

*Proof.* Let  $\varphi = \sum a_{i,j} e^{i,j} \in (C^2_{cl})_k(W)$ . For  $-1 \le i < j \le p-2$ ,  $i+j = k \pmod{p}$  and  $i \ne 0$ , we have

$$(\delta_{\rm cl}^2)_k \varphi(e_{0,i,j}) = k a_{i,j} - (j-i) a_{0,k}$$

If  $k \neq 0$  and  $\varphi$  is a cocycle, then all coefficients  $a_{i,j}$  are determined by  $a_{0,k}$  so that  $\dim_{\mathbb{K}}(\ker(\delta_{cl}^2)_k) \leq 1$ . The rank of  $(\delta_{cl}^1)_k$  is 1 so we must have  $\dim_{\mathbb{K}}(\ker(\delta_{cl}^2)_k) = 1$  as claimed.

**Lemma 4.3.** dim<sub>K</sub>(ker( $\delta_{cl}^2$ )<sub>0</sub>) = 2.

Proof. Let

$$\varphi = a_{-1,1}e_{-1,1} + \sum_{i=2}^{(p-1)/2} a_{i,p-i}e^{i,p-i} \in C_0^2$$

be a cochain. If  $\varphi$  is a cocycle, then for  $3 \le j \le (p-1)/2$  we must have

$$\delta_0^2 \varphi(e_{-1,j,p-j+1}) = (j+1)a_{j-1,p+j-1} + (p-j+2)a_{j,p-j} + (2j-1)a_{-1,1} = 0.$$

Shifting the index with n = j + 2, we have for  $1 \le n \le (p - 5)/2$ ,

(5) 
$$na_{n+2,p-n-2} = (n+3)a_{n+1,p-n-1} + (2n+3)a_{-1,1}.$$

This shows recursively that  $a_{n+2,p-n-2}$  is determined by  $a_{-1,1}$  and  $a_{2,p-2}$ , and therefore  $\dim_{\mathbb{K}}(\ker \delta_0^2) \leq 2$ . If we set  $a_{-1,1} = 1$  and  $a_{2,p-2} = 0$ , then (5) reduces to

$$na_{n+2,p-n-2} = (n+3)a_{n+1,p-n-1} + (2n+3).$$

This recursion equation has the solution

$$a_{n+2,p-n-2} = \frac{1}{3}n(n+2)(n+4)$$

and therefore

(6) 
$$\varphi_{1,0} = \sum_{n=1}^{(p-1)/2} \frac{n(n^2 - 4)}{3} e^{n,p-n}.$$

Now, for any basis vector  $e_{i,j,p-i-j} \in (\wedge^3 W)_0$ , we have by definition

$$\delta_{\rm cl}^2\varphi_{1,0}(e_{i,j,p-i-j}) = \varphi_{1,0}((j-i)e_{i+j,p-i-j} - (p-2i-j)e_{p-j,j} + (p-i-2j)e_{p-i,i})$$

This together with (6) shows  $\varphi_{1,0}$  is a cocycle. If we set  $a_{-1,1} = 2$  and  $a_{2,p-2} = p-4$ , then (5) reduces to the recursion equation

$$na_{n+2,p-n-2} = (n+3)a_{n+1,p-n-1} + (4n+6),$$

which has the solution

$$a_{n+2,p-n-2} = -2(n+2).$$

So by Lemma 4.1 we have

$$\varphi_{2,p-4} = \sum_{n=1}^{(p-1)/2} -2ne^{n,p-n} = \delta^1(e^0)$$

showing  $\varphi_{2,p-4}$  is also a cocycle. Clearly  $\varphi_{2,p-4}$  is not a multiple of  $\varphi_{1,0}$  because  $p-4 \neq 0$ , which means  $\dim_{\mathbb{K}}(\ker \delta_0^2) \geq 2$ .

**Theorem 4.4.** dim<sub>K</sub>( $H^2_{cl}(W)$ ) = 1 and the cocycle  $\varphi_{1,0}$  in Lemma 4.3 generates  $H^2_{cl}(W)$ .

*Proof.* Lemma 4.2 shows that  $(H^2_{\rm cl})_k(W) = 0$  if  $k \neq 0$  and Lemma 4.3 shows that  $(H^2_{\rm cl})_0(W)$  is one dimensional with  $\varphi_{1,0}$  spanning the non-zero cohomology class.

1882

## 5. Restricted Lie Algebra cohomology of W

Since  $\delta^0 = \delta^0_{cl}$  and the restricted coboundary operator  $\delta^1$  is also injective, we have  $H^0(W) = \mathbb{K}$  and  $H^1(W) = 0$ .

