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ON THE NUMBER OF FINITE p/q-SURGERIES

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Abstract. We study finite, non-cyclic knot surgeries, that is, surgeries which give manifolds of finite but not cyclic fundamental group. These manifolds are known to be knot surgeries except for the dihedral manifolds. We show that, for a fixed p, there are finitely many dihedral manifolds that are p/q-surgery, and we place a bound on which manifolds they may be. In the process, we calculate a recursive relationship among the Heegaard Floer d-invariants of dihedral manifolds with a given first homology and calculate a bound on which d-invariants would occur if such a manifold were surgery on a knot in S^3.

1. Introduction

A fundamental question of low-dimensional topology is which manifolds may be constructed by surgery on a knot in S^3. By Seifert's classification of Seifert fibered spaces and Perelman's geometrization, the manifolds of finite fundamental group fall into five types, the lens spaces, the icosahedrals, the octahedrals, the tetrahedrals, and the dihedrals [Sei33, MT07]. All icosahedrals, octahedrals, and tetrahedrals are known to be surgeries on a torus knot (see Proposition 1). The question of which lens spaces are surgeries on a non-trivial knot in S^3 is part of the Berge Conjecture and has been solved recently by J. Greene using an application of Heegaard Floer theory [Gre13]. We also use Heegaard Floer theory to show that, for a fixed surgery coefficient, only finitely many dihedral manifolds may be surgery.

Modeled on Seifert, we shall use the notation

$$\big(e_0; (\alpha_1, \omega_1), (\alpha_2, \omega_2), (\alpha_3, \omega_3)\big)$$

to represent the Seifert fibered space given by taking the S^1 bundle over S^2 with euler number e_0 and then performing $-\alpha_i/\omega_i$ Dehn surgeries on three regular fibers. These descriptions are not unique; for example, $\big(e_0; (\alpha_1, \omega_1)\big) = \big(e_0 + 1; (\alpha_1, \omega_1 - \alpha_1)\big)$. Reversing orientation reverses the signs on all the Seifert invariants.

By Perelman [MT07], all manifolds with finite fundamental group are Seifert fibered, which Seifert classified.

Theorem ([Sei33]). *Excluding lens spaces, the closed, oriented Seifert fibered spaces Y with finite fundamental group are*

(1) *Icosahedral with $H_1(Y) = \mathbb{Z}_m$ and $(m, 30) = 1$.*
(2) *Octahedral with $H_1(Y) = \mathbb{Z}_{2m}$ and $(m, 6) = 1$.*
(3) *Tetrahedral with $H_1(Y) = \mathbb{Z}_{3m}$ and $(m, 2) = 1$.*
(4) *Dihedral with $H_1(Y) = \mathbb{Z}_{4m}$ or $\mathbb{Z}_2 \times \mathbb{Z}_{2m}$.*

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Further, the first three kinds are uniquely determined (up to orientation) by m, but each choice of m corresponds to infinitely many dihedral manifolds with $H_1(Y) = \mathbb{Z}_{4m}$ and infinitely many with $\mathbb{Z}_2 \times \mathbb{Z}_{2m}$.

The dihedral manifolds with non-cyclic first homology may not, of course, be knot surgeries. Many of the others, however, are known to be surgeries as a result of Moser's work on torus knots [Mos71]. An explicit calculation shows (see, e.g., [Doi15, Corollary 8]):

Proposition 1. *Every icosahedral, octahedral, and tetrahedral manifold is surgery on a torus knot. Of each infinite family of dihedral manifolds with first homology $\mathbb{Z}_{4m}$, only finitely many are surgeries on torus knots, and they are $\big(-1; (2,1), (2,1), (n,m)\big)$ where n divides $2m+1$ or $2m-1$.*

In addition to these torus knot surgeries, some finite surgeries arise from iterated torus knots, which Bleiler and Hodgson explicitly listed [BH96, Theorem 7] based on Gordon's classification of satellite knots (all of the resulting manifolds are also surgeries on torus knots) [Gor83, Theorem 7.5], and Boyer and Zhang proved that no other satellite knots have finite surgeries [BZ96, Corollary 1.4].

The hyperbolic case is more complicated. For example, there are a variety of examples of finite, non-cyclic surgeries on hyperbolic knots. Fintushel and Stern mentioned the $(-2,3,7)$ pretzel knot, on which -17-surgery is finite [FS80, K_2 in Section 4], and Bleiler and Hodgson found the 22- and 23-surgeries on the $(-2,3,9)$ pretzel knot [BH96], although all three resulting manifolds are also surgery on torus knots. Mattman et al. showed that there are no other finite, non-cyclic surgeries on pretzel knots [Mat02, Theorem 1.2], [FIK$^+$09, Theorem 1]. There are other restrictions on which hyperbolic knots have finite surgeries, too. Boyer and Zhang showed that any hyperbolic knot has at most five finite or cyclic surgeries, with at most one half-integral and the others integral [BZ01, Theorems 1.1, 1.2]. Li and Ni showed that any such half-integral surgery is also half-integral surgery on an iterated torus knot with the same knot Floer homology [LN].

