

AN OPTIMIZATION PROBLEM AND ITS APPLICATION IN POPULATION DYNAMICS

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ABSTRACT. This paper is concerned with a diffusive logistic model in population ecology. As observed by Y. Lou, in a spatially heterogeneous environment, this model can always support a total population at equilibrium greater than the total carrying capacity. In other words, the ratio of the total population at equilibrium to the total carrying capacity is always larger than 1. Our goal is to find the supremum of this ratio taken over all possible choices of spatial distributions of resources and the species' dispersal rate. A conjecture proposed by W.-M. Ni is that, in the one-dimensional case, the supremum is 3. We settle this conjecture and then apply our result to study the global dynamics of a heterogeneous Lotka-Volterra competition-diffusion system.

1. INTRODUCTION

Over the past few decades, it has been well accepted, by both mathematicians and ecologists, that spatial characteristics play a significant role in population ecology. In an attempt to understand the joint effects of diffusion and spatial heterogeneity in population dynamics, Lou [11] first investigated the diffusive logistic equation

$$(1.1) \quad \begin{cases} u_t = d\Delta u + u(m(x) - u) & \text{in } \Omega \times \mathbb{R}^+, \\ \partial u / \partial \nu = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(x, 0) \geq 0, u(x, 0) \not\equiv 0, & \text{in } \Omega, \end{cases}$$

where $u(x, t)$ represents the population density of a species at location x and time t , which is therefore assumed to be non-negative, d is the random dispersal rate of the species which is assumed to be a positive constant, the habitat Ω is a smooth bounded domain in \mathbb{R}^N , $\Delta = \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2}$ is the usual Laplace operator and ν is the outward unit normal vector on $\partial\Omega$. We impose the zero-flux boundary condition to ensure that no individual crosses the boundary of the habitat. The function $m(x)$ is the intrinsic growth rate or carrying capacity, which reflects the environmental

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influence on the species u . Throughout this paper, we assume that $m(x)$ satisfies the following condition:

$$(M) \quad m(x) \in L^\infty(\Omega), \quad m(x) \geq 0 \text{ and } m \not\equiv \text{const on } \bar{\Omega}.$$

It follows from [1] and the references therein that the stationary problem

$$(1.2) \quad d\Delta\theta + \theta(m(x) - \theta) = 0 \text{ in } \Omega, \quad \partial\theta/\partial\nu = 0 \text{ on } \partial\Omega$$

has a unique positive solution $\theta_{d,m}$ for each $d > 0$ and $\theta_{d,m} \in W^{2,p}(\Omega)$ for every $p \geq 1$. The following interesting property concerning $\theta_{d,m}$ was first observed by Lou [11]:

$$(1.3) \quad \int_{\Omega} \theta_{d,m}(x) dx > \int_{\Omega} m(x) dx, \quad \text{for all } d > 0.$$

Indeed, dividing the equation of $\theta_{d,m}$ by $\theta_{d,m}$ itself and integrating over Ω , we obtain that

$$(1.4) \quad \int_{\Omega} (m - \theta_{d,m}) dx = -d \int_{\Omega} \frac{|\nabla \theta_{d,m}|^2}{\theta_{d,m}^2} dx < 0,$$

where the last inequality follows from the fact that $\theta_{d,m} \not\equiv \text{const}$, as $m \not\equiv \text{const}$. Biologically, (1.3) means that when coupled with diffusion, a heterogeneous environment can support a total population larger than the total carrying capacity of the environment, which is quite different from the case when $m(x) \equiv \text{const}$.

Define

$$(1.5) \quad E(m) := \sup_{d>0} \frac{\int_{\Omega} \theta_{d,m} dx}{\int_{\Omega} m dx}.$$

Then by (1.3), $E(m) > 1$ for any m satisfying condition (M). The following question was initially proposed by W.-M. Ni:

Question. Is $E(m)$ bounded above independent of m ? If so, what is the optimal bound?

One motivation of the above question is to understand how much more the total population could be supported by the same total resources, if the resources were distributed in an “optimal” way, and what would that “optimal” distribution be, if it exists.

