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A KAM THEOREM FOR SOME PARTIAL DIFFERENTIAL EQUATIONS IN ONE DIMENSION

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ABSTRACT. We prove an infinite-dimensional KAM theorem with dense normal frequencies. In this theorem, we relax the separation condition on normal frequencies which is required by the KAM theorem.

1. Introduction

The infinite-dimensional KAM theorem is a powerful tool for constructing quasi-periodic solutions of PDEs. Wayne [24], Pöschel [22] and Kuksin [19] pioneered this research, requiring first-order and second-order Melnikov conditions. Following these works, there have been many important works in this field. On the other hand, the construction of quasi-periodic solutions of PDEs can also be done by imposing only first-order Melnikov conditions. This approach has been developed by Bourgain [4–7], extending the work of Craig-Wayne [9] for periodic solutions. However, the KAM theorem will provide more information about the linear stability of the quasi-periodic solutions. We are more interested in the infinite-dimensional KAM theorem.

The KAM theorem is composed of infinite steps of KAM iteration; to finish one KAM iteration step on the hamiltonian

$$H = \sum_{1\le j\le b} \omega_j(\xi) I_j + \sum_{n\in\mathbb{Z}_1^d} \Omega_n(\xi)|z_n|^2 + P(\xi, I, \theta, z, \bar{z}), \tag{1.1}$$

we need to solve a homological equation $\{N, F\} + \hat{N} = R$. The big problem is to get a lower bound

$$|\langle \omega, k\rangle + \Omega_m - \Omega_n| \ge \gamma(|k|+1)^{-\tau}\,,\ \forall k \in \mathbb{Z}^b\,. \tag{1.2}$$

This leads to the problem of measure estimation. Mathematicians focus on the property of perturbation, the dimension of space, new techniques, etc.

In the beginning, to get the lower bound above, the separation condition $|\Omega_n - \Omega_m| \ge \alpha$ on normal frequencies was required. This requirement restricted

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us to construct quasi-periodic solutions for PDEs under Dirichlet boundary conditions. Then, Chierchia-You [8] obtained quasi-periodic solutions for wave equations under periodic boundary conditions. Using their theorem, we can get KAM tori under asymptotic double normal frequencies. However, there are some great differences between the hamiltonians for PDEs with $d=1$ and with $d>1$. It is easy to see that the Ω_n asymptotically form finite clusters of uniform size and structure if $d=1$. For $d>1$, the cluster sizes may be of arbitrary large dimension.

Then Geng-You [16] proved a KAM result for higher-dimensional PDEs; in their work the perturbation satisfies momentum conservation and a decay restriction. With momentum conservation, we need to solve fewer terms than usual, and the normal form is much easier; with the decay condition on perturbation, by iteration the normal frequencies take the form $\Omega_n = |n|^2 + O(\frac{\varepsilon}{|n|^\ell})$, which makes the measure estimate simple. Then Eliasson-Kuksin [12] obtained a more general result for higher-dimensional PDEs, where the perturbation does not satisfy the condition above as it did in [16]. In their work, the most important thing is that they find a relatively weak decay property, *Töplitz-Lipschitz*; this condition is preserved by KAM iteration. With this property, they overcome the measure estimation problem. Different from [16], the normal form is $N = \langle \omega, I\rangle + \langle \Omega z, \bar{z}\rangle + \langle \mathcal{H} z, \bar{z}\rangle$, where the cluster in $\mathcal{H}$ is growing quickly. Geng-Xu-You [15] gave an understanding of this property. Following Eliasson-Kuksin's work, Procesi-Xu [23] gave another description of the perturbation, which they named *quasi-Töplitz*. Their results relax the decay restriction in [16].

The development of the KAM theorem also focused on unbounded perturbation. The first KAM result on this subject was by Kuksin [20], and then Kappeler-Pöschel [17] for hamiltonians with analytic perturbations given by KdV. In their work, one can find the normal frequencies Ω dependent on the angle variable θ; this makes it hard to solve the homological equation. To solve the homological equation in this problem, one needs Kuksin's Lemma, which is applicable in the case $d=1$. Their result is improved by Liu-Yuan [25] for 1-dimensional derivative NLS (DNLS) equations. Liu-Yuan extend Kuksin's Lemma and obtain a more general KAM theorem.

Recently there have been many interesting works on other PDEs. Grébert-Thomann [13] consider semilinear quantum harmonic Schrödinger equations, corresponding to a generalized hamiltonian. Kappeler-Liang [14] consider the existence of a quasi-periodic solution with large energy for the Schrödinger equation, Berti-Biasco-Procesi [3] consider equations with quasi-differential operators, etc.

In any event, one can find that the normal frequencies of the hamiltonian share a separation condition in all the literature above, that is, $\Omega_n = |n|^\chi + \cdots, \chi > 1, n \in \mathbb{Z}^d, d > 1$, and $\Omega_n = |n|^\chi + \cdots, \chi \geq 1, n \in \mathbb{Z}$. For $0 < \chi < 1$, this usually leads to the density of normal frequency; a famous example is given by the higher-dimensional *wave* equation. A similar problem is found when one considers the *water wave* equation. There is little progress on the existence of quasi-periodic solutions for the water wave equation. This field remains largely open and it is hard for us to use the KAM method; one of the main problems is that the order $\chi = \frac{1}{2}$ (see [10]).