In [Doi15], we showed that many dihedral manifolds cannot be realized as surgery on any knot in S^3. In particular, finite, non-cyclic surgeries on hyperbolic knots must have surgery coefficient $|p| \geq 7$. We classified the elliptic manifolds Y with $|H_1(Y)| \leq 32$ which are surgery on a knot in S^3, and we conjectured that, for a fixed order of first homology, there are at most finitely many [Doi15, Conjecture 17].

Theorem 2. *Of the [potentially infinite] family of manifolds with finite fundamental group and a given first homology, only finitely many are surgery on a knot in S^3.*

For a fixed order of homology, there are only finitely many lens, tetrahedral, octahedral, and icosahedral manifolds. To prove the result, we use the Heegaard Floer d-invariants to obstruct all but finitely many dihedral manifolds with fixed order of homology from being surgery on a knot in S^3. First, we calculate the Heegaard Floer d-invariants for the dihedral manifolds Y with homology $\mathbb{Z}_{4m}$ by applying an algorithm of Némethi [Ném05] to the known Seifert invariants. This calculation may be of independent interest, given the recent applications of Heegaard Floer theory to other questions in low-dimensional topology and knot theory. The d-invariants for the manifolds in question obey a recursive relationship.

Theorem 3. *Fix an integer $m > 0$. Consider the family of dihedral manifolds*

$$Y_n = (-1; (2,1), (2,1), (n,m))$$

with $n > 2m$. There is an ordering of $Spin^c(Y_n) = \{\sigma_n^0, \sigma_n^1, \cdots, \sigma_n^{4m-1}\}$ so that

$$d(Y_{n+m}, \sigma_{n+m}^i) = \begin{cases} d(Y_n, \sigma_n^i) - \frac{1}{4} & \text{if } 0 \le i < 2m, \\ d(Y_n, \sigma_n^i) & \text{if } 2m \le i < 4m. \end{cases}$$

Similarly, there is an ordering on $Spin^c(Y_{-n}) = \{\sigma_{-n}^0, \sigma_{-n}^1, \cdots, \sigma_{-n}^{4m-1}\}$ for all n so that

$$d(Y_{-(n+m)}, \sigma_{-(n+m)}^i) = \begin{cases} d(Y_{-n}, \sigma_{-n}^i) + \frac{1}{4} & \text{if } 0 \le i < 2m, \\ d(Y_{-n}, \sigma_{-n}^i) & \text{if } 2m \le i < 4m. \end{cases}$$

In particular, for a fixed natural number N, there are only finitely many Y_n with $|d(Y_n, \sigma)| \le N$ for all σ. This theorem is actually independent of the parity of n; however, when searching for manifolds which are surgery on a knot in S^3, we may ignore even n because $H_1(Y_n)$ is not cyclic.

We then use the fact that dihedral manifolds are L-spaces, rational homology spheres with Heegaard Floer homologies as simple as possible, e.g., $\widehat{HF}(Y, \sigma) \cong \widehat{HF}(S^3)$ [OS05, Proposition 2.3]. Any positive p/q-surgery on a knot K which produces an L-space obeys three rules: K is fibered [Ni07, Corollary 1.3]; $p/q \ge 2g(K) - 1$ [OS11, Corollary 1.4]; and the d-invariants of the resulting manifold may be calculated from the Alexander polynomial and the d-invariants of $L(p,q)$ [OS03a, Proposition 4.8], which are themselves given by a recursive formula [OS11, Theorem 1.2]. In particular, for a given order of the first homology, it is possible to place a bound on the d-invariants:

Theorem 4. *For a fixed $m > 0$ and for any K so that $S^3_{4m}(K)$ is an L-space, the d-invariants are bounded:*

$$|d(S^3_{4m}(K), \sigma)| \le 4m$$

for all $\sigma \in Spin^c(S^3_{4m}(K))$. In fact, $-4m \le d(S^3_{4m}(K), \sigma) \le m$.

2. The d-invariants of Y_n

We begin with a calculation of the recursive relationship of the d-invariants for dihedral manifolds. For a fixed m, the dihedral manifolds $\{Y_n\}$ are often described with the Seifert invariants $(-1; (2,1), (2,1), (n,m))$ with $gcd(n,m) = 1$. First homology is $\mathbb{Z}_{4m}$ for odd n and $\mathbb{Z}_{2m} \times \mathbb{Z}_2$ for even n. The techniques below require a negative definite intersection form, so, if $n > 0$, we will reverse orientation and massage the Seifert invariants below so that Y_{-n} bounds a negative definite four-manifold:

$$-Y_n = \big(-2; (2,1), (2,1), (n, n-m)\big).$$

Observe that the Seifert invariants for $-Y_n$ are equal to those for Y_{-n}.

Recall the Heegaard Floer *d-invariants* or *correction terms* assigned to a rational homology sphere and choice of spin-c structure. The value $d(Y, \sigma)$ is the minimal grading of an element in $HF^+(Y, \sigma)$ coming from $HF^\infty(Y, \sigma)$. Two standard methods to calculate the d-invariants are that of Ozsváth and Szabó, based on a plumbing diagram [OS03b], and Némethi, based on graded roots, although, for elliptic manifolds, it can be written in terms of the Seifert invariants alone [Ném05].