Another motivation for studying the above question is the important role that $E(m)$ plays in the following two-species Lotka-Volterra competition-diffusion system:

$$(1.6) \quad \begin{cases} U_t = d_1 \Delta U + U(m(x) - U - cV) & \text{in } \Omega \times (0, \infty), \\ V_t = d_2 \Delta V + V(m(x) - bU - V) & \text{in } \Omega \times (0, \infty), \\ \partial U/\partial\nu = \partial V/\partial\nu = 0 & \text{on } \partial\Omega \times (0, \infty), \\ U(x, 0) = U_0(x), \quad V(x, 0) = V_0(x) & \text{in } \Omega, \end{cases}$$

where $U(x, t)$ and $V(x, t)$ represent the population densities of two competing species; $d_1, d_2 > 0$ are the (random) dispersal rates of U and V respectively. For simplicity, we assume that the initial data U_0 and V_0 are non-negative and non-trivial, i.e., not identically zero. The function $m(x)$ represents the common carrying capacity or intrinsic growth rate for both species U and V . The constants $b, c > 0$

represent inter-specific competition coefficients, while both intra-specific competition coefficients in (1.6) have been normalized to be 1. There has been considerable work focusing on studying two-species Lotka-Volterra competition models; see [1–12, 14] and references therein.

Concerning the global dynamics of (1.6), Lou [11] proved a remarkable result, which says that, if $b \in (1/E(m), 1)$ and c is sufficiently small, then there exist $d_2 > d_1 > 0$ such that $(\theta_{d_1, m}, 0)$ is globally asymptotically stable, although in such ranges of b and c , for each $x \in \Omega$, the reaction terms of (1.6) indicate co-existence. We refer the readers to [11] for the precise statement of Lou's result. More importantly, Lou proposed the following conjecture in [11]:

Conjecture A. *Assume that $m(x)$ satisfies condition **(M)** in (1.6). For all $b \in (1/E(m), 1)$ and $c \in (0, 1]$,*

- (i) *$(\theta_{d_1, m}, 0)$ is globally asymptotically stable for all $(d_1, d_2) \in \Sigma_U$;*
- (ii) *there exists a unique co-existence steady state which is globally asymptotically stable if $(d_1, d_2) \notin \Sigma_U$ and $d_1 \leq d_2$,*

where

$$(1.7) \quad \Sigma_U := \{(d_1, d_2) \in (0, \infty) \times (0, \infty) \mid (\theta_{d_1, m}, 0) \text{ is linearly stable}\}.$$

For c small but independent of $b \in (0, 1)$, Lam and Ni [9] established Conjecture A. More recently, He and Ni [5] settled Lou's above conjecture completely and their results indicate that $E(m)$ is a key quantity in characterizing the global behavior of solutions to (1.6). To be more specific, it is proved in [5] that (i) if $b, c \in (0, 1/E(m)]$, then for all $d_1, d_2 > 0$, (1.6) has a unique co-existence steady state that is globally asymptotically stable; (ii) if $1/E(m) < b \leq 1$ and $c \leq 1/E(m)$, then either U wipes out V (i.e., $(\theta_{d_1, m}, 0)$ is globally asymptotically stable) or U and V coexist (i.e., (1.6) has a unique co-existence steady state that is globally asymptotically stable), where which alternative will happen depends solely on the choice of their dispersal rates (d_1, d_2) and is regardless of their initial values. In other words, for the heterogeneous Lotka-Volterra competition-diffusion model, the region $0 < b, c < 1/E(m)$ seems to define the weak competition case.

Furthermore, if the intrinsic growth rates of U and V are not identical in (1.6), but replaced by $m_1(x)$ and $m_2(x)$ respectively, i.e.,

$$(1.8) \quad \begin{cases} U_t = d_1 \Delta U + U(m_1(x) - U - cV) & \text{in } \Omega \times (0, \infty), \\ V_t = d_2 \Delta V + V(m_2(x) - bU - V) & \text{in } \Omega \times (0, \infty), \\ \partial U / \partial \nu = \partial V / \partial \nu = 0 & \text{on } \partial \Omega \times (0, \infty), \\ U(x, 0) = U_0(x), \quad V(x, 0) = V_0(x) & \text{in } \Omega, \end{cases}$$

where m_i satisfies condition **(M)**, then it is proved in [5] that (1.8) has a unique co-existence steady state which is globally asymptotically stable for all $d_1, d_2 > 0$, if

$$b \leq \frac{1}{E(m_1)} \cdot \frac{\int_{\Omega} m_2}{\int_{\Omega} m_1} \quad \text{and} \quad c \leq \frac{1}{E(m_2)} \cdot \frac{\int_{\Omega} m_1}{\int_{\Omega} m_2}.$$

Hence, a better understanding of $E(m)$ and its supremum over all m satisfying condition **(M)** would enable us to obtain a deeper understanding of global dynamics of the above two-species competition-diffusion systems.