In this paper, we relax this condition to be $\chi > 0$. In any event, we can only prove the KAM theorem for $n \in \mathbb{Z}$ when perturbation satisfies momentum conservation.

2. An infinite-dimensional KAM theorem

For given b vectors $S = \{n_1 = 0, n_2, \cdots, n_b\}$ in $\mathbb{Z}$, called *tangential sites*, denote $\mathbb{Z}_1 =: \mathbb{Z} \setminus S$. Now we consider small perturbations of an infinite hamiltonian

$$N = \langle \omega(\xi), I \rangle + \sum_{j \in \mathbb{Z}_1} \Omega_j |z_j|^2 \tag{2.1}$$

on phase space

$$D(r,s) = \{(\theta, I, z, \bar{z}) : |\mathrm{Im}\theta| < r, |I| < s^2, \|z\|_{\rho,p} < s, \|\bar{z}\|_{\rho,p} < s\},$$

which is a neighborhood of $\mathbb{T}^b \times \{I = 0\} \times \{z = 0\} \times \{\bar{z} = 0\}$. Let $z = (\cdots, z_n, \cdots)_{n \in \mathbb{Z}_1}$, and its complex conjugate $\bar{z} = (\cdots, \bar{z}_n, \cdots)_{n \in \mathbb{Z}_1}$; the weighted norm is defined to be

$$\|z\|_{\rho,p} = \sum_{n \in \mathbb{Z}_1} |z_n| e^{2\rho|n|} n^{2p},$$

where $|\cdot|$ denotes the sup-norm of complex vectors.

Let $\mathcal{O}$ be a positive-measure parameter set in $\mathbb{R}^b$. We consider the functions $F(I, \theta, z, \bar{z}; \xi) : D(r,s) \times \mathcal{O} \to \mathbb{C}$, where F is analytic in I, θ, z and of class C^1_W (in the sense of Whitney) in ξ. We expand F in Taylor-Fourier series:

$$F(\theta, I, z, \bar{z}; \xi) = \sum_{k,l,\alpha,\beta} F_{lk\alpha\beta}(\xi) I^l e^{\mathrm{i}\langle k, \theta \rangle} z^\alpha \bar{z}^\beta, \tag{2.2}$$

where the coefficients $F_{lk\alpha\beta}(\xi)$ are of class C^1_W, the vectors $\alpha \equiv (\cdots, \alpha_n, \cdots)_{n \in \mathbb{Z}_1}$, $\beta \equiv (\cdots, \beta_n, \cdots)_{n \in \mathbb{Z}_1}$ have finitely many non-zero components $\alpha_n, \beta_n \in \mathbb{N}$, $z^\alpha \bar{z}^\beta$ denotes $\prod_n z_n^{\alpha_n} \bar{z}_n^{\beta_n}$ and $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{C}^b$.

We use the following weighted norm for F:

$$\|F\|_{r,s} = \|F\|_{D(r,s),\mathcal{O}} \equiv \sup_{\substack{\|z\|_{\rho,p} < s \\ \|\bar{z}\|_{\rho,p} < s}} \sum_{k,l,\alpha,\beta} |F_{kl\alpha\beta}|_{\mathcal{O}} \; s^{2|l|} e^{|k|r} \, |z^\alpha||\bar{z}^\beta|, \tag{2.3}$$

$$|F_{kl\alpha\beta}|_{\mathcal{O}} \equiv \sup_{\xi \in \mathcal{O}} (|F_{kl\alpha\beta}| + |\frac{\partial F_{kl\alpha\beta}}{\partial \xi}|) \tag{2.4}$$

(the derivatives with respect to ξ are in the sense of Whitney). To an analytic function F, we associate a Hamiltonian vector field with coordinates

$$X_F = (F_I, -F_\theta, \{\mathrm{i}F_{z_n}\}_{n \in \mathbb{Z}_1}, \{-\mathrm{i}F_{\bar{z}_n}\}_{n \in \mathcal{Z}_1}).$$

Consider a vector function $G : D(r,s) \times \mathcal{O} \to \ell_\rho$, with

$$G = \sum_{kl\alpha\beta} G_{kl\alpha\beta}(\xi) I^l e^{\mathrm{i}\langle k, \theta \rangle} z^\alpha \bar{z}^\beta,$$

where $G_{kl\alpha\beta} = (\cdots, G^{(i)}_{kl\alpha\beta}, \cdots)_{i \in \mathbb{Z}_1}$. Its norm is similarly defined as

$$\|G\|_{D(r,s),\mathcal{O}} = \sup_{\substack{\|z\|_{\rho,p} < s \\ \|\bar{z}\|_{\rho,p} < s}} \|\mathcal{M}G\|_{\rho,\bar{p}}, \; \bar{p} > p,$$

where

$$\mathcal{M}G = (\cdots, \mathcal{M}G^{(i)}, \cdots)_{i \in \mathbb{Z}_1}, \qquad \mathcal{M}G^{(i)} = \sum_{\alpha,\beta,k,l} |G^{(i)}_{kl\alpha\beta}|_{\mathcal{O}} \; s^{2|l|} e^{|k|r} \, z^\alpha \bar{z}^\beta$$

is a majorant of $G^{(i)}$. The weighted norm of X_F is defined by[1]

$$\|X_F\|_{r,s} =: \|X_F\|_{D(r,s),\mathcal{O}} \equiv \sum_{j=1}^{b} \|F_{I_j}\|_{D(r,s),\mathcal{O}} + \frac{1}{s^2}\sum_{j=1}^{b} \|F_{\theta_j}\|_{D(r,s),\mathcal{O}} + \frac{1}{s}(\|\partial_z F\|_{D(r,s),\mathcal{O}} + \|\partial_{\bar z} F\|_{D(r,s),\mathcal{O}}). \tag{2.5}$$

