PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 144, Number 5, May 2016, Pages 2109–2120
http://dx.doi.org/10.1090/proc/12963
Article electronically published on November 20, 2015

THE CONSTRUCTION OF A COMPLETELY SCRAMBLED SYSTEM BY GRAPH COVERS

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(Communicated by Nimish A. Shah)

Abstract. In this paper, we define a new construction of completely scrambled 0-dimensional systems using the inverse limit of sequences of directed graph covers. These examples are transitive and are not locally equicontinuous. Moreover, any point that is not the unique fixed point is not a point of local equicontinuity.

1. Introduction

Let X be a compact metrizable space, and $f : X \to X$ be a continuous surjective map. In this paper, we call (X, f) a topological dynamical system, and consider the case in which X is 0-dimensional. In this case, we call (X, f) a 0-dimensional system. If X is homeomorphic to the Cantor set, (X, f) is called a Cantor system. Akin, Glasner and Weiss [1] made use of a special sequence of directed graph covers to construct a special homeomorphism that has the generic conjugacy class in the space of all Cantor systems, while Gambaudo and Martens [4] employed special sequences of directed graph covers to study ergodic measures of Cantor minimal systems. In [10], we generalized this latter construction to arbitrary 0-dimensional systems. In this paper, we use a sequence of graph covers to construct examples that are transitive, completely scrambled, and not locally equicontinuous. A subset $S \subseteq X$ is called a *scrambled set* if, for every $x \neq y \in S$,

$$\limsup_{n\to+\infty} d(f^n(x), f^n(y)) > 0$$

and

$$\liminf_{n\to+\infty} d(f^n(x), f^n(y)) = 0.$$

Since Li and Yorke developed the notion of scrambled sets in the study of chaotic systems [8], there has been some discussion as to how large such sets can be. In 1997, Mai reported a non-compact example that is completely scrambled [9], i.e., the scrambled set is the whole space, and conjectured that there was no compact example. Huang and Ye [7] later disputed this conjecture. They constructed a compact, 0-dimensional completely scrambled system. By taking the product of the identity map with any other compact set, and collapsing some subspaces to a point, their example indicated the existence of others on a variety of spaces. In the same paper, they also announced the existence of a transitive example. In 2000, Glasner and Weiss [5] introduced the notion of *local equicontinuity*. Let (X, f) be

Received by the editors May 10, 2014 and, in revised form, June 8, 2015.
2010 *Mathematics Subject Classification.* Primary 37B05, 54H20.
Key words and phrases. Completely scrambled, graph covers, 0-dimensional.

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a homeomorphism on a compact metric space. This is said to be *locally equicontinuous* if every $x \in X$ is an equicontinuity point on the orbit closure of x itself. Blanchard and Huang [2] announced the existence of a number of examples that are transitive, locally equicontinuous, and completely scrambled. In this paper, we construct another set of examples that are 0-dimensional, transitive, and completely scrambled, but not locally equicontinuous. Moreover, every point that is not the unique fixed point is not a point of local equicontinuity. We shall make use of the inverse limit of sequences of graph covers.

2. Preliminaries

In this section, we repeat the construction of graph covers for 0-dimensional systems originally given in Section 3 of [10]. We also describe some notation for later use. A pair $G = (V, E)$ consisting of a finite set V and a relation $E \subseteq V \times V$ on V can be considered as a directed graph with vertices V and an edge from u to v when $(u, v) \in E$. We assume that G is edge surjective, i.e., for every vertex $v \in V$ there exist edges $(u_1, v), (v, u_2) \in E$. Let $G_i = (V_i, E_i)$ with $i = 1, 2$ be directed graphs. A map $\varphi : V_1 \to V_2$ is said to be a graph homomorphism if every edge is mapped to an edge; we describe this as $\varphi : G_1 \to G_2$. Suppose that a graph homomorphism $\varphi : G_1 \to G_2$ satisfies the following condition:

$$(u, v), (u, v') \in E_1 \text{ implies that } \varphi(v) = \varphi(v').$$

In this case, φ is said to be *+directional*. Suppose that a graph homomorphism φ satisfies both of the following conditions:

$$\begin{gathered}(u, v), (u, v') \in E_1 \text{ implies that } \varphi(v) = \varphi(v') \text{ and} \\ (u, v), (u', v) \in E_1 \text{ implies that } \varphi(u) = \varphi(u').\end{gathered}$$

Then, φ is said to be *bidirectional*.