Now, returning to the aforementioned question regarding the supremum of $E(m)$, in the one-dimensional case, i.e., when $N = 1$ and Ω is an open interval, W.-M. Ni

conjectured that the supremum of $E(m)$ over all m 's satisfying condition **(M)** is 3. This paper confirms Ni's conjecture.

Theorem 1.1. *Assume that $N = 1$, i.e., Ω is an open interval and that $m(x)$ satisfies condition **(M)**. Then*

$$(1.9) \quad \sup\{E(m) \mid m \text{ satisfies condition } \mathbf{(M)}\} = 3.$$

Moreover, the supremum 3 in (1.9) is not attainable.

Therefore, in the one-dimensional case, the total population that model (1.1) can support at equilibrium can be as close to three times the total carrying capacity as possible, if the function $m(x)$ is chosen appropriately. (See Theorem 2.3 below for more details.)

According to previous discussions, based on the results in [5] and Theorem 1.1, we obtain the following corollaries immediately.

Corollary 1.2. *Assume that $N = 1$, i.e., Ω is an open interval and that $m(x)$ satisfies condition **(M)**. If $b, c \in (0, 1/3]$, then system (1.6) has a unique co-existence steady state that is globally asymptotically stable for all $d_1, d_2 > 0$.*

Corollary 1.3. *Assume that $N = 1$, i.e., Ω is an open interval and that $m_i(x)$ satisfies **(M)**, $i = 1, 2$. If*

$$b \leq \frac{1}{3} \cdot \frac{\int_{\Omega} m_2}{\int_{\Omega} m_1} \quad \text{and} \quad c \leq \frac{1}{3} \cdot \frac{\int_{\Omega} m_1}{\int_{\Omega} m_2},$$

then system (1.8) has a unique co-existence steady state that is globally asymptotically stable for all $d_1, d_2 > 0$.

We now proceed to prove Theorem 1.1.

2. PROOF OF THEOREM 1.1

First of all, we recall some important properties of $\theta_{d,m}$ when d goes to 0 or ∞ .

Lemma 2.1. *Suppose that condition **(M)** holds.*

- (i) $d \mapsto \theta_{d,m}$ is continuous from $(0, \infty)$ to $W^{2,p}(\Omega) \cap C^{1,\alpha}(\bar{\Omega})$ for every $p \geq 1$ and $\alpha \in (0, 1)$.
- (ii) As $d \rightarrow 0^+$, the solution $\theta_{d,m} \rightarrow m$ in $L^p(\Omega)$ for every $p \geq 1$.
- (iii) As $d \rightarrow \infty$, the solution $\theta_{d,m} \rightarrow \frac{1}{|\Omega|} \int_{\Omega} m(x) dx$ in $W^{2,p}(\Omega)$ for every $p \geq 1$.

Part (i) can be proved by an application of the Implicit Function Theorem. (See Proposition 3.6 in [1] and remarks there.) For proofs of parts (ii) and (iii), see [1, 11].

Now we are ready to show the following:

Theorem 2.2. *Assume that $N = 1$, i.e., Ω is an open interval. Then for any m satisfying condition **(M)**,*

$$E(m) < 3.$$

Proof. By suitable rescaling, we can always assume that $\Omega = (0, 1)$. We now divide our proof into five steps. In Steps 1-4, we show that for any m satisfying condition **(M)** and $d > 0$,

$$\int_0^1 \theta_{d,m}(x) dx \leq 3 \int_0^1 m(x) dx,$$

which implies that $E(m) \leq 3$ for all m satisfying condition **(M)**. Finally in Step 5, we show that actually equality cannot hold.