A function F is said to satisfy *momentum conservation* if $\{F,\mathbb{M}\} = 0$ with $\mathbb{M} = \sum_{i=1}^{b} n_i I_i + \sum_{m\in\mathbb{Z}_1} m|z_m|^2$. This implies that

$$F_{kl\alpha\beta} = 0\,, \qquad \text{if } \pi(k,\alpha,\beta) := \sum_{i=1}^{b} n_i k_i + \sum_{m\in\mathbb{Z}_1} m(\alpha_m - \beta_m) \neq 0. \tag{2.6}$$

By Jacobi's identity, momentum conservation is preserved by Poisson bracket.

As one can see, the hamiltonian equations of motions of N are

$$\dot\theta = \omega, \dot I = 0, \dot z = \Omega \bar z, \dot{\bar z} = \Omega z.$$

For each $\xi \in \mathcal{O}$, there is a solution $(\theta,0,0,0) \to (\theta+\omega t,0,0,0)$ which corresponds to an invariant torus in the phase space. Our aim is to prove that, under suitable assumptions, there is a Cantor set $\mathcal{O}^\infty \subset \mathcal{O}$ with positive Lebesgue measure, such that, for any $\xi \in \mathcal{O}^\infty$ the hamiltonian H still admits invariant tori. The following assumptions are made.

$(A1)$*Nondegeneracy:* The map $\xi \to \omega(\xi)$ is a C^1_W diffeomorphism between $\mathcal{O}$ and its image with $|\omega|_{C^1_W}, |\nabla\omega^{-1}|_{\mathcal{O}} \le M$.

$(A2)$*Asymptotics of normal frequencies:*

$$\Omega_n = |n|^\chi + \tilde\Omega_n,\ \tilde\Omega_n = o(|n|^{-\iota}), 0<\chi<1, \iota>0, \tag{2.7}$$

where $|n|^\iota\tilde\Omega_n$ are C^1_W functions of ξ with C^1_W-norm uniformly bounded by some small positive constant L with $LM<1$.

$(A3)$ *Momentum conservation:* The function P satisfies momentum conservation, $\{P,\mathbb{M}\} = 0$.

$(A4)$ *Regularity of P:* P is real analytic in $I,\theta,z,\bar z$ and C^1_W Whitney smooth in ξ; in addition $\|X_P\|_{D(r,s),\mathcal{O}} < \infty$ with $\bar p = p + \iota$.

Now we are ready to state an infinite-dimensional KAM theorem.

Theorem 2.1. *Let $H = N + P$ satisfy assumptions $(A1)-(A4)$. Let $\gamma > 0$ be small enough. Then there is a positive constant $\varepsilon = \varepsilon(b,\gamma,r,s,\iota,L,M)$ such that if $\|X_P\|_{D(r,s),\mathcal{O}} < \varepsilon$, the following holds: There exist a Cantor subset $\mathcal{O}_\gamma \subset \mathcal{O}$ with* $\mathrm{meas}(\mathcal{O}\setminus\mathcal{O}_\gamma) = O(\gamma)$ *and two maps (analytic in θ and C^1_W in ξ)*

$$\Psi : \mathbb{T}^b \times \mathcal{O}_\gamma \to D(r,s), \qquad \tilde\omega : \mathcal{O}_\gamma \to \mathbb{R}^b,$$

where Ψ is $\frac{\varepsilon}{\gamma^2}$-close to the trivial embedding $\Psi_0 : \mathbb{T}^b \times \mathcal{O} \to \mathbb{T}^b \times \{0,0,0\}$ and $\tilde\omega$ is ε-close to the unperturbed frequency ω, such that for any $\xi \in \mathcal{O}_\gamma$ and $\theta \in \mathbb{T}^b$, the curve $t \to \Psi(\theta + \tilde\omega(\xi)t, \xi)$ is a quasi-periodic solution of the hamiltonian equations governed by $H = N + P$.

[1]The norm $\|\cdot\|_{D(r,s),\mathcal{O}}$ for scalar functions is defined in (2.3).

As an application, we consider the equation

$$u_{tt} + A^2 u = f(u), x \in \mathbb{T}, t \in \mathbb{R}, \tag{2.8}$$

where $A = |\partial_x|^{\frac{1}{2}} + M_\xi$. As one can see under periodic boundary conditions, the operator A has eigenfunction $\phi_n = e^{inx}$ and the eigenvalue is assumed to be

$$\begin{cases} \omega_j &= |j|^{\frac{1}{2}} + \xi_j, j \in S, \\ \Omega_n &= |n|^{\frac{1}{2}},\ n \in \mathbb{Z}_1. \end{cases} \tag{2.9}$$

