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# CLASSIFICATION OF ISOPARAMETRIC HYPERSURFACES IN SPHERES WITH $(g, m) = (6, 1)$

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ABSTRACT. We classify the isospectral families $L(t) = \cos(t)L_0 + \sin(t)L_1 \in \mathrm{Sym}(5, \mathbb{R})$, $t \in \mathbb{R}$, with $L_0 = \mathrm{diag}(\sqrt{3}, \frac{1}{\sqrt{3}}, 0, -\frac{1}{\sqrt{3}}, -\sqrt{3})$. Using this result we provide a classification of isoparametric hypersurfaces in spheres with $(g, m) = (6, 1)$ and thereby give a simplified proof of the fact that any isoparametric hypersurface with $(g, m) = (6, 1)$ is homogeneous. This result was first proven by Dorfmeister and Neher in 1985.

## INTRODUCTION

The principal result of this paper is the following theorem.

**Theorem.** *Let $L(t) = \cos(t)L_0 + \sin(t)L_1 \in Sym(5, \mathbb{R})$, $t \in \mathbb{R}$, be isospectral where $L_0 = diag(\sqrt{3}, \frac{1}{\sqrt{3}}, 0, -\frac{1}{\sqrt{3}}, -\sqrt{3})$. Up to conjugation by an element $A \in \mathrm{O}(5)$ with $AL_0A^{-1} = L_0$, the matrix $L_1$ is given by one of the following matrices:*

$$L_1 = \frac{1}{3\sqrt{3}} \begin{pmatrix} 0&5&0&2&0 \\ 5&0&4&0&2 \\ 0&4&0&4&0 \\ 2&0&4&0&5 \\ 0&2&0&5&0 \end{pmatrix}, \ L_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 0&0&0&0&3\sqrt{2} \\ 0&0&1&0&0 \\ 0&1&0&1&0 \\ 0&0&1&0&0 \\ 3\sqrt{2}&0&0&0&0 \end{pmatrix}, \ L_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 0&0&3&0&0 \\ 0&0&0&\sqrt{2}&0 \\ 3&0&0&0&3 \\ 0&\sqrt{2}&0&0&0 \\ 0&0&3&0&0 \end{pmatrix},$$

$$L_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 0&1&0&2&0 \\ 1&0&0&0&2 \\ 0&0&0&0&0 \\ 2&0&0&0&1 \\ 0&2&0&1&0 \end{pmatrix}, \ L_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 0&\sqrt{3}&0&0&0 \\ \sqrt{3}&0&0&2&0 \\ 0&0&0&0&0 \\ 0&2&0&0&\sqrt{3} \\ 0&0&0&\sqrt{3}&0 \end{pmatrix}, \ L_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 0&0&0&0&3 \\ 0&0&0&1&0 \\ 0&0&0&0&0 \\ 0&1&0&0&0 \\ 3&0&0&0&0 \end{pmatrix}.$$

Using this result we classify isoparametric hypersurfaces in spheres with six different principal curvatures $g = 6$ all of multiplicity $m = 1$ and thereby give a simplified proof of a result of Dorfmeister and Neher [3].

In [4, 6] Miyaoka claims to reprove the result of Dorfmeister and Neher. Based on the idea of [4, 6] Miyaoka [5] proposed how to establish homogeneity for isoparametric hypersurfaces in spheres with six different principal curvatures $g = 6$ all of multiplicity $m = 2$, which is the only remaining open case with $g = 6$. Using (parts of) our main result we give a counterexample to Miyaoka's proof [4, 6].

The present paper is organized as follows: the above theorem is proved in Section 1 and used in Section 2 to classify isoparametric hypersurfaces in $\mathbb{S}^7$ with $g = 6$. Finally, the counterexample to the proof of Miyaoka [4, 6] can be found in the Appendix.

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## 1. Classification of the isospectral families

Subsections 1.1–1.4 of this section serve as preparation for Subsection 1.5 in which we prove the theorem stated in the introduction.

1.1. **Minimal polynomial equation.** In what follows we consider $L(t) \in \mathrm{Sym}(5,\mathbb{R})$, $t \in \mathbb{R}$, with

$$\mathrm{spec}(L(t)) = \left\{-\sqrt{3}, -\tfrac{1}{\sqrt{3}}, 0, \tfrac{1}{\sqrt{3}}, \sqrt{3}\right\} \quad \text{for all } t \in \mathbb{R},$$

where the eigenvalues arise with multiplicity $m$. Below we use the shorthand notation $L$ for $L(t)$. Thus we obtain the minimal polynomial equation

$$0 = (L^2 - 3\cdot \mathbb{1})\,(L^2 - \tfrac{1}{3}\cdot \mathbb{1})\,L = (L^4 - \tfrac{10}{3}\,L^2 + \mathbb{1})\,L.$$

We introduce the complexified operators $L_\pm \in \mathrm{End}(\mathbb{R}^{5m}\otimes\mathbb{C})$ by $L_\pm = \frac{1}{2}\,(L_0 \mp i\,L_1)$. Since $L_0, L_1 \in \mathrm{End}\,\mathbb{R}^{5m}$ are symmetric, $L_+, L_- \in \mathrm{End}(\mathbb{R}^{5m}\otimes\mathbb{C})$ are also symmetric. Plugging $L(t) = \exp(it)\,L_+ + \exp(-it)\,L_-$ into the above equation and sorting by different frequencies yields

(1)
$$L_+^5 = 0, \qquad L_-^5 = 0,$$
(2)
$$15\,\sigma(L_+^4\,L_-) - 10\,L_+^3 = 0, \qquad 15\,\sigma(L_+\,L_-^4) - 10\,L_-^3 = 0,$$
(3)
$$10\,\sigma(L_+^3\,L_-^2) - 10\,\sigma(L_+^2\,L_-) + L_+ = 0, \qquad 10\,\sigma(L_+^2\,L_-^3) - 10\,\sigma(L_+\,L_-^2) + L_- = 0,$$

where $\sigma(L_+^i\,L_-^j) \in \mathrm{Sym}(\mathbb{R}^{5m}\otimes\mathbb{C})$ is given by the sum of all possible words of $L_+^i\,L_-^j$ divided by the number of possible words, for example

$$\sigma(L_+^3\,L_-) = \tfrac{1}{4}\,(L_+^3\,L_- + L_+^2\,L_-\,L_+ + L_+\,L_-\,L_+^2 + L_-\,L_+^3).$$

It suffices to consider the first equation in each of the above rows, since the remaining equations are obtained from these by complex conjugation.

1.2. **The projector onto the kernel of $L(t)$.**

**Lemma 1.1.** *For $t \in \mathbb{R}$ the map $P(t) : \mathbb{R}^{5m} \to \mathbb{R}^{5m}$ given by $P(t) = L(t)^4 - \frac{10}{3}\,L(t)^2 + \mathbb{1}$ is the projector onto the $m$-dimensional kernel of $L(t)$.*

*Proof.* Below we use the shorthand notation $P = P(t)$.

On the one hand, we have $L\,P = P\,L = 0$ by the minimal polynomial equation, i.e., $\mathrm{im}\,P \subset \ker L$. On the other hand, $x \in \ker L$ implies $P\,x = x$, i.e., $\ker L \subset \mathrm{im}\,P$. Consequently, $\mathrm{im}\,P = \ker L$. Finally,

$$P^2 - P = (P - \mathbb{1})P = (L^4 - \tfrac{10}{3}L^2)P = (L^3 - \tfrac{10}{3}L)LP = 0,$$

i.e., $P(t)$ is a projector for all $t \in \mathbb{R}$. □

Substituting $L(t) = \exp(it)\,L_+ + \exp(-it)\,L_-$ in the formula for $P(t)$ yields

$$P(t) = \exp(4it)\,P_4 + \exp(2it)\,P_2 + P_0 + \exp(-2it)\,P_{-2} + \exp(-4it)\,P_{-4},$$

where $P_4, P_2, P_0, P_{-2}, P_{-4} \in \mathrm{Sym}(\mathbb{R}^{5m}\otimes\mathbb{C})$ are given by

$$P_4 = L_+^4, \qquad P_{-4} = \overline{P_4}, \qquad P_2 = 4\,\sigma(L_+^3\,L_-) - \tfrac{10}{3}\,L_+^2, \qquad P_{-2} = \overline{P_2},$$

and $P_0 = 6\,\sigma(L_+^2\,L_-^2) - \frac{20}{3}\,\sigma(L_+\,L_-) + \mathbb{1}$. Clearly, $P_0 = \overline{P_0}$.