Step 1: Monotone $\theta_{d,m}$. We now prove that

$$\int_0^1 \theta_{d,m}(x) dx < 3 \int_0^1 m(x) dx$$

whenever $\theta_{d,m}(x)$ is monotone. Without loss of generality, we may assume that $\theta_{d,m}$ is increasing in x (otherwise consider $\theta_{d,m}(1-x)$).

By standard elliptic regularity, $\theta_{d,m} \in W^{2,p}(\Omega) \cap C^{1,\alpha}(\bar{\Omega})$ for every $p \geq 1$ and $\alpha \in (0,1)$. If $\theta'_{d,m} \equiv \text{const}$ on Ω , then it is obvious that $m \equiv \text{const}$ and $\theta_{d,m} \equiv m$, which implies that $E(m) = 1 < 3$. So from now on, we may assume that $\theta'_{d,m} \geq (\neq) 0$. Since $\theta_{d,m}$ is a weak solution to (1.2), we have

$$(2.1) \quad d \int_0^x \theta''_{d,m} \varphi dx = \int_0^x \theta_{d,m} (\theta_{d,m} - m(x)) \varphi dx$$

for any $x \in (0,1]$ and $\varphi \in C^1([0,x])$. Setting $\varphi = \theta'_{d,m}$ in (2.1) and using $\theta'_{d,m}(0) = 0$, we get

$$(2.2) \quad \frac{d}{2} (\theta_{d,m}(x)')^2 = \frac{\theta_{d,m}^3(x) - \theta_{d,m}^3(0)}{3} - \int_0^x \theta_{d,m} \theta'_{d,m} m < \frac{\theta_{d,m}^3(x)}{3},$$

where we used the fact that $\theta'_{d,m} \geq 0$ by our assumption and $\theta_{d,m} > 0$ on $[0,1]$ by the Maximum Principle. Using the above inequality, we conclude that

$$\int_0^1 (\theta_{d,m} - m) dx = d \int_0^1 \frac{\theta''_{d,m}}{\theta_{d,m}} dx = d \int_0^1 \frac{(\theta'_{d,m})^2}{\theta_{d,m}^2} dx < \frac{2}{3} \int_0^1 \theta_{d,m} dx.$$

Hence $\int_0^1 \theta_{d,m} < 3 \int_0^1 m$ and we finish our proof.

Step 2: Piecewise constant m . Suppose

$$(2.3) \quad m = \begin{cases} a_1, & x \in [x_0, x_1], \\ a_2, & x \in (x_1, x_2], \\ \dots \\ a_n, & x \in (x_{n-1}, x_n], \end{cases}$$

where $0 = x_0 < x_1 < x_2 < \dots < x_n = 1$, $a_i \geq 0$ and $a_j \neq a_{j+1}$, $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, n-1$. Define the set

$$\mathcal{J}_i := \{x \in [x_{i-1}, x_i] \mid \theta'_{d,m}(x) = 0\}$$

for $i = 1, 2, \dots, n$. We claim that

(S) for each i , either \mathcal{J}_i contains at most a single point or $\mathcal{J}_i = [x_{i-1}, x_i]$ and $\theta_{d,m}|_{[x_{i-1}, x_i]} \equiv a_i$.

We now prove the claim. Assume that \mathcal{J}_i contains more than one point. Denote

$$y_{1,i} := \inf\{x \in [x_{i-1}, x_i] \mid \theta'_{d,m}(x) = 0\}$$

and

$$y_{2,i} := \sup\{x \in [x_{i-1}, x_i] \mid \theta'_{d,m}(x) = 0\};$$

then $x_{i-1} \leq y_{1,i} < y_{2,i} \leq x_i$. It suffices to show that $y_{1,i} = x_{i-1}$ and $y_{2,i} = x_i$. Since $\theta'_{d,m}(y_{1,i}) = \theta'_{d,m}(y_{2,i}) = 0$, $\theta_{d,m}|_{(y_{1,i}, y_{2,i})}$ is also a solution to (1.2) with $\Omega = (y_{1,i}, y_{2,i})$ and $m \equiv a_i$. This implies that $\theta_{d,m}|_{(y_{1,i}, y_{2,i})} \equiv a_i$. By interior

elliptic regularity, $\theta_{d,m}|_{(x_{i-1}, x_i)}$ is $C^\infty((x_{i-1}, x_i))$. Moreover, $(\theta_{d,m}, \theta'_{d,m})$ satisfies the following ordinary differential equation in (x_{i-1}, x_i) :