Introducing $v = u_t$, (2.8) is written as

$$\begin{cases} u_t = v, \\ v_t = -A^2 u - f(u). \end{cases} \tag{2.10}$$

Let $q = \frac{1}{\sqrt{2}} A^{\frac{1}{2}} u - \mathrm{i} \frac{1}{\sqrt{2}} A^{-\frac{1}{2}} v$; thus we obtain

$$\frac{1}{\mathrm{i}} q_t = Aq + \frac{1}{\sqrt{2}} A^{-\frac{1}{2}} f(A^{-\frac{1}{2}}(\frac{q+\bar{q}}{\sqrt{2}})). \tag{2.11}$$

Equation (2.11) can be rewritten as the hamiltonian equation

$$q_t = \mathrm{i}\frac{\partial H}{\partial \bar{q}}, \tag{2.12}$$

and the corresponding hamiltonian is

$$H = \frac{1}{2}\langle Aq, q\rangle + \int_0^{2\pi} g(A^{-\frac{1}{2}}(\frac{q+\bar{q}}{\sqrt{2}}))\ dx, \tag{2.13}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in L^2 and g is a primitive function of f.

It is easy to check that the hamiltonian (2.13) satisfies all the assumptions of Theorem 2.1. One has the following result at once.

Theorem 2.2. *There exists a positive-measure Cantor set $\mathcal{C}$ such that for $\xi = (\xi_1, \cdots, \xi_b) \in \mathcal{C}$, the non-linear equation* (2.8) *admits small amplitude analytic quasi-periodic solutions. These solutions are linearly stable.*

3. Proof of Theorem 2.1

Theorem 2.1 will be proved by a KAM iteration which involves an infinite sequence of change of variables. Each KAM iteration step makes the perturbation smaller in a narrow parameter set and analytic domain. We have to prove the convergence of the iteration sequence and estimate the measure of the excluded set with infinite KAM steps.

At the ν-step of the KAM iteration, we consider a hamiltonian vector field with

$$H_\nu = N_\nu + P_\nu = \langle \omega_\nu, I\rangle + \sum_{n\in\mathbb{Z}_1} \Omega_n^\nu |z_n|^2 + P_\nu,$$

where P_ν is defined in $D(r_\nu, s_\nu) \times \mathcal{O}_\nu$ and satisfies $(A1)$–$(A4)$. We will construct a symplectic change of variables

$$\Phi_\nu : D(r_{\nu+1}, s_{\nu+1}) \times \mathcal{O}_{\nu+1} \to D(r_\nu, s_\nu)$$

such that the vector field $X_{H_\nu \circ \Phi_\nu}$ defined on $D(r_{\nu+1}, s_{\nu+1})$ satisfies

$$\|X_{P_{\nu+1}}\|_{D(r_{\nu+1}, s_{\nu+1}), \mathcal{O}_{\nu+1}} \le \varepsilon_\nu^\kappa$$

with some fixed $\kappa > 1$. Moreover, the new hamiltonian still satisfies $(A1)$-$(A4)$.

For simplicity, in the following the quantities without subscripts refer to quantities at the ν^{th} step, while the quantities with subscripts $+$ denote the corresponding quantities at the $(\nu+1)^{\text{th}}$ step. Thus we consider the hamiltonian

$$\begin{aligned} H &= N + P \\ &\equiv e + \langle \omega(\xi), I\rangle + \sum_{n\in\mathbb{Z}_1} \Omega_n(\xi) z_n \bar{z}_n + P(\theta, I, z, \bar{z}, \xi, \varepsilon) \end{aligned} \tag{3.1}$$

defined in $D(r,s)\times\mathcal{O}$.

We assume that for $\xi\in\mathcal{O}$ and $|k|\le K$, there is

$$\begin{aligned} &|\langle k, \omega(\xi)\rangle| \ge \frac{\gamma}{K^\tau}, \quad k\neq 0, \\ &|\langle k, \omega\rangle + \Omega_n| \ge \frac{\gamma}{K^\tau}, \\ &|\langle k, \omega\rangle + \Omega_n + \Omega_m| \ge \frac{\gamma}{K^{\tau+\sigma}}, \\ &|\langle k, \omega\rangle + \Omega_n - \Omega_m| \ge \frac{\gamma}{K^{3\tau+4\sigma+2b}}, \quad |k| + ||n|-|m|| \neq 0, \end{aligned} \tag{3.2}$$

where $\sigma = \max\{\frac{\tau+1}{1-\chi}, \frac{\tau}{\iota}\}$.

Expanding P into the Fourier-Taylor series $P = \sum_{k,l,\alpha,\beta} P_{kl\alpha\beta} I^l e^{\mathrm{i}\langle k,\theta\rangle} z^\alpha \bar{z}^\beta$, $(A3)$ means

$$P_{kl\alpha\beta} = 0 \quad \text{if} \quad \sum_{j=1}^{b} k_j n_j + \sum_{n\in\mathbb{Z}_1} (\alpha_n - \beta_n) n \neq 0. \tag{3.3}$$

We now let $0 < r_+ < r$ and define

$$s_+ = \frac{1}{4} s \varepsilon^{\frac{1}{3}}, \quad \varepsilon_+ = c\gamma^{-2} K^{6\tau+8\sigma+2b} \varepsilon^{\frac{4}{3}}. \tag{3.4}$$

Here and later, the letter c denotes a suitable (possibly different) constant independent on the iteration steps.