For the course of this paper, we use the standard identification of the spin-c structures of the lens space $S^3_{p/q}(U)$ with $\mathbb{Z}_p$ based on a Heegaard diagram as in [OS03a, Proposition 4.8]; for a more general surgery $S^3_{p/q}(K)$, we also use the extension of this identification of the spin-c structures with $\mathbb{Z}_p$ based on a surgery cobordism as in [OS11, Theorem 1.2]. Recall that $d(Y,\sigma) = -d(-Y,\sigma)$, and the d-invariants for a lens space admit a $\mathbb{Z}_2$-action that preserves the spin structure and is compatible with the chosen identification, that is, if s is a spin structure, then $d(L(p,q), s+i) = d(L(p,q), s-i)$ [OS03a].

Proof of Theorem 3. We will assign a numbering to the spin-c structures for Y_n and show (for $n > 2m$) that

$$d(-Y_{n+m}, \sigma^i_{n+m}) - d(-Y_n, \sigma^i_n) = \begin{cases} \frac{1}{4} & \text{if } 0 \leq i < 2m, \\ 0 & \text{if } 2m \leq i < 4m. \end{cases}$$

Némethi [Ném05, Section 11.13] demonstrates one method for calculating $d(Y,\sigma)$ for Seifert fibered spaces with negative definite plumbing diagrams (all terms will be defined below):

$$d(Y,\sigma) = \frac{K^2+s}{4} - 2\chi(\sigma) - 2\min_{i\geq 0} \tau(i). \tag{1}$$

1. Assign variables to the Seifert invariants $\big(e_0; (\alpha_1,\omega_1), (\alpha_2,\omega_2), (\alpha_3,\omega_3)\big)$ where $0 < \omega_i < \alpha_i$. Choose ω_i' such that $0 < \omega_i' < \alpha_i$ and $\omega_i\omega_i' \equiv 1 \bmod \alpha_i$. That is, for Y_n,

$$\begin{aligned} &e_0 = -2, \\ &\alpha_1 = 2, \quad \omega_1 = 1, \quad \omega_1' = 1, \\ &\alpha_2 = 2, \quad \omega_2 = 1, \quad \omega_2' = 1, \\ &\alpha_3 = n, \quad \omega_3 = n-m, \quad \omega_3' \equiv -m^{-1} \bmod n. \end{aligned}$$

Consider also the values $e = e_0 + \sum_{l=1}^3 \frac{\omega_l}{\alpha_l}$, and $\varepsilon = (2 - 3 + \sum_{l=1}^3 \frac{1}{\alpha_l})/e$. Then

$$e = -\frac{m}{n}, \qquad \varepsilon = -\frac{1}{m}.$$

Similarly, substituting $n+m$ for n, it is possible to see that $-Y_{n+m}$ has the same invariants with the exception of

$$\begin{aligned} &e = -\tfrac{m}{n+m}, \\ &\alpha_3 = n+m, \quad \omega_3 = n, \quad \omega_3' \equiv -m^{-1} \bmod (n+m). \end{aligned}$$

2. $K^2 + s$ is defined

$$K^2 + s = \varepsilon^2 e + e + 5 - 12\sum_{l=1}^{3} \mathbf{s}(\omega_l, \alpha_l)$$

where $\mathbf{s}(\omega_l, \alpha_l)$ is a Dedekind sum. Therefore, by Proposition 5, K^2+s differs for $-Y_{n+m}$ and $-Y_n$ by

$$\begin{aligned} &(K^2+s)_{n+m} - (K^2+s)_n \\ &\qquad = -\frac{1}{m(n+m)} - \frac{m}{n+m} + \frac{1}{mn} + \frac{m}{n} \\ &\qquad\qquad - 12\big(\mathbf{s}(n, n+m) - \mathbf{s}(n-m, n)\big) = 1. \end{aligned}$$

3. Each $\sigma_n^i \in Spin^c(-Y_n)$ corresponds to one of the $4m$ distinct integral vectors (a_0, a_1, a_2, a_3) which satisfy

$$(2) \qquad \begin{cases} 0 \leq a_0;\ 0 \leq a_l < \alpha_l, & l = 1,2,3, \\ s(i) = 1 + a_0 + ie_0 + \displaystyle\sum_{l=1}^{3} \left\lfloor \frac{i\omega_l + a_l}{\alpha_l} \right\rfloor \leq 0 & \forall i > 0. \end{cases}$$

See Proposition 6 for the solutions in the case of $-Y_n$ if $n > 2m$:

$$\begin{array}{ll} (0,0,0,a_3) & \text{for } a_3 = 0, 1, \ldots, 2m-1, \\ (0,0,1,a_3) & \text{for } a_3 = 0, 1, \ldots, m-1, \\ (0,1,0,a_3) & \text{for } a_3 = 0, 1, \ldots, m-1. \end{array}$$

4. Then each (a_0, a_1, a_2, a_3) corresponds to a spin-c structure σ and gives

$$-\chi(\sigma) = \sum_{l=0}^{3} \frac{a_l}{2} + \frac{\varepsilon\tilde{a}}{2} + \frac{\tilde{a}^2}{2e} - \sum_{l=1}^{3} \sum_{i=1}^{a_l} \left\{ \frac{i\omega_l'}{\alpha_l} \right\}$$

if $\tilde{a} = a_0 + \sum_{l=1}^{3} \frac{a_l}{\alpha_l}$ and $\{x\}$ is the fractional part of x.