$$\begin{pmatrix} \theta_{d,m} \\ \theta'_{d,m} \end{pmatrix}' = \begin{pmatrix} \theta'_{d,m} \\ \theta_{d,m}(m - \theta_{d,m}) \end{pmatrix} \text{ in } (x_{i-1}, x_i),$$

$$\begin{pmatrix} \theta_{d,m}(y_{\ell,i}) \\ \theta'_{d,m}(y_{\ell,i}) \end{pmatrix} = \begin{pmatrix} a_i \\ 0 \end{pmatrix}, \quad \ell = 1, 2.$$

By uniqueness, we must have $\theta_{d,m} \equiv a_i$ on $[x_{i-1}, x_i]$. This finishes the proof of the claim.

Denote

$$\mathcal{J} := \bigcup \{ \mathcal{J}_i \mid \mathcal{J}_i = [x_{i-1}, x_i], i = 1, 2, \dots, n \}$$

and

$$S := \left(\bigcup_{i=1}^n \mathcal{J}_i \setminus \mathcal{J} \right) \cup \partial \mathcal{J}.$$

It is obvious that S is a finite set. Rewrite it as $S := \{b_0, b_1, \dots, b_k\}$, where $0 = b_0 < b_1 < b_2 < \dots < b_k = 1$. Now each section $[b_{i-1}, b_i], i = 1, 2, \dots, k$, boils down to Step 1, where either $\theta_{d,m}$ is strictly increasing or is constant on $[b_{i-1}, b_i]$, and we get the conclusion by a simple summation.

Step 3: Riemann integrable m . Fix a Riemann integrable m ; for any $\varepsilon > 0$ we can choose a piecewise constant function m_ε such that: $m_\varepsilon \geq m$ and $\int_0^1 m_\varepsilon \leq \int_0^1 m + \varepsilon$. It is easy to see that $\theta_{d,m} \leq \theta_{d,m_\varepsilon}$ by the comparison principle. Thus

$$\int_0^1 \theta_{d,m} dx \leq \int_0^1 \theta_{d,m_\varepsilon} dx \leq 3 \int_0^1 m_\varepsilon dx < 3 \int_0^1 m dx + 3\varepsilon.$$

Letting $\varepsilon \rightarrow 0$ gives the conclusion we need.

Step 4: From Riemann integrable m to any m in $L^\infty((0, 1))$. Since $m \in L^\infty$, by Lusin's Theorem we can choose a continuous function $m_\varepsilon(x)$ such that $\|m_\varepsilon\|_{L^\infty} \leq \|m\|_{L^\infty}$ and $|\{x | m_\varepsilon(x) \neq m(x)\}| < \varepsilon$. Thus, we have

$$\|m_\varepsilon - m\|_{L^1} \leq \varepsilon \|m\|_{L^\infty}.$$

Using the estimate (2.4) in [11], we get

$$\|\theta_{d,m_\varepsilon} - \theta_{d,m}\|_{L^1} \leq C \|m_\varepsilon - m\|_{L^1}^{\frac{1}{3}} \leq C \varepsilon^{\frac{1}{3}} \|m\|_{L^\infty}^{\frac{1}{3}} \rightarrow 0, \text{ as } \varepsilon \rightarrow 0^+.$$

The conclusion in Step 3 implies that

$$\int_0^1 \theta_{d,m_\varepsilon}(x) dx \leq 3 \int_0^1 m_\varepsilon(x) dx,$$

and it follows by letting $\varepsilon \rightarrow 0^+$ that

$$\int_0^1 \theta_{d,m}(x) dx \leq 3 \int_0^1 m(x) dx.$$

Step 5: $E(m) < 3$ for all m satisfying condition (\mathbf{M}) . We prove Step 5 by contradiction. Assume that there exists some $g \in L^\infty(\Omega)$ with $g(x) \geq 0$ and $g \not\equiv \text{const}$ on $\bar{\Omega}$ such that $E(g) = 3$. By Lemma 2.1, the function $d \mapsto \int_\Omega \theta_{d,g} / \int_\Omega g$ attains its minimum value 1 when $d \rightarrow 0^+$ or ∞ . Hence there exists some $\tilde{d} > 0$ such that