We will construct a set $\mathcal{O}_+\subset\mathcal{O}$ and a change of variables $\Phi : D_+\times\mathcal{O}_+ = D(r_+, s_+)\times\mathcal{O}_+ \to D(r,s)\times\mathcal{O}$ such that the transformed hamiltonian $H_+ = N_+ + P_+ \equiv H\circ\Phi$ satisfies all the above iterative assumptions with new parameters s_+, ε_+, r_+ and with $\xi\in\mathcal{O}_+$.

3.1. **Solving the linearized equations.** Expand P into the Fourier-Taylor series

$$P = \sum_{k,l,\alpha,\beta} P_{kl\alpha\beta} e^{\mathrm{i}\langle k,\theta\rangle} I^l z^\alpha \bar{z}^\beta,$$

where $k\in\mathbb{Z}^b, l\in\mathbb{N}^b$ and the multi-indices α and β run over the set of all infinite-dimensional vectors $\alpha\equiv(\cdots,\alpha_n,\cdots)_{n\in\mathbb{Z}_1}$ with finitely many non-zero components of positive integers.

We define

$$R := \sum_{k, 2|l|+|\alpha|+|\beta|\le 2} P_{kl\alpha\beta} e^{\mathrm{i}(k,\theta)} I^l z^\alpha \bar{z}^\beta, \quad \langle R\rangle := \sum_{i=1}^{b} P_{0e_i00} I_i + \sum_{j\in\mathbb{Z}_1} P_{00e_je_j} |z_j|^2.$$

The generating function of our symplectic transformation, denoted by F, solves the “homological equation”:

$$\{N, F\} + \hat{N} = R. \tag{3.5}$$

It is well known (and immediate) that F is uniquely defined by a homological equation for those ξ such that $\langle \omega(\xi), k\rangle + \Omega(\xi)\cdot l \neq 0$. In order to have quantitative bounds, we restrict to a set $\mathcal{O}$ that has the bound (3.2).

To solve this homological equation with condition $(A3)$ (momentum conservation), one can refer to [16]. The key point is that with $(A3)$, we only need to solve fewer terms than before; we only give the estimation below.

3.2. **Estimation on the coordinate transformation.** With the previous section, we give the estimate to X_F and ϕ_F^1.

Lemma 3.1. *Let* $D_i = D(r_+ + \frac{i}{4}(r - r_+), \frac{i}{4}s)$, $0 < i \leq 4$. *Then*

$$\|X_F\|_{D_3,\mathcal{O}} \leq c(\gamma^{-1}K^{6\tau+8\sigma+2b})\varepsilon. \tag{3.6}$$

Lemma 3.2. *Let* $\eta = \varepsilon^{\frac{1}{3}}, D_{i\eta} = D(r_+ + \frac{i}{4}(r - r_+), \frac{i}{4}\eta s), 0 < i \leq 4$. *If* $\varepsilon \ll (\frac{1}{2}\gamma K^{-\tau})^6$, *we then have*

$$\phi_F^t : D_{2\eta} \to D_{3\eta}, \quad -1 \leq t \leq 1. \tag{3.7}$$

Moreover,

$$\|D\phi_F^t - Id\|_{D_{1\eta}} \leq c(\gamma^{-1}K^{6\tau+8\sigma+2b})\varepsilon. \tag{3.8}$$

Momentum conservation is preserved by KAM iteration since momentum conservation is preserved by the Poisson bracket.

Lemma 3.3. P_+ *satisfies momentum conservation.*

3.3. **Estimation for the new perturbation.** The map X_F^1 defined above transforms H into $H_+ = N_+ + P_+$, where

$$P_+ = \int_0^1 \{R(t), F\} \circ \phi_F^t dt + (P - R) \circ \phi_F^1,$$

with $R(t) = (1 - t)(N_+ - N) + tR$. Hence

$$X_{P_+} = \int_0^1 (\phi_F^t)^* X_{\{R(t),F\}} dt + (\phi_F^1)^* X_{(P-R)}.$$

Lemma 3.4. *The new perturbation* P_+ *satisfies the estimate*

$$\|X_{P_+}\|_{D(r_+,s_+)} \leq c\eta\varepsilon + c\gamma^{-1}K^{6\tau+8\sigma+2b}\eta^{-2}\varepsilon^2 \leq \varepsilon_+.$$

3.4. **Iteration lemma and convergence.** In order to make the KAM machine work fluently, for any given $s, \varepsilon, r, \gamma, \bar{p}, p, \delta$, let $\sigma = \max\{\frac{\tau+1}{1-\chi}, \frac{\tau}{\iota}\}$, and for all $\nu \geq 1$ we define the sequences

$$\begin{aligned} r_\nu &= r(1 - \sum_{i=2}^{\nu+1} 2^{-i}), \\ s_\nu &= \frac{1}{4}\eta_{\nu-1}s_{\nu-1} = 2^{-2\nu}(\prod_{i=0}^{\nu-1}\varepsilon_i)^{\frac{1}{3}}s_0, \\ \varepsilon_\nu &= c\gamma^{-2}K_{\nu-1}^{6\tau+8\sigma+2b}\varepsilon_{\nu-1}^{\frac{4}{3}}, \quad \eta_\nu = \varepsilon_\nu^{\frac{1}{3}}, \\ M_\nu &= M_{\nu-1} + \varepsilon_{\nu-1}, \quad L_\nu = L_{\nu-1} + \varepsilon_{\nu-1}, \\ K_\nu &= c\ln\varepsilon_\nu^{-1}, \end{aligned} \tag{3.9}$$

where c is a constant, and the parameters $r_0, \varepsilon_0, L_0, s_0$ and K_0 are defined to be r, ε, L, s and $\ln \frac{1}{\varepsilon}$ respectively.