**Lemma 1.2.** *The minimal polynomial equation is equivalent to*

$$L_+ P_4 = 0, \qquad L_+ P_2 + L_- P_4 = 0, \qquad L_+ P_0 + L_- P_2 = 0.$$

**Corollary 1.3.** $P_i L_\pm P_j = 0$ *for all* $i, j \in I := \{-4, -2, 0, 2, 4\}$.

*Proof.* We have to establish $5\times 2\times 5 = 50$ equations. Obviously, given one equation, the transposed and the conjugate equation are also true, which has to be considered when counting equations. Applying $P_i$ with $i \in I$ from the left to $L_+ P_4 = 0$ we obtain $P_j L_+ P_4 = 0$ for $j \in I$. These are 18 equations. Using this result and Lemma 1.2 we get

$$P_4 L_- P_4 = P_4 (-L_+ P_2) = -(L_+ P_4) P_2 = 0, P_4 L_- P_2 = P_4 (-L_+ P_0) = 0,$$
$$P_{-4} L_+ P_2 = P_{-4} (-L_- P_4) = 0, P_{-4} L_+ P_0 = P_{-4} (-L_- P_2) = 0.$$

Hence we proved $2 + 4 + 4 + 4 = 14$ additional equations. These identities again together with the identity $L_+ P_2 + L_- P_4 = 0$ of Lemma 1.2 imply $P_0 L_+ P_2 = P_0 (-L_- P_4) = 0$ and similarly $P_2 L_+ P_2 = 0$ and $P_{-2} L_+ P_2 = 0$, which are $4 + 2 + 4 = 10$ additional equations. Combining these identities with Lemma 1.2 yields $P_2 L_- P_2 = P_2 (-L_+ P_0) = 0$ and $P_{-2} L_+ P_0 = P_{-2} (-L_- P_2) = 0$, which are $2 + 4 = 6$ additional equations. The two remaining equations, $P_0 L_\pm P_0 = 0$, are obtained by combining $P_{-2} L_+ P_0 = 0$ and $L_+ P_0 + L_- P_2 = 0$. □

1.3. **The span of the kernel over time.** Following Miyaoka [4] we introduce

$$E = \operatorname{span}_{t\in\mathbb{R}} \ker L(t) \subset \mathbb{R}^{5m}.$$

Obviously, the independence of $\ker L(t)$ of $t \in \mathbb{R}$ is equivalent to $\dim E = m$.

**Lemma 1.4.** $E = \sum_{i\in I} \operatorname{im} P_i$ *and* $\dim E \le 3m$.

*Proof.* Since $\operatorname{im} P(t) = \ker L(t)$ we have to prove $\operatorname{span}_{t\in\mathbb{R}} \operatorname{im} P(t) = \sum_{i\in I} \operatorname{im} P_i$. Clearly, $\operatorname{span}_{t\in\mathbb{R}} \operatorname{im} P(t) \subseteq \sum_{i\in I} \operatorname{im} P_i$. Hence the first claim follows from the identities

$$\exp(4it)P_4 + P_0 + \exp(-4it)P_{-4} = \tfrac{1}{2}(P(t) + P(t + \tfrac{\pi}{2})), \tag{4}$$
$$P_0 = \tfrac{1}{3}(P(t) + P(t + \tfrac{\pi}{3}) + P(t + \tfrac{2\pi}{3})), \tag{5}$$
$$\exp(2it)P_2 + \exp(-2it)P_{-2} = \tfrac{1}{2}(P(t) - P(t + \tfrac{\pi}{2})). \tag{6}$$

In order to prove the second claim let $d = \dim E$. Using $\dim(\ker L(t)) = m$ for $t \in \mathbb{R}$, we get $\dim(L(t)E) \ge d - m$. Corollary 1.3 implies $L(t) E \perp E$ for all $t \in \mathbb{R}$ and thus $L_\pm E \perp E$. Combining $L(t)E \subset E^\perp$ and $\dim(L(t)E) \ge d - m$ we obtain $\dim E^\perp \ge d - m$. From $E \oplus E^\perp = \mathbb{R}^{5m}$ we have $\dim E + \dim E^\perp = \dim \mathbb{R}^{5m}$. Thus we get $5m = \dim \mathbb{R}^{5m} = \dim E + \dim E^\perp \ge 2d - m$, hence the claim. □

**Corollary 1.5.** $L(t) E \perp E$ *for all* $t \in \mathbb{R}$ *and thus* $L_\pm E \perp E$.

**Lemma 1.6.** *The following five statements are equivalent:* (i) $\ker L(t)$ *is constant,* (ii) $\dim E = m$, (iii) $L(t) E = 0$ *for* $t \in \mathbb{R}$, (iv) $L_+ E = 0$, (v) $L_+ P_i = 0$ *for all* $i \in \{-4, -2, 0, 2, 4\}$.

*Proof.* The equivalence of (iv) and (v) follows from Lemma 1.4, the rest is obvious. □

*Remark* 1.7. We will see below (see, e.g., Lemma 1.19) that the minimal polynomial equation of one focal manifold is not sufficient to prove $\dim E = m$: we construct explicitly isospectral families which satisfy the minimal polynomial equation but have a nonconstant kernel.

1.4. **Some linear algebra.** In this subsection we provide some linear algebra results which we will need for the proofs in Subsection 1.5.

We denote by $\{e_1, e_2\}$ and $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ the standard basis of $\mathbb{C}^2$ and the usual almost complex structure of $\mathbb{C}^2$, respectively. Below we work with the basis $\{e_+, e_-\}$ of $\mathbb{C}^2$ built by the isotropic vectors $e_\pm = \frac{1}{\sqrt{2}}(e_1 \pm i\, e_2)$. A basis of $M_2(\mathbb{C})$ is given by $\{\rho, \overline{\rho}, \sigma, \overline{\sigma}\}$, where $\rho = e_+\, e_-^{tr} = \frac{1}{2}(\mathbb{1} + i\, J)$, $\overline{\rho} = e_-\, e_+^{tr} = \frac{1}{2}(\mathbb{1} - i\, J)$, $\sigma = e_+\, e_+^{tr}$, and $\overline{\sigma} = e_-\, e_-^{tr}$.

**Lemma 1.8.** *The following identities hold:*

(1) $\rho^2 = \rho$, $\overline{\rho}^2 = \overline{\rho}$, $\rho\overline{\rho} = 0$, $\overline{\rho}\rho = 0$, $\rho + \overline{\rho} = \mathbb{1}$, $iJ = \rho - \overline{\rho}$, $\rho^{tr} = \overline{\rho}$, $e_+^{tr}\, \rho = 0$, $e_-^{tr}\, \overline{\rho} = 0$.
(2) $\sigma^2 = 0$, $\overline{\sigma}^2 = 0$, $\sigma^{tr} = \sigma$, $\overline{\sigma}^{tr} = \overline{\sigma}$, $e_+^{tr}\, \sigma = 0$, $e_-^{tr}\, \overline{\sigma} = 0$.
(3) $\rho\, \sigma = \sigma = \sigma\, \overline{\rho}$, $\overline{\rho}\, \sigma = \sigma\, \rho = 0$, $\sigma\, \overline{\sigma} = \rho$.

**Lemma 1.9.** *For $B \in \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^{2\,l})$ the statement $(e_+^{tr} \otimes \mathbb{1}_l)B = 0$ is equivalent to $B = e_+ \otimes B_0$ for some $B_0 \in \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^l)$. Furthermore, $B_0$ is given by $B_0 = (e_-^{tr} \otimes \mathbb{1}_l)B$ and is thus uniquely determined by $B$.*

**Corollary 1.10.** *For $B \in \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^{2\,l}), c \in \mathbb{C}^*$ and an injective $A_0 \in \mathrm{End}\, \mathbb{C}^l$ with $(c\, e_+^{tr} \otimes A_0)B = 0$ we have $B = e_+ \otimes B_0$ where $B_0 = (e_-^{tr} \otimes \mathbb{1}_l)B$.*

Note that a change of the basis in $\mathrm{O}_n(\mathbb{R})$ is compatible with the structure of the problem: let $U \in \mathrm{O}_n(\mathbb{R}) = \mathrm{O}_n(\mathbb{C}) \cap \mathrm{U}(n)$ be given and set $L_+^{'} = UL_+U^{tr}$, and $L_-^{'} = UL_-U^{tr}$. Thus $L_\pm^{'}$ satisfy the same identities as $L_\pm$.