$$\int_\Omega \theta_{\tilde{d},g} = 3 \int_\Omega g.$$

By Step 1, $\theta'_{\tilde{d},g}(x) \not\equiv 0$. Hence without loss of generality, we may assume that there exists some $x_0 \in (0, 1)$ such that $\theta'_{\tilde{d},g}(x_0) > 0$. By the continuity of $\theta'_{\tilde{d},g}$, we can choose $\delta > 0$ such that $\theta'_{\tilde{d},g}(x) > 0$, for all $x \in (x_0 - \delta, x_0 + \delta)$. Denote

$$\tau_1 := \inf\{x \mid \theta'_{\tilde{d},g}(y) > 0, \forall y \in (x, x_0)\}$$

and

$$\tau_2 := \sup\{x \mid \theta'_{\tilde{d},g}(y) > 0, \forall y \in (x_0, x)\}.$$

Then we have $\theta'_{\tilde{d},g}(\tau_1) = \theta'_{\tilde{d},g}(\tau_2) = 0$ and $\theta'_{\tilde{d},g}(x) > 0$, for all $x \in (\tau_1, \tau_2)$. By Step 1, we have

$$(2.4) \quad \int_{\tau_1}^{\tau_2} \theta_{\tilde{d},g}(x) dx < 3 \int_{\tau_1}^{\tau_2} g(x) dx.$$

Since $\theta_{\tilde{d},g}|_{(0,\tau_1)}$ and $\theta_{\tilde{d},g}|_{(\tau_2,1)}$ are solutions to (1.2) with $d = \tilde{d}$, $\Omega = (0, \tau_1)$ and $(\tau_2, 1)$ and $m = g|_{(0,\tau_1)}$ and $g|_{(\tau_2,0)}$ respectively, Steps 1-4 imply that

$$(2.5) \quad \int_0^{\tau_1} \theta_{\tilde{d},g}(x) dx \leq 3 \int_0^{\tau_1} g(x) dx$$

and

$$(2.6) \quad \int_{\tau_2}^1 \theta_{\tilde{d},g}(x) dx \leq 3 \int_{\tau_2}^1 g(x) dx.$$

Adding the above three inequalities together, we obtain that

$$(2.7) \quad \int_0^1 \theta_{\tilde{d},g}(x) dx < 3 \int_0^1 g(x) dx,$$

which is a contradiction. This finishes the proof of the theorem. \square

Theorem 2.3. *The upper bound “3” in Theorem 2.2 is optimal. In other words, there exist $m_\varepsilon(x)$ and d_ε such that*

$$\frac{\int_{\Omega} \theta_{d_\varepsilon, m_\varepsilon}}{\int_{\Omega} m_\varepsilon} \rightarrow 3, \text{ as } \varepsilon \rightarrow 0^+.$$

Proof. Similar to Theorem 2.2, by suitable rescaling, we can always assume that $\Omega = (0, 1)$. Now choose $d_\varepsilon = \sqrt{\varepsilon}$, and

$$(2.8) \quad m_\varepsilon(x) = \begin{cases} 0, & x \in [0, 1 - \varepsilon], \\ 1/\varepsilon, & x \in (1 - \varepsilon, 1]. \end{cases}$$

For notational convenience, in the rest of this proof we omit the subscript ε in d_ε , m_ε and $\theta_{d_\varepsilon, m_\varepsilon}$. By (1.4), we have

$$\begin{aligned} \int_0^1 (\theta_{d,m} - m) dx &= d \int_0^1 \frac{(\theta'_{d,m})^2}{\theta_{d,m}^2} dx \\ &= d \int_0^1 \frac{\int_0^x [(\theta'_{d,m}(y))^2]' dy}{\theta_{d,m}^2} dx \\ &= 2 \int_0^1 \frac{\int_0^x \theta_{d,m}(y) [\theta_{d,m}(y) - m(y)] \theta'_{d,m}(y) dy}{\theta_{d,m}^2} dx \\ &= \frac{2}{3} \int_0^1 \theta_{d,m} - \frac{2}{3} \int_0^1 \frac{\theta_{d,m}^3(0)}{\theta_{d,m}^2} dx \\ &\quad - \frac{1}{\varepsilon} \int_{1-\varepsilon}^1 \int_{1-\varepsilon}^x \frac{\theta_{d,m}(y) \theta'_{d,m}(y)}{\theta_{d,m}^2(x)} dy dx \\ &=: \frac{2}{3} \int_0^1 \theta_{d,m} - I - II. \end{aligned}$$

Hence, to finish the proof of the theorem, we only need to prove that both I and II are $o(1)$ as $\varepsilon \rightarrow 0^+$.