We iterate the KAM step and get the iteration sequence.

Lemma 3.5. *Suppose $H_\nu = N_\nu + P_\nu$ is well defined in $D(r_\nu, s_\nu) \times \mathcal{O}_\nu$, where*

$$N_\nu = \langle \omega_\nu(\xi), I\rangle + \langle \Omega^\nu z, \bar{z}\rangle,$$

the functions ω_ν and Ω^ν are C^1_W smooth and

$$|\omega_\nu|_{C^1_W}, |\nabla \omega_\nu^{-1}|_{\mathcal{O}_\nu} \leq M_\nu, ||n|^\iota \tilde{\Omega}^\nu_n|_{C^1_W} \leq L_\nu M_\nu, \quad |\Omega^\nu_n - \Omega^{\nu-1}_n|_{\mathcal{O}_\nu} \leq \frac{\varepsilon_{\nu-1}}{|n|^\iota};$$

what's more

$$\|X_{P_\nu}\|_{D(r_\nu, s_\nu), \mathcal{O}_\nu} \leq \varepsilon_\nu.$$

Then there exists a symplectic change of variables $\Phi_\nu : D(r_{\nu+1}, s_{\nu+1}) \times \mathcal{O}_{\nu+1} \to D(r_\nu, s_\nu)$, such that on $D(r_{\nu+1}, s_{\nu+1}) \times \mathcal{O}_{\nu+1}$ we have

$$H_{\nu+1} = H_\nu \circ \Phi_\nu = e_{\nu+1} + N_{\nu+1} + P_{\nu+1} = e_{\nu+1} + \langle \omega_{\nu+1}, I\rangle + \langle \Omega^{\nu+1} z, \bar{z}\rangle + P_{\nu+1},$$

with $\omega_{\nu+1} = \omega_\nu + \sum\limits_{|l|=1} l P_{0l00}$, $\Omega^{\nu+1}_n = \Omega^\nu_n + P^\nu_{00e_ne_n}$.

The functions $\omega_{\nu+1}$ and $\Omega^{\nu+1}_n$ are C^1_W smooth with

$$|\omega_{\nu+1}|_{C^1_W}, |\nabla \omega_{\nu+1}^{-1}|_{\mathcal{O}} \leq M_{\nu+1}, ||n|^\iota \tilde{\Omega}^{\nu+1}_n|_{C^1_W} \leq L_{\nu+1} M_{\nu+1}, \ |\Omega^{\nu+1}_n - \Omega^\nu_n|_{\mathcal{O}_{\nu+1}} \leq \frac{\varepsilon_\nu}{|n|^\iota};$$

$$\|X_{P_{\nu+1}}\|_{D(r_{\nu+1}, s_{\nu+1}), \mathcal{O}_{\nu+1}} \leq \varepsilon_{\nu+1}.$$

3.4.1. *Convergence.* Suppose that the assumptions of Theorem 2.1 are satisfied. Recall that

$$\varepsilon_0 = \varepsilon, r_0 = r, s_0 = s, \rho_0 = \rho, L_0 = L,$$

and $\mathcal{O}$ is a bounded positive-measure set. The assumptions of the iteration lemma are satisfied when $\nu = 0$ if ε_0 and γ are sufficiently small. Inductively, we obtain the following sequences:

$$\mathcal{O}_{\nu+1} \subset \mathcal{O}_\nu,$$
$$\Psi^\nu = \Phi_0 \circ \Phi_1 \circ \cdots \circ \Phi_\nu : D(r_{\nu+1}, s_{\nu+1}) \times \mathcal{O}_{\nu+1} \to D(r_0, s_0), \nu \geq 0,$$
$$H \circ \Psi^\nu = H_{\nu+1} = N_{\nu+1} + P_{\nu+1}.$$

Let $\tilde{\mathcal{O}} = \bigcap_{\nu=0}^\infty \mathcal{O}_\nu$. As in [21, 22], thanks to Lemma 3.2, we conclude that $N_\nu, \Psi^\nu, D\Psi^\nu, \omega_\nu$ converge uniformly on $D(\frac{1}{2}r, 0) \times \tilde{\mathcal{O}}$ with

$$N_\infty = e_\infty + \langle \omega_\infty, I\rangle + \sum_n \Omega^\infty_n z_n \bar{z}_n.$$

Since

$$\varepsilon_{\nu+1} = c(\gamma^{-1} K_\nu^{6\tau+8\sigma+2b})\varepsilon_\nu^{\frac{4}{3}},$$

it follows that $\varepsilon_{\nu+1} \to 0$ provided that ε is sufficiently small. And we also have $\sum\limits_{\nu=0}^\infty \varepsilon_\nu \leq 2\varepsilon$.