Definition 2.1. A graph homomorphism $\varphi : G_1 \to G_2$ is called a *cover* if it is a +directional edge-surjective graph homomorphism.

Let $\mathcal{G}$ be a sequence $G_1 \xleftarrow{\varphi_1} G_2 \xleftarrow{\varphi_2} \cdots$ of graph homomorphisms.

Notation 2.2. For $m > n$, let $\varphi_{m,n} := \varphi_n \circ \varphi_{n+1} \circ \cdots \circ \varphi_{m-1}$.

Then, $\varphi_{m,n}$ is a graph homomorphism. If all φ_i $(i \in \mathbb{N}^+)$ are edge surjective, then every $\varphi_{m,n}$ is edge surjective. Similarly, if all φ_i $(i \in \mathbb{N}^+)$ are covers, every $\varphi_{m,n}$ is a cover. Let us write $G_i = (V_i, E_i)$ for $i \in \mathbb{N}$. Define

$$\begin{gathered}V_{\mathcal{G}} := \{\, (x_0, x_1, x_2, \dots) \in \prod_{i=0}^{\infty} V_i \mid x_i = \varphi_i(x_{i+1}) \text{ for all } i \in \mathbb{N} \,\} \text{ and} \\ E_{\mathcal{G}} := \{\, (x, y) \in V_{\mathcal{G}} \times V_{\mathcal{G}} \mid (x_i, y_i) \in E_i \text{ for all } i \in \mathbb{N} \,\},\end{gathered}$$

each equipped with the product topology.

Notation 2.3. For each $n \in \mathbb{N}$, the projection from $V_{\mathcal{G}}$ to V_n is denoted by $\varphi_{\infty,n}$.

Notation 2.4. Let X be a compact metrizable 0-dimensional space. A finite partition of X by non-empty clopen sets is called a *decomposition*. The set of all decompositions of X is denoted by $\mathscr{D}(X)$. Each $\mathcal{U} \in \mathscr{D}(X)$ is endowed with the discrete topology.

Notation 2.5. Let $f : X \to X$ be a continuous surjective mapping from a compact metrizable 0-dimensional space X onto itself. Let $\mathcal{U}$ be a decomposition of X. Then, a map $\kappa_{\mathcal{U}} : X \to \mathcal{U}$ is defined as $\kappa_{\mathcal{U}}(x) = U \in \mathcal{U}$ if $x \in U \in \mathcal{U}$. A surjective relation $f^{\mathcal{U}}$ on $\mathcal{U}$ is defined as

$$f^{\mathcal{U}} := \{ (u, v) \mid f(u) \cap v \neq \emptyset \}.$$

In general, $(\mathcal{U}, f^{\mathcal{U}})$ is a graph, because f is a surjective relation.

We can state the following:

Lemma 2.6. *Let $\mathcal{G}$ be a sequence $G_0 \xleftarrow{\varphi_0} G_1 \xleftarrow{\varphi_1} G_2 \xleftarrow{\varphi_2} \cdots$ of covers. Then, $V_{\mathcal{G}}$ is a compact metrizable 0-dimensional space, and the relation $E_{\mathcal{G}}$ determines a continuous mapping from $V_{\mathcal{G}}$ onto itself. In addition, if the sequence is bidirectional, then the relation $E_{\mathcal{G}}$ determines a homeomorphism.*

Proof. See Lemma 3.5 of [10]. □

Let $\mathcal{G}$ be a sequence $G_0 \xleftarrow{\varphi_0} G_1 \xleftarrow{\varphi_1} G_2 \xleftarrow{\varphi_2} \cdots$ of covers. Then, by the above lemma, $E_{\mathcal{G}}$ defines a continuous surjective mapping from $V_{\mathcal{G}}$ onto itself. The 0-dimensional system $(V_{\mathcal{G}}, E_{\mathcal{G}})$ is called the inverse limit of $\mathcal{G}$, and is denoted by G_∞. We write $G_\infty = (V_{\mathcal{G}}, E_{\mathcal{G}}) = (X, f)$.