**Lemma 1.11.** *For $A \in \mathrm{End}\, \mathbb{C}^d$ with $A^{tr} = A$ and $A^2 = 0$ there exist $U \in \mathrm{O}_d(\mathbb{R})$ and a positive definite, diagonal matrix $A_0 \in \mathrm{End}\, \mathbb{R}^{d_0} \subset \mathrm{End}\, \mathbb{C}^{d_0}$ such that $UAU^{tr} = \left(\begin{smallmatrix} \sigma\otimes A_0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$.*

*Proof.* The real and symmetric matrices $\mathrm{Re}(A) = \frac{1}{2}(A + A^c)$ and $\mathrm{Im}(A) = \frac{1}{2i}(A - A^c)$ satisfy $\mathrm{Re}(A)^2 = \frac{1}{4}\{A, A^c\} = \mathrm{Im}(A)^2$, where $A^c$ denotes the conjugate of $A$. Consequently, $Q := \frac{1}{4}\{A, A^c\}$ is a positive semidefinite matrix and therefore $\ker Q^\perp = \mathrm{im}\, Q$. Since $A$ and $A^c$ commute with $Q$, the endomorphism $A$ and $A^c$ map the subspace $\ker Q^\perp = \mathrm{im}\, Q$ onto itself. Moreover, using $\mathrm{Re}(A)^2 = \frac{1}{4}\{A, A^c\} = \mathrm{Im}(A)^2$, we prove easily that $A$ and $A^c$ vanish on the subspace $\ker Q$. By a straightforward computation we verify that $J_0 := -\frac{1}{4}\, i\,(A\, Q^{-1}\, A^c - A^c\, Q^{-1}\, A)$ defines an almost complex structure on the subspace $\mathrm{im}\, Q$ and thus there exists a $d_0 \in \mathbb{N}$ such that $\dim(\mathrm{im}\, Q) = 2d_0$. We can choose a basis of $\mathrm{im}\, Q$ such that $\mathrm{Re}(A)$ is diagonal. Moreover, we have $J_0\mathrm{Re}(A) = \mathrm{Im}(A)$ and $J_0\mathrm{Im}(A) = -\mathrm{Re}(A)$. Let $\mathrm{Re}(A) = \mathrm{diag}(A_1, A_2)$ where $A_1, A_2 \in \mathrm{diag}(d_0, \mathbb{R})$. We thus get $A_2 = -A_1$ and we can choose a basis of $\mathrm{im}\, Q$ such that $\mathrm{Re}(A) = \mathrm{diag}(A_0, -A_0)$, where $A_0 \in \mathrm{diag}(d_0, \mathbb{R})$ is positive definite. ∎

**Convention 1.12.** Let a symmetric matrix $A$ with $A^2 = 0$ be given. Below we write for short that Lemma 1.11 implies that there exists a diagonal matrix $A_0$, which is positive definite or the null matrix such that $A = \left(\begin{smallmatrix} \sigma\otimes A_0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, i.e., we will not mention that this identity only holds up to conjugation by an element of the orthogonal group.

**Lemma 1.13.** *For $A = \sigma \otimes A_0 \in \mathrm{Mat}(2n_1, \mathbb{C})$ and $B = \sigma \otimes B_0 \in \mathrm{Mat}(2n_2, \mathbb{C})$, where $A_0 \in \mathrm{Mat}(n_1, \mathbb{R})$ and $B_0 \in \mathrm{Mat}(n_2, \mathbb{R})$ are positive definite, diagonal matrices, we have*

1. *$CA=0$ for $C \in \mathrm{Mat}(n_3 \times 2n_1, \mathbb{C}) \Rightarrow C = e_+^{tr} \otimes C_0$ with $C_0 \in \mathrm{Mat}(n_3 \times n_1, \mathbb{C})$,*
2. *$BC=0$ for $C \in \mathrm{Mat}(2n_2 \times n_4, \mathbb{C}) \Rightarrow C = e_+ \otimes C_0$ with $C_0 \in \mathrm{Mat}(n_2 \times n_4, \mathbb{C})$,*
3. *$CA = 0$ and $BC = 0$ for $C \in \mathrm{Mat}(2n_2 \times 2n_1, \mathbb{C}) \Rightarrow C = \sigma \otimes C_0$ where $C_0 \in \mathrm{Mat}(n_2 \times n_1, \mathbb{C})$.*

*Proof.* We just prove (1) since (2) follows similarly and (3) is a consequence of (1) and (2). Let $A_0 = \mathrm{diag}(a_1, \ldots, a_{n_1})$ and denote the first row of $C$ by $(c_1, \ldots, c_{2n_1})$. Multiplication of the first row of $C$ with the first column of $A$ yields $(c_1 + ic_{n_1+1})a_1 = 0$. Since $A$ is positive definite we get $c_{n_1+1} = ic_1$. Analogously, we obtain $c_{n_1+j} = ic_j$ for $1 \leq j \leq n_1$. The claim is established by proceeding analogously for the remaining rows of $C$. $\square$

1.5. **Isospectral families of focal shape operators for the case $m = 1$.** In this subsection we prove our main theorem and assume $(g, m) = (6, 1)$ throughout.

**Theorem 1.14.** *Let $L(t) = \cos(t)L_0 + \sin(t)L_1 \in Sym(5, \mathbb{R})$, $t \in \mathbb{R}$, be isospectral where $L_0 = diag(\sqrt{3}, \frac{1}{\sqrt{3}}, 0, -\frac{1}{\sqrt{3}}, -\sqrt{3})$. Up to conjugation by an element $A \in \mathrm{O}(5)$ with $AL_0A^{-1} = L_0$, the matrix $L_1$ is given by one of the following matrices:*

$$L_1 = \frac{1}{3\sqrt{3}} \begin{pmatrix} 0&5&0&2&0 \\ 5&0&4&0&2 \\ 0&4&0&4&0 \\ 2&0&4&0&5 \\ 0&2&0&5&0 \end{pmatrix}, \quad L_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 0&0&0&0&3\sqrt{2} \\ 0&0&1&0&0 \\ 0&1&0&1&0 \\ 0&0&1&0&0 \\ 3\sqrt{2}&0&0&0&0 \end{pmatrix}, \quad L_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 0&0&3&0&0 \\ 0&0&0&\sqrt{2}&0 \\ 3&0&0&0&3 \\ 0&\sqrt{2}&0&0&0 \\ 0&0&3&0&0 \end{pmatrix},$$

$$L_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 0&1&0&2&0 \\ 1&0&0&0&2 \\ 0&0&0&0&0 \\ 2&0&0&0&1 \\ 0&2&0&1&0 \end{pmatrix}, \quad L_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 0&\sqrt{3}&0&0&0 \\ \sqrt{3}&0&0&2&0 \\ 0&0&0&0&0 \\ 0&2&0&0&\sqrt{3} \\ 0&0&0&\sqrt{3}&0 \end{pmatrix}, \quad L_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 0&0&0&0&3 \\ 0&0&0&1&0 \\ 0&0&0&0&0 \\ 0&1&0&0&0 \\ 3&0&0&0&0 \end{pmatrix}.$$

*For these cases $\dim(E)$ is given by $3, 2, 2, 1, 1$ and $1$, respectively.*

The proof of this theorem consists of the lemmas of this subsection.

*Remark* 1.15. For the case $(g, m) = (6, 2)$, in which the matrices are 10 by 10, there does not yet exist a classification of the isospectral families of focal shape operators.

**Lemma 1.16.** *Up to conjugation by an element of $\mathrm{O}_5(\mathbb{R})$ the matrix $P_4 = L_+^4$ is of the form $P_4 = \left(\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right)$ with $A = \sigma \otimes A_0$, where $A_0 \in \mathbb{R}$.*

*Proof.* Since $P_4^2 = L_+^8 = 0$, Lemma 1.11 implies $P_4 = \left(\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right)$ with $A = \sigma \otimes A_0$, where $A_0 \in \mathrm{Sym}(d_1, \mathbb{R})$ is a diagonal and positive definite matrix or the null matrix with $2d_1 \leq 5$. In what follows we assume $d_1 = 2$. Thus by $L_+P_4 = P_4L_+ = 0$ and Lemma 1.13 we have $L_+ = \left(\begin{smallmatrix} \sigma \otimes U_0 & e_+ \otimes V_0 \\ e_+^{tr} \otimes {V_0}^{tr} & W_0 \end{smallmatrix}\right)$, where $U_0 \in \mathrm{Mat}(2, \mathbb{C})$, $V_0 \in \mathrm{Mat}(2 \times 1, \mathbb{C})$, and $W_0 \in \mathbb{C}$. Thus we get $P_4 = L_+^4 = \left(\begin{smallmatrix} W_0^2(\sigma \otimes V_0 V_0^{tr}) & W_0^3(e_+ \otimes V_0) \\ W_0^3(e_+^{tr} \otimes {V_0}^{tr}) & W_0^4 \end{smallmatrix}\right)$ and therefore $W_0 = 0$. However this implies $P_4 = 0$, which contradicts our assumption. Thus $d_1 \in \{0, 1\}$. $\square$

By Lemma 1.2 we have $L_+^5 = 0$. Below we consider successively the four cases $L_+^{j+1} = 0$, $L_+^j \neq 0$, $j \in \{1, 2, 3, 4\}$, and determine the possible $L_1$ for each case.