Let ϕ^t_H be the flow of X_H. Since $H \circ \Psi^\nu = H_{\nu+1}$, we have

$$\phi^t_H \circ \Psi^\nu = \Psi^\nu \circ \phi^t_{H_{\nu+1}}. \tag{3.10}$$

The uniform convergence of $\Psi^\nu, D\Psi^\nu, \omega_\nu$ and X_{H_ν} implies that the limits can be taken on both sides of (3.10). Hence, on $D_{\frac{1}{2}\rho}(\frac{1}{2}r, 0) \times \tilde{\mathcal{O}}$ we get

$$\phi^t_H \circ \Psi^\infty = \Psi^\infty \circ \phi^t_{H_\infty} \tag{3.11}$$

and

$$\Psi^\infty : D(\frac{1}{2}r, 0) \times \tilde{\mathcal{O}} \to D(r,s) \times \mathcal{O}.$$

It follows from (3.11) that

$$\phi_H^t(\Psi^\infty(\mathbb{T}^b \times \{\xi\})) = \Psi^\infty(\mathbb{T}^b \times \{\xi\})$$

for $\xi \in \tilde{\mathcal{O}}$. This means that $\Psi^\infty(\mathbb{T}^b \times \{\xi\})$ is an embedded torus which is invariant for the original perturbed hamiltonian system at $\xi \in \tilde{\mathcal{O}}$. We remark here that the frequencies $\omega^\infty(\xi)$ associated to $\Psi^\infty(\mathbb{T}^b \times \{\xi\})$ are slightly different from $\omega(\xi)$. The normal behavior of the invariant torus is governed by normal frequencies Ω_n^∞. □

3.5. **Measure estimates.** For notational convenience, let $\mathcal{O}_0 = \mathcal{O}$, $K_0 = 0$. Then at the ν^{th} KAM iteration step, we define $\mathcal{O}_{\nu+1} = \mathcal{O}_\nu \backslash \mathcal{R}^\nu$; the resonant set $\mathcal{R}^\nu$ is defined to be

$$\mathcal{R}^\nu = \bigcup_{\substack{|k| \le K_\nu, \\ n,m \in \mathbb{Z}_1}} (\mathcal{R}_k^\nu \cup \mathcal{R}_{kn}^\nu \cup \mathcal{R}_{knm}^\nu), \tag{3.12}$$

where

$$\mathcal{R}_k^\nu = \{\xi \in \mathcal{O}_\nu : |\langle k, \omega_\nu(\xi)\rangle| < \frac{\gamma}{K_\nu^\tau}\}, \tag{3.13}$$

$$\mathcal{R}_{kn}^\nu = \{\xi \in \mathcal{O}_\nu : |\langle k, \omega_\nu\rangle + \Omega_n^\nu| < \frac{\gamma}{K^\tau}\}, \tag{3.14}$$

$$\mathcal{R}_{knm}^\nu = \{\xi \in \mathcal{O}_\nu : |\langle k, \omega_\nu\rangle \pm \Omega_n^\nu \pm \Omega_m^\nu| < \frac{\gamma}{K^{3\tau+4\sigma+b}}\}. \tag{3.15}$$

Lemma 3.6 (Lemma 8.4 of [2]). *Let $g : \mathcal{I} \to \mathbb{R}$ be $b+3$ times differentiable, and assume that*

(1) $\forall \sigma \in \mathcal{I}$ there exists $s \le b+2$ such that $g^{(s)}(\sigma) > B$.

(2) There exists A such that $|g^{(s)}(\sigma)| \le A$ for $\forall \sigma \in \mathcal{I}$ and $\forall s$ with $1 \le s \le b+3$. Define

$$\mathcal{I}_h \equiv \{\sigma \in \mathcal{I} : |g(\sigma)| \le h\}.$$

Then

$$\frac{\text{meas}(\mathcal{I}_h)}{\text{meas}(\mathcal{I})} \le \frac{A}{B} 2(2+3+\cdots+(b+3)+2B^{-1})h^{\frac{1}{b+3}}.$$

For the measure estimates, given $\varrho > 0$ we define

$$\mathcal{R}_{k,l}^\varrho := \left\{\xi \in \mathcal{O} : |\langle \omega, k\rangle + \Omega \cdot l| < \gamma K^{-\varrho}\right\}.$$

Lemma 3.7. *For all $(k,l) \neq (0,0)$, $|k| \le K$ and $|l| \le 2$, which satisfy momentum conservation, one has* $\text{meas}(\mathcal{R}_{k,l}^\varrho) \le C\gamma K^{-\varrho}$.

Proof. By assumption $\mathcal{O}$ is contained in some open set of diameter D.

Choose a to be a vector such that $\langle k, a\rangle = |k|$. We have

$$|\partial_t(\langle k, \omega(\xi + ta)\rangle + \Omega \cdot l)| \ge M(|k| - ML) \ge \frac{M}{2},$$

which leads to

$$\int_{R_{k,l}^\varrho} d\xi \le 2M^{-1}\gamma K^{-\varrho} \int_{\xi+ta \cap R_{k,l}^\varrho} dt \int d\xi_2 \dots d\xi_b \le 2M^{-1} D^{b-1} \gamma K^{-\varrho}.$$

□

For a proof see [2].