Fix $(0,0,0,a_3)$ considered as a solution for (2) for both $-Y_{n+m}$ and $-Y_n$. First,

$$(\tilde{a})_n = \frac{a_3}{n} \quad \text{and} \quad (\tilde{a})_{n+m} = \frac{a_3}{n+m}.$$

Next, compare $-\chi(\sigma_{n+m})$ for $-Y_{n+m}$ and $\chi(\sigma_n)$ for $-Y_n$ using Proposition 7

$$\begin{aligned} &-2\big(\chi(\sigma_{n+m}) - \chi(\sigma_n)\big) \\ &\qquad = \frac{a_3^2 + a_3}{n(n+m)} - 2\sum_{i=1}^{a_3} \left\{ \frac{i(\omega_3')_{n+m}}{n+m} \right\} + 2\sum_{i=1}^{a_3} \left\{ \frac{i(\omega_3')_n}{n} \right\} \\ &\qquad = \frac{a_3^2 + a_3}{n(n+m)} - 2\sum_{i=1}^{a_3} \frac{i}{n(n+m)} = 0. \end{aligned}$$

Finally, the solutions $(0,1,0,a_3)$ and $(0,0,1,a_3)$ give

$$(\tilde{a})_n = \frac{a_3}{n} + \frac{1}{2} \quad \text{and} \quad (\tilde{a})_{n+m} = \frac{a_3}{n+m} + \frac{1}{2},$$

so that

$$-2\big(\chi(\sigma_{n+m}) - \chi(\sigma_n)\big) = \frac{a_3^2 + a_3}{n(n+m)} - \frac{1}{4} - 2\sum_{i=1}^{a_3} \frac{i}{n(n+m)} = -\frac{1}{4}.$$

5. Proposition 8 shows that $\min \tau(i) = 0$.
6. Finally, applying the results of paragraphs 2, 4, and 5 above to equation (1) implies

$$d(-Y_{n+m}, \sigma_{n+m}) = d(-Y_n, \sigma_n) + \frac{1}{4}$$

for σ_{n+m} and σ_n the spin-c structures that correspond to $(0,0,0,a_3)$, and

$$d(-Y_{n+m}, \sigma_{n+m}) = d(-Y_n, \sigma_n)$$

for σ_{n+m} and σ_n corresponding to $(0,0,1,a_3)$ or $(0,1,0,a_3)$. Reversing orientation and making a reasonable choice of ordering on spin-c structures gives the theorem statement. $\square$

Calculations suggest that the above result holds for $m < n < 2m$ as well, although the (a_0, a_1, a_2, a_3) from paragraph 3 are different for some spin-c structures, and the corresponding calculations involving $\chi(\sigma)$ are problematic. See Conjecture 10.

We prove assorted results required in the previous proof.

Proposition 5. *If* $\gcd(m, n) = 1$,

$$-12\big(\boldsymbol{s}(n, n+m) - \boldsymbol{s}(n-m, n)\big) = 3 - \frac{n+m}{n} - \frac{n}{n+m} - \frac{1}{n(n+m)}.$$

Proof. Note that $\mathbf{s}(n+m, n) = \mathbf{s}(m, n)$ and $\mathbf{s}(n-m, n) = \mathbf{s}(-m, n) = -\mathbf{s}(m, n)$. The result now follows from the Dedekind sum reciprocity formula:

$$\mathbf{s}(a, b) + \mathbf{s}(b, a) = \frac{1}{12}\left(\frac{a}{b} + \frac{b}{a} + \frac{1}{ab}\right) - \frac{1}{4}.$$

□

Proposition 6. *Consider*

$$\begin{cases} 0 \le a_0; \quad 0 \le a_1 < 2; \quad 0 \le a_2 < 2; \quad 0 \le a_3 < n; \\ s(i) = 1 + a_0 - 2i + \left\lfloor \frac{i+a_1}{2} \right\rfloor + \left\lfloor \frac{i+a_2}{2} \right\rfloor + \left\lfloor \frac{i(n-m)+a_3}{n} \right\rfloor \le 0 \quad \forall i > 0. \end{cases}$$

If $n > 2m$, *the integral solutions are*

$$\begin{aligned} &(0,0,0,a_3) && \textit{for } a_3 = 0, 1, \ldots, 2m-1, \\ &(0,0,1,a_3) && \textit{for } a_3 = 0, 1, \ldots, m-1, \\ &(0,1,0,a_3) && \textit{for } a_3 = 0, 1, \ldots, m-1. \end{aligned}$$