Notation 2.7. Let $\mathcal{G}$ be a sequence $G_0 \xleftarrow{\varphi_0} G_1 \xleftarrow{\varphi_1} G_2 \xleftarrow{\varphi_2} \cdots$ of covers. Let $G_i = (V_i, E_i)$ for $i \in \mathbb{N}$. For each $i \in \mathbb{N}$, we define

$$\mathcal{U}_i := \{ \varphi_{\infty,i}^{-1}(u) \mid u \in V_i \},$$

which we can identify with V_i itself.

From [10], we have the following:

Theorem 2.8. *A topological dynamical system is 0-dimensional if and only if it is topologically conjugate to G_∞ for some sequence of covers $G_0 \xleftarrow{\varphi_0} G_1 \xleftarrow{\varphi_1} G_2 \xleftarrow{\varphi_2} \cdots$. In addition, if all of the covers are bidirectional, then the resulting 0-dimensional system is a homeomorphism.*

We now give some notation that will be used later in the paper.

(N-1) We write $G_\infty = (X, f)$,
(N-2) we fix a metric d on X,
(N-3) for each $i \in \mathbb{N}$, we write $G_i = (V_i, E_i)$,
(N-4) for each $i \in \mathbb{N}$, we define $U(v) := \varphi_{\infty,i}^{-1}(v)$ for $v \in V_i$ and $\mathcal{U}_i := \{ U(v) \mid v \in V_i \} \in \mathscr{D}(X)$, and
(N-5) for each $i \in \mathbb{N}$, there exists a bijective map $V_i \ni v \leftrightarrow U(v) \in \mathcal{U}_i$. By this bijection, we obtain a graph isomorphism $G_i \cong (\mathcal{U}_i, f^{\mathcal{U}_i})$.

Let $G = (V, E)$ be a surjective directed graph. A sequence of vertices $w = (v_0, v_1, \dots, v_l)$ of G is said to be a *walk* of length l if $(v_i, v_{i+1}) \in E$ for all $0 \leq i < l$. We denote $l(w) := l$ and $V(w) := \{ v_0, v_1, \dots, v_l \}$. We say that a walk $w = (v_0, v_1, \dots, v_l)$ is a *path* if v_i $(0 \leq i \leq l)$ are mutually distinct. A walk $c = (v_0, v_1, \dots, v_l)$ is said to be a *cycle* of period l if $v_0 = v_l$, and a cycle $c = (v_0, v_1, \dots, v_l)$ is a *circuit* of period l if the v_i $(0 \leq i < l)$ are mutually distinct. Let $w_1 = (u_0, u_1, \dots, u_l)$ and $w_2 = (v_0, v_1, \dots, v_{l'})$ be walks such that $u_l = v_0$. Then, we denote $w_1 + w_2 := (u_0, u_1, \dots, u_l, v_1, v_2, \dots, v_{l'})$. Note that $l(w_1 + w_2) = l + l'$.

3. Construction

Let G_0 be a singleton graph with only one vertex $v_{0,0}$ and one edge $(v_{0,0}, v_{0,0})$. We shall construct a sequence of graph covers $G_0 \xleftarrow{\varphi_0} G_1 \xleftarrow{\varphi_1} G_2 \xleftarrow{\varphi_2} \cdots$ such that every G_n $(n \geq 1)$ is a generalized figure-8 with a special vertex $v_{n,0} \in V_n$ and a special edge $e_{n,0} = (v_{n,0}, v_{n,0}) \in E_n$ for each $n \in \mathbb{N}$. We assume that $\varphi_n(v_{n+1,0}) = v_{n,0}$ for each $n \in \mathbb{N}$. Thus, $\varphi_n(e_{n+1,0}) = e_{n,0}$ for each $n \in \mathbb{N}$. With this setting, we construct a class of examples. We assume that, for each $n \geq 1$, G_n consists of circuits $\{ c_{n,1}, c_{n,2}, \ldots, c_{n,n}, e_{n,0} \}$ such that for each $1 \leq i < j \leq n$, $V(c_{n,i}) \cap V(c_{n,j}) = \{ v_{n,0} \}$. We write $l(n,l) := l(c_{n,l})$ and $c_{n,l} = (v_{n,l,0} = v_{n,0}, v_{n,l,1}, v_{n,l,2}, \ldots, v_{n,l,l(n,l)} = v_{n,0})$. Let us construct a cover. We assume that $\varphi_n(V(c_{n+1,n+1})) = v_{n,0}$ for each $n \in \mathbb{N}$, and that, for each $1 \leq i \leq n$, $\varphi_n(c_{n+1,i}) = e_{n,0} + 2c_{n,i} + 2c_{n,i+1} + \cdots + 2c_{n,n} + e_{n,0}$. These are bidirectional covers, and the resulting continuous surjection (X, f) is a homeomorphism. The length of each $c_{n+1,i}$ with $1 \leq i \leq n$ is determined by the length of each $c_{n,j}$ with $1 \leq j \leq n$, and we can take $l(c_{n+1,n+1}) > 1$ arbitrarily. If we include $l(c_{n+1,n+1}) = 1$, we cannot distinguish $c_{n+1,n+1}$ with $e_{n+1,0}$. Therefore, we avoid this case. As stated in the previous section, G_∞ is written as (X, f), and we now use the notation described earlier. It is clear that X has no isolated points. Thus, (X, f) is a Cantor system.