**Lemma 1.17.** *Let $P_4 = \left(\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right)$ with $A = \sigma \otimes A_0$, where $A_0 \in \mathbb{R} - \{0\}$. Up to conjugation, by an element of $\mathrm{O}_5(\mathbb{R})$, the matrix $L_+^2$ is of the form*

$$L_+^2 = \begin{pmatrix} \sigma\otimes B_0 & \sigma\otimes C_1 & e_+\otimes C_2 \\ \sigma\otimes C_1^{tr} & \sigma\otimes D_0 & 0 \\ e_+^{tr}\otimes C_2^{tr} & 0 & 0 \end{pmatrix},$$

*where $B_0 \in \mathbb{C}$, $D_0 \in \mathrm{Sym}(d_2, \mathbb{R})$ is a diagonal, positive definite matrix or the null matrix with $2d_2 \leq 3$, $C_1 \in Mat(d_2, \mathbb{C})$, $C_2 \in \mathrm{Mat}(1\times(3-2d_2), \mathbb{C})$, and $A_0 = C_2\, C_2^{tr}$.*

*Proof.* Introduce the notation $L_+^2 = \left(\begin{smallmatrix} B & C \\ C^{tr} & D \end{smallmatrix}\right)$ where $B \in \mathrm{Sym}(2d_1, \mathbb{C})$. Hence $L_+^2\, P_4 = P_4\, L_+^2 = 0$ imply $B\,A = 0$, $A\,B = 0$, and $A\,C = 0$. By Lemma 1.13 we get $B = \sigma \otimes B_0$ with $B_0 \in \mathbb{C}$ and $C = e_+ \otimes C_0$. Calculating $(L_+^2)^2$ and using Lemma 1.16 we get $A = CC^{tr}$, $DC^{tr} = 0$, and $D^2 = 0$. In particular, $A_0 = C_0\, C_0^{tr}$. Since $D^2 = 0$, Lemma 1.11 implies $D = \left(\begin{smallmatrix} \sigma\otimes D_0 & \\ & 0 \end{smallmatrix}\right)$, where $D_0 \in \mathrm{Sym}(d_2, \mathbb{R})$ is a diagonal and positive definite matrix. From $D \in \mathrm{Sym}(3, \mathbb{C})$ we get $2d_2 \leq 3$. Lemma 1.13 yields $C^{tr} = \begin{pmatrix} \sigma\otimes C_1^{tr} \\ e_+^{tr}\otimes C_2^{tr} \end{pmatrix}$, where $C_1 \in \mathrm{Mat}(1\times d_2, \mathbb{C})$ and $C_2 \in \mathrm{Mat}(1\times(3-2d_2), \mathbb{C})$. Finally, $\sigma \otimes A_0 = C\,C^{tr} = \sigma \otimes (C_2\, C_2^{tr})$ implies $A_0 = C_2\, C_2^{tr}$. $\square$

**Lemma 1.18.** *Assume $P_4 \neq 0$. Up to conjugation by an element of $\mathrm{O}_5(\mathbb{R})$ the matrix $L_+$ is of the form*

$$L_+ = \left(\begin{smallmatrix} 0 & e_+\otimes F_0 \\ e_+^{tr}\otimes F_0^{tr} & G \end{smallmatrix}\right),$$

*where $G = \left(\begin{smallmatrix} \sigma\otimes G_1 & e_+\otimes G_3 \\ e_+^{tr}\otimes G_3 & 0 \end{smallmatrix}\right) \in \mathrm{Sym}(3, \mathbb{C})$ with $G_1 \in \mathbb{C}$. Furthermore,*

$$B_0 = F_0\, F_0^{tr}, \qquad C_0 = F_0\, G, \qquad D = G^2, \qquad A_0 = F_0\, D\, F_0^{tr},$$

*and $d_2 = 1$. Finally, $D_0 = G_3^2$.*

*Proof.* Using Lemma 1.16 and $L_+P_4 = P_4L_+ = L_+^5 = 0$ we deduce

$$L_+ = \left(\begin{smallmatrix} \sigma\otimes E_0 & e_+\otimes F_0 \\ e_+^{tr}\otimes F_0^{tr} & G \end{smallmatrix}\right)$$

for an $E_0 \in \mathbb{C}$. By Lemma 1.3 we get $P_4L_-P_4 = 0$ which is equivalent to $E_0 = 0$. Calculating $L_+^2$ and using Lemma 1.17, we obtain the first three of the claimed identities. Plugging $C_0 = F_0G$ into $A_0 = C_0\, C_0^{tr}$ and using $D = G^2$ we obtain the fourth equation, which implies that $D_0$ cannot vanish, i.e., $d_2 = 1$. Decomposing $G$ corresponding to $D$ and evaluating $D = G^2$ yields that $G$ is of the stated form. $\square$

**Lemma 1.19.** *If $\mathrm{rk}\, P_4 = 1$ then there exists an $A \in \mathrm{O}(5)$ such that $AL(t)A^{-1} = \cos(t)L_0 + \sin(t)L_1$ with $L_0 = diag(\sqrt{3}, \frac{1}{\sqrt{3}}, 0, -\frac{1}{\sqrt{3}}, -\sqrt{3})$ and*

$$L_1 = \frac{1}{3\sqrt{3}} \begin{pmatrix} 0 & 5 & 0 & 2 & 0 \\ 5 & 0 & 4 & 0 & 2 \\ 0 & 4 & 0 & 4 & 0 \\ 2 & 0 & 4 & 0 & 5 \\ 0 & 2 & 0 & 5 & 0 \end{pmatrix}.$$

*In particular,* $\dim E = 3$.

*Proof.* Introduce the notation $F_0 = (e_+^{tr} \otimes F_1 + e_-^{tr} \otimes F_2, F_3)$ with $F_1, F_2, F_3 \in \mathbb{C}$. Then $A_0 = F_0\, D\, F_0^{tr}$ is equivalent to $A_0 = F_2^2\, D_0$ which implies $F_2 \in \mathbb{R}^*$. From the $(4,5)$-component of $L_+\, P_2 + L_-\, P_4 = 0$ and $F_2, G_3 \in \mathbb{R}^*$ we have $F_1 = 0$. Thus the $(5,1)$-component of $L_+\, P_2 + L_-\, P_4 = 0$ yields $G_1 = -2\frac{F_2F_3}{G_3}$. Therefore the $(5,5)$-component of $L_+\, P_0 + L_-\, P_2 = 0$ implies $F_3 = 0$. Hence $L_+\, P_2 + L_-\, P_4 = 0$ is equivalent to $3G_3^2 + 3F_2^2 - 5 = 0$ and $L_+P_0 + L_-P_2 = 0$ reduces to $3 - 10F_2^2 + 3F_2^4 = 0$. Consequently, $F_2 = \pm\frac{1}{\sqrt{3}}$ or $F_2 = \pm\sqrt{3}$. If $F_2 = \pm\sqrt{3}$ we have

$\mathrm{Im}G_3 \neq 0$ which contradicts $G_3 \in \mathbb{R}^*$. Thus $F_2 = \pm\frac{1}{\sqrt{3}}$. Consequently, $(F_2, G_3) \in \Big\{(\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}), (\frac{1}{\sqrt{3}}, -\frac{2}{\sqrt{3}}), (-\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}), (-\frac{1}{\sqrt{3}}, -\frac{2}{\sqrt{3}})\Big\}$. We determine $L_0$ and $L_1$ for each of these cases and perform a change of the basis such that the basis consists of unit eigenvectors of $L_0$. If $(F_2, G_3) = (\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}})$ or $(F_2, G_3) = (-\frac{1}{\sqrt{3}}, -\frac{2}{\sqrt{3}})$ we obtain the above $L_1$ with the $+$-sign. For the remaining two cases the sign of $L_1$ changes, which corresponds to a change of orientation of $(M, g_0)$. Conjugating $\cos(t)L_0 + \sin(t)L_1$ by $\mathrm{diag}(-1, 1, -1, 1, -1)$ the claim follows. $\square$

**Lemma 1.20.** *For $P_4 = 0$ and $L_+^3 \neq 0$ there exists an $A \in \mathrm{O}(5)$ such that $AL(t)A^{-1} = \cos(t)L_0 + \sin(t)L_1$ with $L_0 = diag(\sqrt{3}, \frac{1}{\sqrt{3}}, 0, -\frac{1}{\sqrt{3}}, -\sqrt{3})$ and*

$$L_1 = \frac{1}{\sqrt{3}}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & \sqrt{3} & 0 \end{pmatrix}.$$

*In particular,* $\dim E = 1$.