**Lemma 5.1.** If  $(\varphi, \omega) \in C^2(W)$ , then  $\operatorname{ind}^2(\varphi, \omega) = 0$  and hence  $(\varphi, \omega) \in \ker \delta^2$  if and only if  $\varphi \in \ker \delta^2_{cl}$ .

*Proof.* It suffices to show that for all  $g, h \in W$ ,

$$g \wedge h^{[p]} - [g, h, \cdots, h] \wedge h = 0$$

where, in this proof, the bracket [g, h, ..., h] always means p - 1 factors of h. For  $q, h \in W$ , we have

$$g \wedge h^{[p]} = g \wedge \gamma(h)h = \gamma(h)(g \wedge h).$$

The algebra W is of rank one so there is a non-empty Zariski open subset  $U \subset W$  such if  $y \in U, x \in W$  and [x, y] = 0, then  $x \in \mathbb{K}y$ . Moreover,

$$[[g, h, \dots, h], h] = [g, h^{[p]}] = [g, \gamma(h)h] = [\gamma(h)g, h]$$

so that

$$[[g, h, \dots, h] - \gamma(h)g, h] = 0.$$

This shows that there is a scalar  $\gamma'(g,h) \in \mathbb{K}$  with

$$[g, h, \dots, h] - \gamma(h)g = \gamma'(g, h)h,$$

at least for  $h \in U$ . However, the mapping  $g \mapsto [g, h, \dots, h] - \gamma(h)g$  is algebraic, so

$$[g, h, \dots, h] = \gamma(h)g + \gamma'(g, h)h$$

for all  $h \in W$  and hence

$$[g,h,\ldots,h] \wedge h = \gamma(h)(g \wedge h) + \gamma'(g,h)(h \wedge h) = \gamma(h)(g \wedge h)$$

proving the lemma.

If  $\varphi_{1,0} \in C^2_{cl}(W)$  is the cocycle from Lemma 4.3 and  $\omega : W \to \mathbb{K}$  is any map with the \*-property with respect to  $\varphi_{1,0}$ , Lemma 5.1 implies that  $(\varphi_{1,0}, \omega) \in C^2(W)$  is a restricted cocycle. For  $-1 \leq i \leq p-2$ , let  $\omega_i : W \to \mathbb{K}$  be defined by

$$\omega_i(\alpha_{-1}e_{-1} + \dots + \alpha_{p-2}e_{p-2}) = \alpha_i^p.$$

**Lemma 5.2.** For  $-1 \leq i \leq p-2$ , the map  $\omega_i$  has the \*-property with respect to 0 and  $(0,\omega_i) \in C^2(W)$  is a cocycle. Moreover, if  $\omega : W \to \mathbb{K}$  is any map with the \*-property with respect to  $\varphi_{1,0}$ , the cohomology classes represented by the  $(0,\omega_i)$  and  $(\varphi_{1,0},\omega)$  comprise a linearly independent subset in  $H^2(W)$ .

*Proof.* If  $-1 \leq i \leq p-2$ , an easy computation shows that  $\omega_i$  is *p*-semilinear and hence has the \*-property with respect to 0. Therefore  $(0, \omega_i) \in C^2(W)$  and  $\delta^2(0, \omega_i) = (0, 0)$  by Lemma 5.1. If  $\alpha_i, \beta \in \mathbb{K}$  and

$$\sum \alpha_i(0,\omega_i) + \beta(\varphi_{1,0},\omega) = \left(\beta\varphi_{1,0}, \sum \alpha_i\omega_i + \beta\omega\right) = (0,0),$$

then  $\beta = 0$ , and evaluating at  $(0, e_j)$  shows  $\alpha_j = 0$ . Therefore

$$\mathcal{B} = \{(0, \omega_{-1}), \dots, (0, \omega_{p-2}), (\varphi_{1,0}, \omega)\}$$

is a linearly independent set in  $C^2(W)$ . Moreover, if

$$\sum \alpha_i(0,\omega_i) + \beta(\varphi_{1,0},\omega) = \left(\beta\varphi_{1,0},\sum \alpha_i\omega_i + \beta\omega\right) = \delta^1\psi = (\delta^1_{\rm cl}\psi,{\rm ind}^1\psi)$$