For I , since $\theta_{d,m}$ is increasing,

$$I \leq \frac{2}{3} \int_0^1 \theta_{d,m}(0) dx = \frac{2}{3} \theta_{d,m}(0).$$

Assume $\varepsilon \in (0, 1/4)$; then $(1/4, 3/4) \subset (0, 1 - \varepsilon)$. Hence

$$\theta''_{d,m}(y) = \frac{1}{d} \theta_{d,m}^2(y) \geq 0 \text{ for } y \in (1/4, 3/4),$$

which implies that

$$\theta_{d,m}(x) - \theta_{d,m}\left(\frac{1}{4}\right) \geq \left[\min_{x \in (1/4, 3/4)} \theta'_{d,m} \right] \cdot \left(x - \frac{1}{4}\right) = \theta'_{d,m}\left(\frac{1}{4}\right) \left(x - \frac{1}{4}\right)$$

for all $x \in (1/4, 3/4)$. Therefore by Theorem 2.2, we have

$$\begin{aligned} 3 &> \int_0^1 \theta_{d,m}(x) dx > \int_{\frac{1}{4}}^{\frac{3}{4}} [\theta_{d,m}(x) - \theta_{d,m}\left(\frac{1}{4}\right)] dx \\ &\geq \theta'_{d,m}\left(\frac{1}{4}\right) \int_{\frac{1}{4}}^{\frac{3}{4}} \left(x - \frac{1}{4}\right) dx = \frac{1}{8} \theta'_{d,m}\left(\frac{1}{4}\right). \end{aligned}$$

Now, $\frac{1}{4} \theta_{d,m}^2(0) < \int_0^{\frac{1}{4}} \theta_{d,m}^2 dx = d \int_0^{\frac{1}{4}} \theta''_{d,m}(x) dx = d \theta'_{d,m}\left(\frac{1}{4}\right) < 24\sqrt{\varepsilon}$. This implies that

$$I \leq \frac{2}{3} \theta_{d,m}(0) < \frac{8\sqrt{6}}{3} \sqrt[4]{\varepsilon}.$$

Next, we consider II . Since $\theta_{d,m}$ is increasing, we have

$$II \leq \int_{1-\varepsilon}^1 \frac{\theta'_{d,m}(y)}{\theta_{d,m}(y)} dy \leq \left(d \int_0^1 \frac{(\theta'_{d,m})^2}{\theta_{d,m}^2} dy \right)^{\frac{1}{2}} \cdot \frac{\sqrt{\varepsilon}}{\sqrt{d}} = \left(d \int_0^1 \frac{(\theta'_{d,m})^2}{\theta_{d,m}^2} dy \right)^{\frac{1}{2}} \cdot \sqrt[4]{\varepsilon}.$$

Combined with the fact that $d \int_0^1 \frac{(\theta'_{d,m})^2}{\theta_{d,m}^2} dy = \int_0^1 (\theta_{d,m} - m) dy \leq 2 \int m$ (Theorem 2.2), we get

$$II \leq \left(\int_0^1 (\theta_{d,m} - m) dy \right)^{\frac{1}{2}} \sqrt[4]{\varepsilon} \leq \left(2 \int_0^1 m dy \right)^{\frac{1}{2}} \sqrt[4]{\varepsilon} \leq \sqrt{2} \sqrt[4]{\varepsilon}. \quad \square$$

Now Theorem 1.1 follows directly from Theorems 2.2 and 2.3.

3. MISCELLANEOUS REMARKS

In the higher-dimensional case, the supremum of $E(m)$ over all m 's satisfying condition **(M)** is necessarily no less than 3. Numerical simulation by T. Mori and S. Yotsutani [13] suggests that, even in the two-dimensional case, the supremum is much larger than 3. We hope to return to the higher-dimensional case in future work.

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