*Proof.* By $(L_+^3)^2 = 0$ and Lemma 1.11 we get $L_+^3 = \begin{pmatrix} \sigma\otimes S_0 & 0 \\ 0 & 0 \end{pmatrix}$ where $S_0 \in \mathrm{Mat}(d_3, \mathbb{R})$ is a positive definite, diagonal matrix. Therefore $d_3 \in \{1, 2\}$. Introduce the notation $L_+ = \begin{pmatrix} T & U \\ U^{tr} & V \end{pmatrix}$ where $T \in \mathrm{Mat}(2d_3, \mathbb{C})$. From $L_+ L_+^3 = 0$ and $L_+^3 L_+ = 0$ we get $T = \sigma \otimes T_0$ with $T_0 \in \mathrm{Mat}(d_3, \mathbb{C})$ and $U = e_+ \otimes U_0$. Furthermore, the identity $L_+ P_2 = 0$ implies $L_+^3 P_2 = 0$, which is equivalent to $T_0 = 0$. Hence $L_+ = \begin{pmatrix} 0 & e_+\otimes U_0 \\ e_+^{tr}\otimes U_0^{tr} & V \end{pmatrix}$. Calculating $L_+^3$ and comparing with $L_+^3 = \begin{pmatrix} \sigma\otimes S_0 & 0 \\ 0 & 0 \end{pmatrix}$ yields $V^3 = 0$ and $(e_+ \otimes U_0)V^2 = 0$. If $d_3 = 2$ we have $V \in \mathbb{C}$ and thus $V = 0$, which yields $S_0 = 0$, contradicting our assumption. Thus $d_3 = 1$. From $(V^2)^2 = 0$ and Lemma 1.11 we have $V^2 = \begin{pmatrix} \sigma\otimes W & \\ & 0 \end{pmatrix}$ where $W \geq 0$.

First let $W = 0$ and thus $V^2 = 0$. Using Lemma 1.11 we get $V = \begin{pmatrix} \sigma\otimes V_0 & 0 \\ 0 & 0 \end{pmatrix}$ for $V_0 \in \mathbb{R}$. Since $V_0 = 0$ would imply $S_0 = 0$, we have $V_0 > 0$. Introduce $u_1, u_2, u_3 \in \mathbb{C}$ by $U_0 = (u_1, u_2, u_3)$. Calculating $L_+^3$ yields $S_0 = \frac{1}{4}(u_1 + iu_2)^2 V_0$. Since $S_0 > 0$, we get $u_1 + iu_2 \in \mathbb{R}^*$. Therefore the $(4, 2)$ equation of $L_+ P_2 = 0$ is equivalent to $\overline{u_1} + i\,\overline{u_2} = 0$. Combining this equation with $u_1 + iu_2 \in \mathbb{R}^*$ yields $u_1 \in \mathbb{R}$ and $u_2 \in i\mathbb{R}$. Hence the $(5, 5)$ equation of $L_+ P_0 + L_- P_2 = 0$ is equivalent to $u_2^2\overline{u_3}^2 = 0$. Since $u_2 = 0$ would imply $L_+^3 = 0$, we get $u_3 = 0$. Thus $L_+ P_2 = 0$ is equivalent to $V_0^2 = (10 + 12u_2^2)/3$. Plugging this into $L_+ P_0 + L_- P_2 = 0$ yields $u_2 = \pm\frac{i}{\sqrt{2}}$ and thus $V_0 = \frac{2}{\sqrt{3}}$. For both possible cases we obtain $-L_1$. Conjugating by $\mathrm{diag}(-1, 1, 1, -1, 1)$ yields the claim.

If $W > 0$, the equation $(e_+ \otimes U_0)V^2 = 0$ implies $u_1 = -i\,u_2$, where the $u_i$ are as above. Since $u_1 = -i\,u_2$ yields $L_+^3 = 0$, the case $W > 0$ cannot occur. $\square$

**Lemma 1.21.** *For $L_+^3 = 0$ and $L_+^2 \neq 0$ there exists an $A \in \mathrm{O}(5)$ such that $AL(t)A^{-1} = \cos(t)L_0 + \sin(t)L_1$ with $L_0 = diag(\sqrt{3}, \frac{1}{\sqrt{3}}, 0, -\frac{1}{\sqrt{3}}, -\sqrt{3})$ and*

$$L_1 = \frac{1}{\sqrt{6}}\begin{pmatrix} 0 & 0 & 0 & 0 & 3\sqrt{2} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 3\sqrt{2} & 0 & 0 & 0 & 0 \end{pmatrix}, L_1 = \frac{1}{\sqrt{6}}\begin{pmatrix} 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 \\ 3 & 0 & 0 & 0 & 3 \\ 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \end{pmatrix}, \quad \textit{or} \ L_1 = \frac{1}{\sqrt{3}}\begin{pmatrix} 0 & 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 & 0 \end{pmatrix}.$$

*In particular,* $\dim E = 2$ *for the first two cases and* $\dim E = 1$ *for the last one.*

*Proof.* Using Lemma 1.11 identity $(L_+^2)^2 = 0$ implies $L_+^2 = \begin{pmatrix} \sigma\otimes S_0 & 0 \\ 0 & 0 \end{pmatrix}$, where $S_0$ is a positive definite, diagonal matrix. Introduce the notation $L_+ = \begin{pmatrix} T & U \\ U^{tr} & W \end{pmatrix}$. From

$L_+^2\, L_+ = 0 = L_+\, L_+^2$ we have $T = \sigma \otimes T_0$ and $U = e_+ \otimes U_0$. Since $L_+\, P_2 = 0$ is equivalent to $T_0 = 0$, we get $L_+ = \left(\begin{smallmatrix} 0 & e_+\otimes U_0 \\ e_+^{tr}\otimes U_0^{tr} & W \end{smallmatrix}\right)$. Calculating $L_+^2$ and using Lemma 1.17 yields $W^2 = 0$, $(e_+ \otimes U_0)W = 0$, and $S_0 = U_0\, U_0^{tr}$.

First we suppose $S_0 \in \mathrm{Mat}(2, \mathbb{R})$ which implies $W \in \mathbb{C}$. Hence $W^2 = 0$ implies $W = 0$ and therefore $\mathrm{rk}\, L(t) \leq 2$ for all $t \in \mathbb{R}$, which is a contradiction.

Next let $S_0 \in \mathbb{R}$ and introduce the notation $U_0 = (u_1, u_2, u_3)$ with $u_i \in \mathbb{C}$. By Lemma 1.11 and $W^2 = 0$ we have $W = \left(\begin{smallmatrix} \sigma\otimes W_0 & \\ & 0 \end{smallmatrix}\right)$, where $W_0 \geq 0$.
First let $W_0 = 0$. There exists an $s_1 \in \mathbb{R}$ such that conjugating $L_\pm$ by $T_1 = \begin{pmatrix} \mathbb{1}_2 & & \\ & D(s_1) & \\ & & 1 \end{pmatrix}$, where $D(t) = \left(\begin{smallmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{smallmatrix}\right)$, transforms $U_0$ into the form $U_0 = (u_1, u_2, u_3)$ with $u_1 \in \mathbb{R}$ and $u_2, u_3 \in \mathbb{C}$. Similarly, there exists an $s_2 \in \mathbb{R}$ such that conjugating $T_1 L_\pm T_1^{-1}$ by $T_2 = \left(\begin{smallmatrix} \mathbb{1}_3 & \\ & D(s_2) \end{smallmatrix}\right)$ transforms $U_0$ into the form $U_0 = (u_1, u_2, u_3)$ with $u_1, u_2 \in \mathbb{R}$ and $u_3 \in \mathbb{C}$. Finally, there exists an $s_3 \in \mathbb{R}$ such that conjugating $T_2 T_1 L_\pm T_1^{-1} T_2^{-1}$ by $T_3 = \begin{pmatrix} \mathbb{1}_2 & & \\ & D(s_3) & \\ & & 1 \end{pmatrix}$ transforms $U_0$ into the form $U_0 = (0, u_2, u_3)$ with $u_2 \in \mathbb{R}$ and $u_3 \in \mathbb{C}$. Since $S_0 = U_0\, U_0^{tr}$ and $S_0 \in \mathbb{R}$, we thus get $u_3 \in \mathbb{R}$ or $u_3 \in i\mathbb{R}$. One proves easily that the eigenvalues of $L(t)$ are given by 0 and $\pm\sqrt{\sum_{i=1}^{3}(u_i \overline{u_i} \pm u_i^2)}$ and thus in the former case at least three eigenvalues vanish. Consequently, $u_3 \in i\mathbb{R}$. Hence $L_+ P_0 + L_- P_2 = 0$ implies $(u_2, u_3) = (\pm\frac{\sqrt{3}}{2}, \pm i \frac{1}{2\sqrt{3}})$. It is straightforward to verify that for each of these cases there exists an $A \in \mathrm{O}(5)$ such that $AL(t)A^{-1} = \cos(t) L_0 + \sin(t) L_1$ with $L_0 = \mathrm{diag}(\sqrt{3}, \frac{1}{\sqrt{3}}, 0, -\frac{1}{\sqrt{3}}, -\sqrt{3})$ and $L_1$ is given by