Proof. Check that $(0, 0, 0, 2m-1)$ satisfies $s(i) \le 0$ if $i > 0$: observe that $s(1) = s(2) = 0$, and $s(i+2) - s(i) \le 0$. Then

$$\begin{aligned} &s(i+2) - s(i) \\ &\qquad = 1 - 2(i+2) + 2\left\lfloor \frac{i+2}{2} \right\rfloor + \left\lfloor \frac{(i+2)(n-m) + 2m - 1}{n} \right\rfloor \\ &\qquad\quad - 1 + 2i - 2\left\lfloor \frac{i}{2} \right\rfloor - \left\lfloor \frac{i(n-m) + 2m - 1}{n} \right\rfloor \\ &\qquad = -2 + \left\lfloor \frac{(i+2)(n-m) + 2m - 1}{n} \right\rfloor - \left\lfloor \frac{i(n-m) + 2m - 1}{n} \right\rfloor \\ &\qquad \le -2 + \left\lceil \frac{2(n-m)}{n} \right\rceil \le 0. \end{aligned}$$

Similarly, $(0, 0, 1, m-1)$ and $(0, 1, 0, m-1)$ are solutions, as $s(1) = 0$ and $s(i+1) - s(i) \le 0$. If (a_0, a_1, a_2, a_3) is a solution, then so is $(a_0 - 1, a_1, a_2, a_3)$, if $a_0 > 0$; $(a_0, a_1 - 1, a_2, a_3)$, if $a_1 > 0$; etc. This produces $4m$ integral solutions, and Némethi's algorithm says that there are exactly $|H_1(Y_n)| = 4m$ solutions. It is also easy to verify directly that $(0, 0, 0, 2m)$, $(0, 0, 1, m)$, $(0, 1, 0, m)$, and $(1, 0, 0, 0)$ are not solutions. □

Proposition 7. *Fix* $0 < m < n$. *For any* i *with* $0 < i < n+m$, *if there are integers* $0 < x < n+m$ *and* $0 < y < n$ *such that*

$$\begin{aligned} xm &\equiv -1 \bmod (n+m), \\ ym &\equiv -1 \bmod n, \end{aligned}$$

then

$$\left\{\frac{ix}{n+m}\right\} - \left\{\frac{iy}{n}\right\} = \frac{i}{n(n+m)}.$$

Proof. Given y, there is a $0 < k < m$ such that $ym = nk - 1$. Note that $(y+k)m = (n+m)k - 1$, and $y + k < n + m$, so $x = y + k$. Therefore,

$$\frac{ix}{n+m} = \frac{ixn}{n(n+m)} = \frac{iyn + ikn}{n(n+m)} = \frac{iyn + iym + i}{n(n+m)},$$
$$\frac{iy}{n} = \frac{iyn + iym}{n(n+m)}$$

so $\frac{ix}{n+m} - \frac{iy}{n} = \frac{i}{n(n+m)}$.

Recall that $\{a\} - \{b\} = \{a - b\}$ as long as $\{b\} \leq \{a\}$. In this case, note that

$$\left\{\frac{iy}{n}\right\} \leq \left\{\frac{iy}{n} + \frac{i}{n(n+m)}\right\}$$

iff

$$\left\{\frac{iy}{n}\right\} \leq \left\{1 - \frac{i}{n(n+m)}\right\},$$

which is true since the left hand side is a multiple of $\frac{1}{n}$, and the right hand side is

$$\left\{1 - \frac{i}{n(n+m)}\right\} \geq \left\{1 - \frac{n+m}{n(n+m)}\right\} = \left\{1 - \frac{1}{n}\right\}.$$

Finally, $0 < i < n + m$, so $\left\{\frac{i}{n(n+m)}\right\} = \frac{i}{n(n+m)}$. □

Proposition 8.

$$\min_{i \geq 0} \tau(i) = \tau(0) = 0.$$

Proof. The $\tau(i)$ are defined by setting $\tau(0) = 0$ and

$$\tau(i+1) - \tau(i) = 1 + a_0 - ie_0 + \sum_{l=1}^{3} \left\lfloor \frac{-i\omega_l + a_l}{\alpha_l} \right\rfloor$$

when $i \geq 0$. For $(a_0, a_1, a_2, a_3) = (0, 0, 0, a_3)$ where $0 \leq a_3 < 2m$,

$$\begin{aligned}\tau(i+1) - \tau(i) &= 1 + 2i + 2\left\lfloor -\frac{i}{2} \right\rfloor + \left\lfloor \frac{-i(n-m) + a_3}{n} \right\rfloor \\ &\geq i + \left\lfloor -\frac{i(n-m)}{n} \right\rfloor + \left\lfloor \frac{a_3}{n} \right\rfloor \geq \left\lfloor \frac{a_3}{n} \right\rfloor = 0 \quad \forall i \geq 0.\end{aligned}$$