Lemma 3.8.

$$\text{meas}(\bigcup_{|k|\le K_\nu} \mathcal{R}_k^\nu) \le K_\nu^b \frac{\gamma}{K_\nu^\tau} = \frac{\gamma^{\frac{1}{4}}}{K_\nu^{\tau-b}},$$

$$\text{meas}(\bigcup_{|k|\le K_\nu, n} \mathcal{R}_{kn}^\nu) \le K_\nu^{b+\sigma} \frac{\gamma}{K_\nu^{\tau+\sigma}} = \frac{\gamma}{K_\nu^{\tau-b}}.$$

Lemma 3.9.

$$\text{meas}(\bigcup_{|k|\le K_\nu, n, m} \mathcal{R}_{knm}^\nu) \le \frac{\gamma}{K_\nu^{2\tau}}.$$

Proof. Notice that for momentum conservation

$$\sum_{j=1}^{b} k_j n_j + \sum_{n\in\mathbb{Z}_1} (\alpha_n - \beta_n) n = 0, \tag{3.16}$$

one has $|n-m| \le C_b|k| \le C_b K_\nu$.

We denote $\pi(k) = \sum_{j=1}^b k_j i_j$; then one has

$$\bigcup_{|k|\le K_\nu, n, m} R_{knm}^\nu = \bigcup_{|k|\le K_\nu, n} R_{kn, n+\pi(k)}^\nu.$$

Recall from Lemma 3.8 that $\forall \xi \notin \bigcup_{|k|\le K_\nu} \mathcal{R}_k^\nu$ and $\forall |k| \le K_\nu$ one has $|\langle k, \omega\rangle| \ge \gamma K_\nu^{-\tau}$. Then if $|n|$ or $|m| \ge K_\nu^{\tau+2\sigma}$ (recall $\sigma = \max\{\frac{\tau+1}{1-\chi}, \frac{\tau}{\iota}\}$), one has

$$\begin{aligned}
& |\langle k, \omega\rangle + \Omega_n^\nu - \Omega_{n+\pi(k)}^\nu| \\
= \; & |\langle k, \omega\rangle + |n|^\chi + \tilde{\Omega}_n^\nu - |n+\pi(k)|^\chi - \tilde{\Omega}_{n+\pi(k)}^\nu| \\
\ge \; & |\langle k, \omega\rangle| - ||n|^\chi - |n+\pi(k)|^\chi| - |\tilde{\Omega}_n^\nu| - |\tilde{\Omega}_{n+\pi(k)}^\nu| \\
\ge \; & \gamma K_\nu^{-\tau} - \chi |\frac{\pi(k)}{|n|^{1-\chi}}| - \frac{\varepsilon_0}{|n|^\iota} - \frac{\varepsilon_0}{|n+\pi(k)|^\iota} \\
\ge \; & \gamma K_\nu^{-\tau} - \chi |\frac{\pi(k)}{|K_\nu^{\tau+2\sigma}|^{1-\chi}}| - \frac{\varepsilon_0}{K_\nu^{\tau+2\sigma}} - \frac{\varepsilon_0}{K_\nu^{\tau+2\sigma}} \\
\ge \; & \gamma K_\nu^{-\tau} - \frac{\gamma}{4} K_\nu^{-\tau} - \frac{\gamma}{4} K_\nu^{-\tau} \\
\ge \; & \frac{1}{2} \gamma K_\nu^{-\tau}.
\end{aligned}$$

With this reduction, we only consider the resonant set to be no more than $K^{2\tau+4\sigma+b}$, and

$$\bigcup_{\substack{|k|\le K_\nu, \\ n,m\in\mathbb{Z}_1}} \mathcal{R}_{knm}^\nu = \bigcup_{\substack{|k|\le K_\nu, \\ |n|,|m|\le K^{\tau+2\sigma}}} \mathcal{R}_{knm}^\nu.$$

With Lemma 3.7,

$$\text{meas}(\bigcup_{\substack{|k|\le K_\nu, \\ n,m\in\mathbb{Z}_1}} \mathcal{R}_{knm}^\nu) \le \frac{\gamma}{K^{3\tau+4\sigma+b}} * K^{2\tau+4\sigma} * K_\nu^b \le \frac{\gamma}{K_\nu^\tau}.$$

one has the final estimate. ∎

Lemma 3.10. *Let $\tau > b$. Then the total measure needed to exclude along KAM iteration is*

$$\begin{aligned} & \text{meas}(\bigcup_{\nu\geq 0}\mathcal{R}^\nu) \\ = \quad & \text{meas}[\bigcup_{\nu\geq 0}(\bigcup_{|k|\leq K_\nu,n,m}\mathcal{R}_k^\nu\cup\mathcal{R}_{kn}^\nu\cup\mathcal{R}_{knm}^\nu)] \\ \leq \quad & \sum_{\nu\geq 0}\frac{\gamma}{K_\nu^\tau}\leq\gamma. \end{aligned}$$

Appendix A

Lemma A.1 (Lemma 2.1 of [23])**.** *For any regular analytic functions f, g in $D(r,s)$ and C_W^1 in $\mathcal{O}$ with finite semi-norm* (2.5)*, one has*

$$\|[X_f,X_g]\|_{r',s'}\leq 2^{2d+1}\delta^{-1}\|X_f\|_{r,s}\|X_g\|_{r,s}\,,$$

$$\|X_{\{f,g\}}\|_{r',s'}\leq 2^{2d+1}\delta^{-1}\|X_f\|_{r,s}\|X_g\|_{r,s}\,,$$

where $\delta=(\frac{r'}{r})^2\min(s-s',1-\frac{r}{r}')$.

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