$$L_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 & 0 \end{pmatrix}.$$

Below we assume $W_0 > 0$. The identity $(e_+ \otimes U_0)W = 0$ yields $u_1 = -i\, u_2$. Thus the $(5,5)$ equation of $L_+\, P_0 + L_-\, P_2 = 0$ is given by $W_0\, \overline{u_2}^2\, u_3^2 = 0$. Since $u_3 = 0$ would imply $L_+^2 = 0$, we have $u_2 = 0$. Consequently, $S_0 = U_0 U_0^{tr}$ is equivalent to $S_0 = u_3^2$ and thus we have $u_3 \in \mathbb{R}^*$. Hence $L_+\, P_0 + L_-\, P_2 = 0$ yields $W_0 \in \left\{1/\sqrt{3}, \sqrt{3}\right\}$ and $u_3 \in \left\{\pm 1/\sqrt{6}, \pm\sqrt{3/2}\right\}$. From $\mathrm{spec}(L(t)) = \left\{0, \pm\sqrt{2} u_3, \pm W_0\right\}$ we thus get (i) $(W_0, u_3) = (1/\sqrt{3}, \sqrt{3/2})$, (ii) $(W_0, u_3) = (1/\sqrt{3}, -\sqrt{3/2})$, (iii) $(W_0, u_3) = (\sqrt{3}, 1/\sqrt{6})$, or (iv) $(W_0, u_3) = (\sqrt{3}, -1/\sqrt{6})$. We determine $L_0$ and $L_1$ for each of these cases and perform a change of the basis such that the basis consists of unit eigenvectors of $L_0$, more precisely $L_0 = \mathrm{diag}(\sqrt{3}, \frac{1}{\sqrt{3}}, 0, -\frac{1}{\sqrt{3}}, -\sqrt{3})$. For the cases (i)–(iv) we get

$$L_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{2} & 0 \\ 3 & 0 & 0 & 0 & 3 \\ 0 & -\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \end{pmatrix}, \qquad L_1 = -\frac{1}{\sqrt{6}} \begin{pmatrix} 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 \\ 3 & 0 & 0 & 0 & 3 \\ 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \end{pmatrix},$$

$$L_1 = -\frac{1}{\sqrt{6}} \begin{pmatrix} 0 & 0 & 0 & 0 & 3\sqrt{2} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 3\sqrt{2} & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad L_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 0 & 0 & 0 & 0 & -3\sqrt{2} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -3\sqrt{2} & 0 & 0 & 0 & 0 \end{pmatrix},$$

respectively. Conjugating the matrices of the first row by the matrices $\mathrm{diag}(1, -1, 1, 1, 1)$ and $\mathrm{diag}(-1, -1, 1, 1, -1)$, respectively, and the matrices of the second row by the matrices diag$(-1, -1, 1, -1, 1)$ and diag$(-1, 1, 1, 1, 1)$, respectively, the claim follows. ☐

**Lemma 1.22.** *If $L_+^2 = 0$ there exists an $A \in \mathrm{O}(5)$ such that $AL(t)A^{-1} = \cos(t)L_0 + \sin(t)L_1$ with $L_0 = diag(\sqrt{3}, \frac{1}{\sqrt{3}}, 0, -\frac{1}{\sqrt{3}}, -\sqrt{3})$ and*

$$L_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 0&0&0&0&3 \\ 0&0&0&1&0 \\ 0&0&0&0&0 \\ 0&1&0&0&0 \\ 3&0&0&0&0 \end{pmatrix}.$$

*In particular,* $\dim E = 1$.

*Proof.* From $L_+^2 = 0$ we have $L_+ = \left(\begin{smallmatrix} & \sigma\otimes S_0 \\ 0 & \end{smallmatrix}\right)$, where $S_0 \in \mathrm{Mat}(d, \mathbb{R})$ is a diagonal and positive definite matrix. Furthermore, $P_{\pm 4} = 0 = P_{\pm 2}$ and thus $P(t) = P_0$, which implies $P_0^2 = P_0$. Introduce the notation $P_0 = \left(\begin{smallmatrix} T & U \\ U^{tr} & V \end{smallmatrix}\right)$, where $T \in \mathrm{Mat}(2d, \mathbb{C})$. By the very definition of $P_0$ we get $U = 0$ and $V = \mathbb{1}$. The equations $L_+P_0 = 0 = P_0L_+$ imply $T = \sigma \otimes T_0$ for $T_0 \in \mathrm{Mat}(d, \mathbb{R})$. Therefore $P_0^2 = P_0$ yields $T_0 = 0$. Consequently, $S_0^4 - \frac{10}{3}S_0^2 + \mathbb{1}_d = 0$. If $d = 1$ we get $\mathrm{rk}\, P_0 = 3$ which contradicts our assumption. Hence $d = 2$. Thus we obtain $S_0 = \mathrm{diag}(\sqrt{3}, \frac{1}{\sqrt{3}})$ or $S_0 = \mathrm{diag}(\frac{1}{\sqrt{3}}, \sqrt{3})$. In the former case the claim follows by conjugation by diag$(-1, -1, 1, 1, 1)$, the remaining case is treated similarly. ☐

By combining the previous results we finally obtain Theorem 1.14.

## 2. Classification of isoparametric hypersurfaces with $(g, m) = (6, 1)$

After giving a very short exposition to isoparametric hypersurfaces in spheres in Subsection 2.1, we explain in Subsection 2.2 the significance of Theorem 1.14 in the context of isoparametric hypersurfaces in spheres. Finally, in Subsection 2.3 we show that all isoparametric hypersurfaces in $\mathbb{S}^7$ with $g = 6$ are homogeneous and thereby reprove a result of Dorfmeister and Neher [3].

2.1. **Isoparametric hypersurfaces in spheres.** Hypersurfaces in spheres with constant principal curvatures are called isoparametric. Münzner [8, 9] showed that the number of distinct principal curvatures $g$ can be only 1, 2, 3, 4, or 6, and gave restrictions for the multiplicities as well. The possible multiplicities of the curvature distributions were classified in [1, 9, 11], and coincide with the multiplicities in the known examples. So far the cases $g = 4$ and $g = 6$ are not yet completely classified. See, e.g., the paper [12] of Thorbergsson for a survey of isoparametric hypersurfaces in spheres.

For the case $g = 6$ all multiplicities coincide and are given either by $m = 1$ or $m = 2$. Furthermore, exactly two examples are known for this case, both of which are homogeneous. They are given as orbits of the isotropy representation of $\mathrm{G}_2/\mathrm{SO}(4)$ or as orbits in the unit sphere $\mathbb{S}^{13}$ of the Lie algebra $\mathfrak{g}_2$ of the adjoint representation of the Lie group $\mathrm{G}_2$ and have multiplicities $m = 1$ and $m = 2$, respectively. Dorfmeister and Neher [3] conjectured that all isoparametric hypersurfaces with $g = 6$ are homogeneous and proved this in the affirmative for the case $m = 1$. Since homogeneous isoparametric hypersurfaces in spheres were classified by Takagi and Takahashi [13], this provides a classification of isoparametric hypersurfaces with $(g, m) = (6, 1)$. The case $m = 2$ is not classified yet.

2.2. **Link of isoparametric hypersurfaces to Theorem 1.14.** Throughout this paper $M$ denotes a connected, smooth manifold of dimension $n$. An embedding $F_0 : M \hookrightarrow \mathbb{S}^{n+1}$ together with a distinguished unit normal vector field $\nu_0 \in \Gamma(\nu M)$ is called an *isoparametric hypersurface in* $\mathbb{S}^{n+1}$ if and only if its principal curvatures are constant. We denote by $A_0$ the shape operator of $F_0$ with respect to $\nu_0$ and by $\lambda_j^0$, $j \in \{1, \ldots, g\}$, the principal curvatures. We further assume without loss of generality $\lambda_1^0 > \cdots > \lambda_g^0$ and define $\theta_j \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $\lambda_j^0 = \cot(\theta_j)$. It is well known that the $j$-th curvature distribution $D_j$, which is given by $D_j(p) = \mathrm{Eig}(A_{0|p}, \lambda_j^0)$ for $p \in M$, is integrable and its leaves $\mathfrak{L}_j$ are small spheres in $\mathbb{S}^{n+1}$.

We consider the *parallel surface* $F_s : M \hookrightarrow \mathbb{S}^{n+1}$ defined via

$$p \mapsto F_s(p) := \exp_{F_0(p)}(s\nu_{0|p}) = \cos(s)F_0(p) + \sin(s)\nu_{0|p},$$

endowed with the orientation $\nu_s(p) = -\sin(s)F_0(p) + \cos(s)\nu_{0|_p}$. If $s \neq \theta_j$, the parallel surface $F_s(M)$ is again an isoparametric hypersurface with principal curvatures $\lambda_j^s = \cot(\theta_j - s)$. For $s_0 = \theta_j + \ell\pi$, $\ell \in \mathbb{Z}_2$, the map $F_{s_0}$ focalizes $\mathfrak{L}_j(p)$ to one point in the $(n - m_j)$-dimensional focal submanifold $M_{j,\ell} := F_{\theta_j + \ell\pi}(M)$.