For $(0, 1, 0, a_3)$ or $(0, 0, 1, a_3)$ with $0 \leq a_3 < m$,

$$\begin{aligned}\tau(i+1) - \tau(i) &= 1 + 2i + \left\lfloor -\frac{i}{2} \right\rfloor + \left\lfloor -\frac{i+1}{2} \right\rfloor + \left\lfloor \frac{-i(n-m) + a_3}{n} \right\rfloor \\ &\geq i + \left\lfloor -\frac{i(n-m)}{n} \right\rfloor + \left\lfloor \frac{a_3}{n} \right\rfloor \geq \left\lfloor \frac{a_3}{n} \right\rfloor = 0 \quad \forall i \geq 0,\end{aligned}$$

which means $\tau(i)$ is increasing and

$$\min_{i \geq 0} \tau(i) = \tau(0) = 0.$$

□

3. The d-invariants obstruct surgery

Theorem 3 provides a recursive formula for $d(Y_n, \sigma)$. In particular, it shows that, for half of the possible σ, $d(Y_n, \sigma) \to -\infty$ as $n \to \infty$. However, L-spaces like the Y_n can only be knot surgeries under certain circumstances. We prove Theorem 4, which shows that the $d(Y_n, \sigma)$ are, in fact, bounded if Y_n is surgery on a knot.

Proof of Theorem 4. Say $S^3_{4m}(K)$ is a dihedral manifold. Then it is an L-space [OS05, Proposition 2.3].

Assume $0 < q < p$. By [OS11, Corollary 1.4], if K admits a positive L-space surgery, then $S^3_{p/q}(K)$ is an L-space iff $\frac{p}{q} \geq 2g(K) - 1$. In this case, $p/q = 4m$, so

$$g(K) \leq 2m.$$

Recall that we chose as assignment of the spin-c structures to $\mathbb{Z}_p$ as in [OS11, Theorem 1.2].

Then [OS11, Theorem 1.2] gives a formula for the d-invariants of a surgery, $d(S^3_{p/q}(K), i) = d(S^3_{p/q}(U), i) - 2\sum_{j=1}^{\infty} j a_{|\lfloor i/q \rfloor|+j}$ for $|i| \leq p/2$ where the a_j are the coefficients of the symmetrized Alexander polynomial. Therefore,

$$d(S^3_{4m}(K), i) = d(S^3_{4m}(U), i) - 2\sum_{j=1}^{\infty} j a_{|i|+j} \tag{3}$$

with i chosen so $-2m < i \leq 2m$. Additionally, [OS03a, Proposition 4.8] gives a formula for the lens space: $d(S^3_{p/q}(U), i) = -\left(\frac{pq-(2i+1-p-q)^2}{4pq}\right) - d(S^3_{q/r}(U), j)$ where $r \equiv p \pmod q$, $j \equiv i \pmod q$, and i and j are chosen so $0 \leq i < p$, $0 \leq j < q$. Since $d(S^3) = 0$,

$$d(S^3_{4m}(U), i) = -\frac{4m - (2i-4m)^2}{16m} = -\frac{1}{4} + \frac{(i-2m)^2}{4m} \tag{4}$$

where $0 \leq i < 4m$. By equation (4), $d(S^3_{4m}(U), i) = d(S^3_{4m}(U), 4m - i)$; because $-i \equiv 4m - i \pmod{4m}$, equation (3) therefore shows $d(S^3_{4m}(K), -i) = d(S^3_{4m}(K), i)$. We will now consider only $0 \leq i \leq \frac{p}{2} = 2m$.

By the second derivative test, the minimum occurs at $i = 2m$ and the maximum at $i = 0$ (recall we need only calculate $0 \leq i \leq 2m$):

$$-\frac{1}{4} \leq d(L_{4m,1}, i) \leq m - \frac{1}{4}.$$

By Proposition 9 below, for $|i| \leq g(K)$,

$$0 \leq \sum_{j=1}^{\infty} j a_{|i|+j} \leq g(K) - 1 \leq 2m - 1.$$

Finally,

$$-4m + \frac{7}{4} \leq d(S^3_{4m}(K), i) \leq m - \frac{1}{4}.$$

□

For the proof above, we only need to bound $\sum_{j=1}^{\infty} j a_{|i|+j}$ by a rational function of m like $g(K) \leq 2m$, but the following may be of independent interest since it appears to be sharpest possible.

Proposition 9. *Let K be a knot with genus $g = g(K)$ and normalized Alexander polynomial*

$$\Delta_K(T) = a_0 + \sum_{j=1}^{g} a_j(T^j + T^{-j})$$

which admits an L-space surgery. If $|i| < g(K) - 1$,

$$0 \le \sum_{j=1}^{\infty} j a_{|i|+j} \le g(K) - |i| - 1. \tag{5}$$

If $|i| = g(K) - 1$, then the sum is 1; if $|i| = g(K)$, then 0.

Proof. If K has L-space surgeries, then the non-zero a_i are alternating $+1$s and -1s [OS05, Corollary 1.3] where the highest non-zero term is $a_g = +1$ (since K is fibered [Ni07, Corollary 1.3]) and the second highest term is $a_{g-1} = -1$ [HW].