 $\square$ 

for some  $\psi \in C^1(W)$ , then  $\beta = 0$ , otherwise  $\varphi_{1,0} \in \operatorname{im} \delta^1_{\operatorname{cl}}$ . Therefore  $\delta^1_{\operatorname{cl}} \psi = 0$  so that  $\psi = 0$  and hence  $\alpha_i = 0$  showing  $\mathcal{B}$  is linearly independent in  $H^2(W)$ .  $\Box$ 

Proof of Theorem 3.1. For  $i = -1, \ldots, p-2$ , let  $\varphi_i = \delta_{cl}^1(e^i)$  and let  $\varphi_{p-1} = \varphi_{1,0}$ so that  $\{\varphi_{-1}, \ldots, \varphi_{p-2}, \varphi_{p-1}\}$  is a basis for ker $(\delta_{cl}^2)$  by Lemmas 4.2 and 4.3. For each *i*, choose a map  $\xi_i : W \to \mathbb{K}$  that has the \*-property with respect to  $\varphi_i$  so by Lemmas 5.1 and 5.2

$$\{(\varphi_{-1},\xi_{-1}),\ldots,(\varphi_{p-1},\xi_{p-1}),(0,\omega_1),\ldots,(0,\omega_p)\}$$

is a linearly independent subset of ker  $\delta^2$ . If  $(\varphi, \omega) \in \ker \delta^2$ , then  $\varphi \in \ker \delta^2_{cl}$ , so there are scalars  $\alpha_i$   $(-1 \leq i \leq p-1)$  such that  $\varphi = \sum \alpha_i \varphi_i$ . If  $\xi = \sum \alpha_i \xi_i$ , then  $\sum \alpha_i(\varphi_i, \xi_i) = (\varphi, \xi)$ . We have  $(0, \omega - \xi) \in C^2(W)$  which means there are scalars  $\beta_j$   $(1 \leq j \leq p)$  such that  $\omega - \xi = \sum \beta_j \omega_j$ . Therefore

$$(\varphi,\omega) = (\varphi,\xi) + \sum \beta_j(0,\omega_j) = \sum \alpha_i(\varphi_i,\xi_i) + \sum \beta_j(0,\omega_j).$$

This shows

$$\{(\varphi_{-1},\xi_{-1}),\ldots,(\varphi_{p-1},\xi_{p-1}),(0,\omega_1),\ldots,(0,\omega_p)\}$$

is a basis for ker  $\delta^2$  and hence  $\dim_{\mathbb{K}} \ker \delta^2 = 2p + 1$ . We have already seen that  $\dim_{\mathbb{K}} \operatorname{im} \delta^1 = p$  so  $\dim_{\mathbb{K}} H^2(W) = p + 1$ , and from Lemma 5.2 it follows that the cohomology classes represented by  $(\varphi_{p-1}, \xi_{p-1}), (0, \omega_1), \ldots, (0, \omega_p)$  form a basis for  $H^2(W)$ . Finally, the subspace spanned by the cohomology classes of  $(0, \omega_i)$  is clearly *p*-dimensional, and the ordinary (non-restricted) one-dimensional central extensions of *W* corresponding to these cohomology classes are trivial as ordinary Lie algebra extensions.

We conclude this section with explicit descriptions of the p + 1 one-dimensional restricted central extensions of W. For  $-1 \leq i \leq p - 2$ , let  $E_i$  denote the onedimensional restricted central extension of W determined by the cohomology class of the cocycle  $(0, \omega_i)$ . Then  $E_i = W \oplus \mathbb{K}c$  as a  $\mathbb{K}$ -vector space, and using (4) we have for all  $-1 \leq j, k \leq p - 2$ ,

$$[e_j, e_k] = (k - j)e_{j+k};$$
  

$$[e_j, c] = 0;$$
  

$$e_j^{[p]} = \delta_{0,j}e_0 + \delta_{i,j}c;$$
  

$$c^{[p]} = 0,$$

where  $\delta$  denotes Kronecker's delta-function.

Let us denote by  $\varphi$  the cocycle  $\varphi_{1,0}$  given in (6) and define  $\omega : W \to \mathbb{K}$  to have the \*-property with respect to  $\varphi$  using (1) and declaring  $\omega(e_j) = 0$  for all  $-1 \leq j \leq p-2$ . Note that  $\omega \neq 0$ , but  $\omega(0) = 0$  by (1). Now, for  $-1 \leq j, k \leq p-2$ , (6) gives

$$\varphi(e_{j,k}) = \frac{j(j^2 - 4)}{3} \delta_{0,j+k}.$$

Therefore, if  $E = W \oplus \mathbb{K}c$  denotes the one-dimensional central extension of W determined by the cocycle  $(\varphi, \omega)$ , we have for all  $-1 \leq j, k \leq p-2$ ,