Let $\ell \in \mathbb{Z}_2$ be given. Münzner [8] proved that the spectrum of the shape operator $\mathcal{A}_{\nu_{|\overline{p}}}$ of $M_{j,\ell}$ is independent of $\nu \in \nu M_{j,\ell}$ and $\overline{p} \in M_{j,\ell}$ and is given by

$$\mathrm{spec}(\mathcal{A}_{\nu_{|\overline{p}}}) = \left\{\cot\left((i-j)\pi/g\right) \mid i \in \{1, \ldots, g\}\,,\, i \neq j\right\}.$$

Thus for each $\overline{p} \in M_{j,\ell}$ and each pair of orthonormal vectors $v_0, v_1 \in \nu_{\overline{p}} M_{j,\ell}$, the family of shape operators $L(t) = \mathcal{A}_{\cos(t)v_0 + \sin(t)v_1} = \cos(t)\mathcal{A}_{v_0} + \sin(t)\mathcal{A}_{v_1}, t \in \mathbb{R}$, is isospectral. We introduce the shorthand notation $L = L(t)$, $L_0 = \mathcal{A}_{v_0}$, $L_1 = \mathcal{A}_{v_1}$. Consequently, if we restrict ourselves to the case $(g, m) = (6, 1)$ we get an isospectral family $L(t)$ with spectrum $\{-\sqrt{3}, -\frac{1}{\sqrt{3}}, 0, \frac{1}{\sqrt{3}}, \sqrt{3}\}$, where all eigenvalues have multiplicity 1 — these families are classified in Theorem 1.14.

2.3. **Proof of homogeneity.** Takagi and Takahashi [13] classified homogeneous isoparametric hypersurfaces in spheres and in particular showed that for $(g, m) = (6, 1)$ there is only one example, namely the isoparametric hypersurface is given by orbits of the isotropy representation of $\mathrm{G}_2/\mathrm{SO}(4)$. Before proving the main result of this section we consider the homogeneous example in more detail.

In [10] the symmetric, trilinear form $\alpha$ was introduced by

$$\alpha(\,\cdot\,,\,\cdot\,,\,\cdot\,) = g_0((\nabla^0_{\cdot} A_0)\,\cdot\,,\,\cdot\,),$$

where $g_0 = F_0^* \langle\cdot,\cdot\rangle_{\mathbb{S}^{n+1}}$ and $\nabla^0$ is the associated Levi-Civita connection. Furthermore, it was shown, using the computations in [7], that for the homogeneous example with $(g, m) = (6, 1)$ the components $\alpha_{i,j,k} := \alpha(e_i, e_j, e_k)$ are given by

$$\alpha_{1,2,3} = \alpha_{3,4,5} = \alpha_{1,5,6} = \sqrt{\tfrac{3}{2}},\ \alpha_{2,4,6} = -\sqrt{\tfrac{3}{2}},\ \alpha_{1,3,5} = -2\sqrt{\tfrac{3}{2}}, \tag{7}$$

and all other $\alpha_{i,j,k}$ with $i \leq j \leq k$ vanish, or by

$$\alpha_{4,5,6} = \alpha_{2,3,4} = \alpha_{1,2,6} = \sqrt{\tfrac{3}{2}},\ \alpha_{1,3,5} = -\sqrt{\tfrac{3}{2}},\ \alpha_{2,4,6} = -2\sqrt{\tfrac{3}{2}}, \tag{8}$$

and all other $\alpha_{i,j,k}$ with $i \leq j \leq k$ vanish. Note that (8) is obtained from (7) by flipping the orientation of the isoparametric hypersurface.

The strategy for proving that all isoparametric hypersurfaces in $\mathbb{S}^7$ with $g = 6$ are homogeneous is as follows: the main step consists in the proof of the fact that for these isoparametric hypersurfaces all $\alpha_{i,j,k}^2$ coincide with those of the homogeneous example, i.e., either with (7) or (8). Then the desired result follows by the following proposition of Abresch.

**Proposition 2.1** (see Proposition 12.5 in [1])**.** *Isoparametric hypersurfaces $M \subset \mathbb{S}^7$ with $g = 6$ are homogeneous if and only if all the functions $\alpha^2_{i,\,j,\,k}$ are constant on $M \subset \mathbb{S}^7$.*

**Theorem 2.2.** *Isoparametric hypersurfaces in $\mathbb{S}^7$ with $(g, m) = (6, 1)$ are homogeneous.*

*Proof.* Let an isoparametric hypersurface $M$ and $p \in M$ be given. Recall that for $s_0 = \theta_i + \ell\pi$, $\ell \in \mathbb{Z}_2$, the map $F_{s_0}$ focalizes $\mathfrak{L}_i(p)$ to one point $p_{i,\ell} \in M_{i,\ell}$. Let the isospectral family of focal shape operators at this point be denoted by $L^{i,\ell}(t) = \cos(t)L_0 + \sin(t)L_1^{i,\ell}$ and assume $L_0 = \operatorname{diag}(\sqrt{3}, \frac{1}{\sqrt{3}}, 0, -\frac{1}{\sqrt{3}}, -\sqrt{3})$. Then it is easy to prove that $L_1^{i,\ell}$ is given by

$$(-1)^\ell \begin{pmatrix} 0 & \sqrt{\frac{2}{3}}\,\alpha_{i,\,i+1,\,i+2} & \frac{1}{\sqrt{2}}\,\alpha_{i,\,i+1,\,i+3} & \sqrt{\frac{2}{3}}\,\alpha_{i,\,i+1,\,i+4} & \sqrt{2}\,\alpha_{i,\,i+1,\,i+5} \\ \sqrt{\frac{2}{3}}\,\alpha_{i,\,i+1,\,i+2} & 0 & \frac{1}{\sqrt{6}}\,\alpha_{i,\,i+2,\,i+3} & \frac{\sqrt{2}}{3}\,\alpha_{i,\,i+2,\,i+4} & \sqrt{\frac{2}{3}}\,\alpha_{i,\,i+2,\,i+5} \\ \frac{1}{\sqrt{2}}\,\alpha_{i,\,i+1,\,i+3} & \frac{1}{\sqrt{6}}\,\alpha_{i,\,i+2,\,i+3} & 0 & \frac{1}{\sqrt{6}}\,\alpha_{i,\,i+3,\,i+4} & \frac{1}{\sqrt{2}}\,\alpha_{i,\,i+3,\,i+5} \\ \sqrt{\frac{2}{3}}\,\alpha_{i,\,i+1,\,i+4} & \frac{\sqrt{2}}{3}\,\alpha_{i,\,i+2,\,i+4} & \frac{1}{\sqrt{6}}\,\alpha_{i,\,i+3,\,i+4} & 0 & \sqrt{\frac{2}{3}}\,\alpha_{i,\,i+4,\,i+5} \\ \sqrt{2}\,\alpha_{i,\,i+1,\,i+5} & \sqrt{\frac{2}{3}}\,\alpha_{i,\,i+2,\,i+5} & \frac{1}{\sqrt{2}}\,\alpha_{i,\,i+3,\,i+5} & \sqrt{\frac{2}{3}}\,\alpha_{i,\,i+4,\,i+5} & 0 \end{pmatrix},$$

where $\alpha_{k_1,\,k_2,\,k_3} = \alpha_{|p}(e_{k_1}, e_{k_2}, e_{k_3})$ and the indices are cyclic of order 6.

First we prove that the 'type' of $L_1^{i,\ell}(p)$ is constant for $p \in M$: $L_1^{i,\ell}(p)$ is of one of the forms given in Theorem 1.14. Since $L_1^{i,\ell}(p)$ depends continuously on $p \in M$ and $M$ is connected, the form of $L_1^{i,\ell}(p)$ is constant for all $p \in M$.

Next we show that $L_1^{i,\ell}(p)$ can only be of the fifth or sixth type listed in Theorem 1.14 and thus coincide with the homogenous example.

Let us first suppose that $L_1^{6,0}(p)$ is of the first form listed in Theorem 1.14. This implies $\alpha_{1,\,2,\,6} = \frac{5}{3\sqrt{2}}$ which in turn yields that the $(1,5)$-entry of $L_1^{1,0}(p)$ is given by $\frac{5}{3}$. However this coefficient does not arise in one of the possible $L_1$ listed in Theorem 1.14 and therefore this case cannot arise.

Next we suppose that $L_1^{6,0}(p)$ is of the second form listed in Theorem 1.14. This implies $\alpha_{2,\,3,\,6} = 1$ which in turn yields that the $(1,4)$-entry of $L_1^{2,0}(p)$ is given by $\sqrt{\frac{2}{3}}$. However this coefficient does not arise in one of the possible $L_1$ listed in Theorem 1.14 and therefore this case cannot arise.

Next we suppose that $L_1^{6,0}(p)$ is of the third form listed in Theorem 1.14. This implies $\alpha_{1,\,3,\,6} = \sqrt{3}$ which in turn yields that the $(2,5)$-entry of $L_1^{1,0}(p)$ is given by $\sqrt{2}$. However this coefficient does not arise in one of the possible $L_1$ listed in Theorem 1.14 and therefore this case cannot arise.