Consider any sequence $\{a_j\}_{j=0}^n$ whose non-zero terms are alternating $+1$s and -1s. Let the subsequence of non-zero terms be $\{a_{j_i}\}_{i=1}^k$. If the top term $a_{j_k} = +1$, and k is even, then $a_{j_{2i}} = +1$ and $a_{j_{2i-1}} = -1$, so

$$\sum_{j=1}^{n} j a_j = \sum_{i=1}^{k} j_i a_{j_i} = \sum_{i=1}^{k/2} (j_{2i} - j_{2i-1}) \ge 0.$$

If $a_{j_k} = +1$ and k is odd, $a_{j_{2i+1}} = +1$ and $a_{j_{2i}} = -1$, so

$$\sum_{j=1}^{n} j a_j = \sum_{i=1}^{k} j_i a_{j_i} = j_1 + \sum_{i=1}^{(k-1)/2} (j_{2i+1} - j_{2i}) \ge 0.$$

Similarly, if the top term a_n is -1 instead of $+1$, then

$$\sum_{j=1}^{n} j a_j \le 0.$$

Therefore, if $a_k = +1$,

$$\sum_{i=1}^{k} j_i a_{j_i} = j_k a_{j_k} + \sum_{i=1}^{k-1} j_i a_{j_i} \le j_k.$$

If $j_k > 1$ and the second highest term is $a_{j_k - 1} = -1$, then

$$\sum_{i=1}^{k} j_i a_{j_i} = j_k - (j_k - 1) + \sum_{i=1}^{k-2} j_i a_{j_i} \le j_k - 1.$$

Finally, if $\{a_j\}_{j=0}^g$ are the coefficients of the symmetrized Alexander polynomial $\Delta_K(T)$, then $\{a_{|i|+j}\}_{j=0}^{g-|i|}$ is a sequence whose non-zero terms are alternating $+1$s and -1s and whose top terms (if they exist) are $a_{|i|+(g-|i|)} = +1$ and $a_{|i|+(g-|i|-1)} = -1$. If $g - |i| \ge 2$,

$$\sum_{j=1}^{\infty} j a_{|i|+j} = \sum_{j=1}^{g-|i|} j a_{|i|+j} \le g - |i| - 1.$$

A direct calculation also shows the sum is 1 if $g - |i| = 1$ and 0 if $g - |i| = 0$. $\square$

Finally, we derive Theorem 2 from Theorems 3 and 4.

Proof of Theorem 2. Recall that the finite surgeries on non-hyperbolic knots are classified. Seifert classified the dihedral manifolds with $|H_1(Y)| = 4m$ as $Y_n = \big(-1; (2,1), (2,1), (n,m)\big)$. If Y_n is $S^3_{4m/q}(K)$ and K is hyperbolic, then $q = 1$ (Boyer and Zhang showed finite exceptional surgeries are integral or half-integral [BZ01, Theorems 1.1, 1.2]).

For a fixed m, the d-invariants may be arbitrarily large: Theorem 3 shows

$$\exists \sigma_n \in Spin^c(Y_n) \quad \text{s.t.} \quad \lim_{n\to\infty} \big|d(Y_n, \sigma_n)\big| = \infty.$$

However, if Y_n is a surgery, its d-invariants are bounded by m: if $Y_n = S^3_{4m}(K)$, Theorem 4 shows

$$\forall \sigma \in Spin^c(Y_n), \qquad |d(Y_n, \sigma)| \leq 4m.$$

□

4. Conjectures

Calculations indicate that the condition that $n > 2m$ in Theorem 3 is unnecessary. The condition $n > 2m$ appears in the proof only in paragraph 3 (using Proposition 6) where some spin-c structures correspond to vectors $(0, 0, 0, a_3)$ with $a_3 < 2m - 1$. For example, if $m < n < 2m$, then $a_3 < n$, and there are additional solutions $(1, 0, 0, a_3)$ for $a_3 < 2m - n$. The proof carries through exactly as before for all cases except the $(1, 0, 0, a_3)$, where the calculations $\chi(\sigma_{n+m}) - \chi(\sigma_n)$ are challenging.

Conjecture 10. *Theorem* 3 *is true for all* n*.*

The above observations indicate also that no two manifolds in the family $\{Y_n\}$ for a fixed n (mod m) share the same set of d-invariants. In fact, additional calculations suggest that the same is true even for $n < 0$. Since the d-invariants are spin-c homology cobordism invariants by [OS03a, Theorem 1.2], it seems that the d-invariants may be useful in classifying the homology cobordism classes of elliptic manifolds, as studied in, e.g., [FS87].

Conjecture 11. *No distinct dihedral manifolds are rational homology cobordant.*

Additionally, calculations indicate (see [Doi15, Corollary 5]) that the bound in Theorem 2 could be stated more explicitly. Improving these bounds would require an explicit calculation of the d-invariants in Theorem 3 rather than a recursive relation. It may also require a better bound for Theorem 4, perhaps coming from a better understanding of the Alexander polynomials of L-space knots and so a better version of Proposition 9.