$$[e_j, e_k] = (k - j)e_{j+k} + \frac{j(j^2 - 4)}{3}\delta_{0,j+k}c;$$
  

$$[e_j, c] = 0;$$
  

$$e_j^{[p]} = \delta_{0,j}e_0;$$
  

$$c^{[p]} = 0.$$

The Lie bracket in the extension E is similar to the bracket in the (characteristic zero) Virasoro algebra ([12], Def. 5.2) insofar as the coefficients in both corresponding cocycles are given by cubic polynomials of the same form. For this reason, it is natural to refer to the extension E as the (restricted) modular Virasoro algebra.

#### References

- Richard E. Block, On the Mills-Seligman axioms for Lie algebras of classical type, Trans. Amer. Math. Soc. 121 (1966), 378–392. MR0188356 (32 #5795)
- [2] D. B. Fuks and Tyler J. Evans, On the structure of the restricted Lie algebra for the Witt algebra in a finite characteristic (Russian, with Russian summary), Funktsional. Anal. i Prilozhen. 36 (2002), no. 2, 69–74, 96, DOI 10.1023/A:1015622607840; English transl., Funct. Anal. Appl. 36 (2002), no. 2, 140–144. MR1922020 (2003f:17020)
- [3] Tyler J. Evans and Dmitry Fuchs, A complex for the cohomology of restricted Lie algebras, J. Fixed Point Theory Appl. 3 (2008), no. 1, 159–179, DOI 10.1007/s11784-008-0060-y. MR2402915 (2009e:17035)
- [4] D. B. Fuks, Cohomology of infinite-dimensional Lie algebras, Contemporary Soviet Mathematics, Consultants Bureau, New York, 1986. Translated from the Russian by A. B. Sosinskiĭ. MR874337 (88b:17001)
- [5] D. B. Fuchs and I. M. Gelfand, Cohomology of the Lie algebra of vector fields on a circle, Functional Analysis and Its Applications, 2:342–343, 1968.
- [6] J. E. Humphreys, Existence of Levi factors in certain algebraic groups, Pacific J. Math. 23 (1967), 543–546. MR0218365 (36 #1452)
- [7] J. E. Humphreys, Restricted Lie algebras (and beyond), Algebraists' homage: papers in ring theory and related topics (New Haven, Conn., 1981), Contemp. Math., vol. 13, Amer. Math. Soc., Providence, R.I., 1982, pp. 91–98. MR685939 (84d:17012)
- [8] Nathan Jacobson, Abstract derivation and Lie algebras, Trans. Amer. Math. Soc. 42 (1937), no. 2, 206–224, DOI 10.2307/1989656. MR1501922
- [9] N. Jacobson, Restricted Lie algebras of characteristic p, Trans. Amer. Math. Soc. 50 (1941), 15–25. MR0005118 (3,103g)
- [10] Jens Carsten Jantzen, Representations of algebraic groups, 2nd ed., Mathematical Surveys and Monographs, vol. 107, American Mathematical Society, Providence, RI, 2003. MR2015057 (2004h:20061)
- [11] George J. McNinch, Levi decompositions of a linear algebraic group, Transform. Groups 15 (2010), no. 4, 937–964, DOI 10.1007/s00031-010-9111-8. MR2753264 (2012b:20110)
- [12] M. Schottenloher, A mathematical introduction to conformal field theory, 2nd ed., Lecture Notes in Physics, vol. 759, Springer-Verlag, Berlin, 2008. MR2492295 (2011a:81219)
- [13] F. V. Weinstein, Filtering bases: a tool to compute cohomologies of abstract subalgebras of the Witt algebra, Unconventional Lie algebras, Adv. Soviet Math., vol. 17, Amer. Math. Soc., Providence, RI, 1993, pp. 155–216. MR1254731 (94k:17042)

[14] N. N. Yakovlev, Cohomologies of Witt algebras, Functional Analysis and Its Applications, 9(3):269–271, July-September 1975.

Department of Mathematics, Humboldt State University, Arcata, California 95521 $E\text{-}mail\ address: evans@humboldt.edu$ 

INSTITUTE OF MATHEMATICS, EÖTVÖS LORÁND UNIVERSITY, H-1117 BUDAPEST HUNGARY *E-mail address*: fialowsk@cs.elte.hu

Department of Mathematics, University of Wisconsin-Eau Claire, Eau Claire, Wisconsin54701

 $E\text{-}mail \ address: \texttt{penkavmrQuwec.edu}$