Next we suppose that $L_1^{6,0}(p)$ is of the fourth form listed in Theorem 1.14. This implies $\alpha_{1,\,2,\,6} = \alpha_{4,\,5,\,6} = 1/\sqrt{2}$ and $\alpha_{1,\,4,\,6} = \alpha_{2,\,5,\,6} = \sqrt{2}$ which in turn yields that the $(1,5)$-entry and the $(3,5)$-entry of $L_1^{1,0}(p)$ are given by 1. However this contradicts Theorem 1.14 and therefore this case cannot arise.

Finally, for the fifth and sixth case one proves easily that everything is consistent and that in these cases all $\alpha(e_i, e_j, e_k)^2$ with $i, j, k \in \{1, \ldots, 6\}$ coincide with those of (8) and (7), respectively.

Therefore for all isoparametric hypersurfaces in $\mathbb{S}^7$ with $(g, m) = (6, 1)$ all $\alpha(e_i, e_j, e_k)^2$ with $i, j, k \in \{1, \ldots, 6\}$ coincide with those of the homogeneous example and are in particular constant. Hence the claim follows from Proposition 2.1, i.e., Proposition 12.5 in [2]. □

## Appendix A. Counterexamples to the proof of Miyaoka [4,6]

We give counterexamples to some of Miyaoka's proofs in [4,6].

A.1. **Proposition 8.1 and Proposition 8.2 in** [6] **are not compatible.** In Paragraph 3 of [6] Miyaoka claims to prove by contradiction that the case $\dim E = 3$ does not occur. Although the statement is true, the proof is incorrect: we show that Proposition 8.1 and Proposition 8.2 are not compatible.

In Proposition 8.1 Miyaoka [4, 6] claims that $\{e_3(t), X_1(t), X_2(t)\}$ and $\{Z_1(t), Z_2(t)\}$ constitute orthonormal frames of $E$ and $E^\perp$, respectively, where

$$\begin{aligned} X_1(t) &= \alpha(t)(e_1(t) + e_5(t)) + \beta(t)(e_2(t) + e_4(t)), \\ X_2(t) &= \tfrac{1}{\sqrt{\sigma(t)}}\left(\tfrac{\beta(t)}{\sqrt{3}}(e_1(t) - e_5(t)) - \sqrt{3}\alpha(t)(e_2(t) - e_4(t)\right), \\ Z_1(t) &= \tfrac{1}{\sqrt{\sigma(t)}}\left(\sqrt{3}\alpha(t)(e_1(t) - e_5(t)) + \tfrac{\beta(t)}{\sqrt{3}}(e_2(t) - e_4(t)\right), \\ Z_2(t) &= \beta(t)(e_1(t) + e_5(t)) - \alpha(t)(e_2(t) + e_4(t)), \end{aligned}$$

and $\alpha, \beta, \sigma$ are differentiable real functions on the interval $[0, 3\pi]$ satisfying $\alpha^2 + \beta^2 = \frac{1}{2}$ and $\sigma = 2(3\alpha^2 + \frac{1}{3}\beta^2)$.

Below we assume that $L(t)$ is given as in Lemma 1.19 where we chose without loss of generality the $L_1$ with the $+$-sign. Consider the following unit eigenvectors of $L(t)$:

$$e_1(t) = (f_1(t), \tfrac{1}{6}(3\sin(t) + \sin(2t)), \tfrac{4}{9}\sin^2(t), \tfrac{1}{6}(3\sin(t) - \sin(2t)), f_1(t+\pi))^{tr},$$

where $f_1(t) = \frac{1}{9}\cos^2(\frac{t}{2})(7 + 2\cos(t))$, is a unit eigenvector of $L(t)$ with eigenvalue $\sqrt{3}$;

$$e_2(t) = (f_2(t), \cos^2(\tfrac{t}{2})(1 - 2\cos(t)), -\tfrac{2}{3}\sin(2t), -(1 + 2\cos(t))\sin^2(\tfrac{t}{2}), f_2(t+\pi))^{tr},$$

where $f_2(t) = \frac{1}{6}(3 + 2\cos(t))\sin(t)$, is a unit eigenvector of $L(t)$ with eigenvalue $\frac{1}{\sqrt{3}}$;

$$e_3(t) = (\tfrac{4}{9}\sin^2 t, -\tfrac{2}{3}\sin(2t), \tfrac{1}{9}(1 + 8\cos(2t)), \tfrac{2}{3}\sin(2t), \tfrac{4}{9}\sin^2 t)^{tr}$$

is an eigenvector of $L(t)$ with eigenvalue 0. Then $e_4(t) = \pm e_2(t+\pi)$ and $e_5(t) = \pm e_1(t+\pi)$ are eigenvectors of $L(t)$ with eigenvalues $-\frac{1}{\sqrt{3}}$ and $-\sqrt{3}$, respectively. Following Miyaoka [6] we assume $e_4(t) = e_2(t+\pi)$ and $e_5(t) = e_1(t+\pi)$. Thus we get

$$E = \operatorname{span}((0, 0, 1, 0, 0)^{tr}, (1, 0, 0, 0, 1)^{tr}, (0, -1, 0, 1, 0)^{tr}).$$

Therefore $e_1(t) + e_5(t), e_2(t) + e_4(t) \in E$ but $e_1(t) - e_5(t), e_2(t) - e_4(t) \in E^\perp$. Consequently, the element $X_2(t)$ does not lie in $E$, contradicting Proposition 8.1 in [4].

One may try to avoid this problem by another choice of the eigenvectors $e_4(t)$ and $e_5(t)$. Note that for any admissible choice of $e_4(t)$ and $e_5(t)$ we have: if $\alpha(t) \neq 0$ and $\beta(t) \neq 0$ at least one of the vectors $X_1(t)$ or $X_2(t)$ does not lie in $E$. Thus either $\alpha \equiv 0$ or $\beta \equiv 0$ and we may assume without loss of generality that $\alpha \equiv 0$. In order for $X_1(t), X_2(t)$ to lie in $E$ we must have $e_4(t)=-e_2(t+\pi)$ and $e_5(t)=e_1(t+\pi)$, which implies $e_4(0)=-e_2(\pi)$ and $e_4(\pi)=-e_2(0)$. However this implies that the proof of Proposition 8.2 [4, 6] does not work anymore. Indeed, we no longer obtain a proof by contradiction: just follow along the lines of this proof and use $e_1(\pi)=e_5(0)$, $e_2(\pi)=-e_4(0)$, $e_3(\pi)=e_3(0)$, $e_4(\pi)=-e_2(0)$, and $e_5(\pi)=e_1(0)$.

Conclusion: the contradiction obtained in [4] and [6] results from the inadmissible assumption that Proposition 8.1 in [4] and $e_4(t)=e_2(t+\pi)$, $e_5(t)=e_1(t+\pi)$ hold. If we change the sign of exactly one of the eigenvectors $e_4(t)$ or $e_5(t)$, Proposition 8.1 is true but then the proof of Proposition 8.2 becomes incorrect.

A.2. **Counterexample to the proof of Proposition 7.1 in** [6]**.** In [6] Proposition 7.1 is used to exclude the case $\dim E=2$.

Below we suppose that $L(t)$ is given as in Lemma 1.21, where we assume that $L_1$ is of the first form stated in this lemma — the argument is similar for the case when $L_1$ is of the second form in that lemma. Then

$$\begin{aligned}
e_1(t) &= (\cos(t/2), 0, 0, 0, \sin(t/2))^{tr},\\
e_2(t) &= (0, \cos^2(t/2), \sin(t)/\sqrt{2}, \sin^2(t/2), 0)^{tr},\\
e_3(t) &= (0, -\sin(t)/\sqrt{2}, \cos(t), \sin(t)/\sqrt{2}, 0)^{tr},\\
e_4(t) &= (0, \sin^2(t/2), -\sin(t)/\sqrt{2}, \cos^2(t/2), 0)^{tr},\\
e_5(t) &= (-\sin(t/2), 0, 0, 0, \cos(t/2))^{tr}
\end{aligned}$$

constitutes an orthonormal basis of eigenvectors of $L(t)$ where the corresponding eigenvalues are given by $\sqrt{3}, \frac{1}{\sqrt{3}}, 0, -\frac{1}{\sqrt{3}}$, and $-\sqrt{3}$, respectively. Hence $e_3(\pi)=-e_3(0)$, $e_2(\pi)=e_4(0)$, $e_4(\pi)=e_2(0)$, $e_1(\pi)=e_5(0)$, and $e_5(\pi)=-e_1(0)$. This example proves that not only the four cases listed in [4, 6], namely $(e_1+e_5)(\pi)=(e_1+e_5)(0)$ and $(e_2+e_4)(\pi)=\pm(e_2+e_4)(0)$ or $(e_1+e_5)(\pi)=-(e_1+e_5)(0)$ and $(e_2+e_4)(\pi)=\pm(e_2+e_4)(0)$ occur. The missing cases cannot be excluded by the argument given in [4, 6].

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