Conjecture 12. *Of the infinite family of dihedral manifolds*

$$Y_n = \big(-1; (2,1), (2,1), (n,m)\big),$$

the only ones which are surgery on a hyperbolic knot have odd $|n| \leq 2m + 1$*.*

References

[BH96] Steven A. Bleiler and Craig D. Hodgson, *Spherical space forms and Dehn filling*, Topology **35** (1996), no. 3, 809–833, DOI 10.1016/0040-9383(95)00040-2. MR1396779 (97f:57007)

[BZ96] S. Boyer and X. Zhang, *Finite Dehn surgery on knots*, J. Amer. Math. Soc. **9** (1996), no. 4, 1005–1050, DOI 10.1090/S0894-0347-96-00201-9. MR1333293 (97h:57013)

[BZ01] Steven Boyer and Xingru Zhang, *A proof of the finite filling conjecture*, J. Differential Geom. **59** (2001), no. 1, 87–176. MR1909249 (2003k:57007)

[Doi15] Margaret I. Doig, *Finite knot surgeries and Heegaard Floer homology*, Algebr. Geom. Topol. **15** (2015), no. 2, 667–690, DOI 10.2140/agt.2015.15.667. MR3342672

[FIK$^+$09] David Futer, Masaharu Ishikawa, Yuichi Kabaya, Thomas W. Mattman, and Koya Shimokawa, *Finite surgeries on three-tangle pretzel knots*, Algebr. Geom. Topol. **9** (2009), no. 2, 743–771, DOI 10.2140/agt.2009.9.743. MR2496889 (2010a:57008)

[FS80] Ronald Fintushel and Ronald J. Stern, *Constructing lens spaces by surgery on knots*, Math. Z. **175** (1980), no. 1, 33–51, DOI 10.1007/BF01161380. MR595630 (82i:57009a)

[FS87] Ronald Fintushel and Ronald Stern, *Rational homology cobordisms of spherical space forms*, Topology **26** (1987), no. 3, 385–393, DOI 10.1016/0040-9383(87)90008-5. MR899056 (89d:57051)

[Gor83] C. McA. Gordon, *Dehn surgery and satellite knots*, Trans. Amer. Math. Soc. **275** (1983), no. 2, 687–708, DOI 10.2307/1999046. MR682725 (84d:57003)

[Gre13] Joshua Evan Greene, *The lens space realization problem*, Ann. of Math. (2) **177** (2013), no. 2, 449–511, DOI 10.4007/annals.2013.177.2.3. MR3010805

[HW] Matthew Hedden and Liam Watson, *On the geography and botany of knot Floer homology*, arXiv:1404.6913.

[LN] Eileen Li and Yi Ni, *Half-integral finite surgeries on knots in S^3*, arXiv:1310.1346.

[Mat02] Thomas W. Mattman, *Cyclic and finite surgeries on pretzel knots*, J. Knot Theory Ramifications **11** (2002), no. 6, 891–902, DOI 10.1142/S0218216502002037. Knots 2000 Korea, Vol. 3 (Yongpyong). MR1936241 (2003j:57013)

[Mos71] Louise Moser, *Elementary surgery along a torus knot*, Pacific J. Math. **38** (1971), 737–745. MR0383406 (52 #4287)

[MT07] John Morgan and Gang Tian, *Ricci flow and the Poincaré conjecture*, Clay Mathematics Monographs, vol. 3, American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2007. MR2334563 (2008d:57020)

[Ném05] András Némethi, *On the Ozsváth-Szabó invariant of negative definite plumbed 3-manifolds*, Geom. Topol. **9** (2005), 991–1042, DOI 10.2140/gt.2005.9.991. MR2140997 (2006c:57011)

[Ni07] Yi Ni, *Knot Floer homology detects fibred knots*, Invent. Math. **170** (2007), no. 3, 577–608, DOI 10.1007/s00222-007-0075-9. MR2357503 (2008j:57053)

[OS03a] Peter Ozsváth and Zoltán Szabó, *Absolutely graded Floer homologies and intersection forms for four-manifolds with boundary*, Adv. Math. **173** (2003), no. 2, 179–261, DOI 10.1016/S0001-8708(02)00030-0. MR1957829 (2003m:57066)

[OS03b] Peter Ozsváth and Zoltán Szabó, *On the Floer homology of plumbed three-manifolds*, Geom. Topol. **7** (2003), 185–224 (electronic), DOI 10.2140/gt.2003.7.185. MR1988284 (2004h:57039)

[OS05] Peter Ozsváth and Zoltán Szabó, *On knot Floer homology and lens space surgeries*, Topology **44** (2005), no. 6, 1281–1300, DOI 10.1016/j.top.2005.05.001. MR2168576 (2006f:57034)

[OS11] Peter S. Ozsváth and Zoltán Szabó, *Knot Floer homology and rational surgeries*, Algebr. Geom. Topol. **11** (2011), no. 1, 1–68, DOI 10.2140/agt.2011.11.1. MR2764036 (2012h:57056)

[Sei33] H. Seifert, *Topologie Dreidimensionaler Gefaserter Räume* (German), Acta Math. **60** (1933), no. 1, 147–238, DOI 10.1007/BF02398271. MR1555366

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