Theorem 3.1. *The Cantor system (X, f) is completely scrambled, topologically transitive, and is not locally equicontinuous.*

Notation 3.2. We denote $p := (v_{1,0}, v_{2,0}, v_{3,0}, \ldots) \in X$, and every $e_{n,0}$ $(n \in \mathbb{N})$ is simply described as e if there is no possibility of confusion.

It is obvious that p is a fixed point. From the construction of the covers, the next lemma follows easily:

Lemma 3.3. *For $m > n$ and $1 \leq l \leq l' \leq n$, it follows that $l(m,l) > l(n,l')$.*

Lemma 3.4. *We have that $p \in X$ is the only fixed point, and this is the only periodic point.*

Proof. The first statement is obvious. Suppose that there exists a periodic point other than p. Then, there exists some $N > 0$ such that, for all $n \geq N$, there exists a circuit c_n of G_n of the same period, and the φ_n are isomorphisms of these circuits. This contradicts the construction of this graph cover. □

In this section, we show that (X, f) is completely scrambled by stating successive lemmas. Broadly, we show, in the following order, that

$\liminf_{i \to +\infty} d(f^k(x), f^k(p)) = 0$ for $x \neq p$,
$\limsup_{i \to +\infty} d(f^k(x), f^k(p)) > 0$ for $x \neq p$,
$\limsup_{i \to +\infty} d(f^k(x), f^k(y)) = 0$ for $x \neq y \in X \setminus \{ p \}$, and
$\limsup_{i \to +\infty} d(f^k(x), f^k(y)) > 0$ for $x \neq y \in X \setminus \{ p \}$ in separate cases.

Lemma 3.5. *For each $x \in X$, it follows that* $\liminf_{k \to +\infty} d(f^k(x), p) = 0$.

Proof. For any $n \in \mathbb{N}$, it follows that the sequence $\varphi_{\infty,n}(f^k(x))$ $(k > 0)$ follows a walk of G_n. It is clear that this walk passes $v_{n,0}$ infinitely many times. Thus, the conclusion is evident. □

Notation 3.6. For $v_{n,i,j} \in V_n$ with $0 < j < l(n,i)$, we denote $\operatorname{remn}(v) := l(n,i) - j$. For an $x \in U(v) \subset X$, $\operatorname{remn}(v)$ steps remain until we reach $U(v_{n,0})$, i.e., $f^i(x) \notin U(v_{n,0})$ for $0 \le i < \operatorname{remn}(v)$ and $f^{\operatorname{remn}(v)}(x) \in U(v_{n,0})$.

Notation 3.7. For $v \in V_n$, we denote the *degree* of v as follows:

$$\deg(v) = \begin{cases} +\infty, & \text{if } v = v_{n,0}, \\ i, & \text{if } v \in V(c_{n,i}) \backslash \{ v_{n,0} \}. \end{cases}$$

Lemma 3.8. *Let $x = (v_0, v_1, v_2, \dots) \in X$. For $n < n'$, it follows that* $\deg v_n \ge \deg v_{n'} \ge 1$.

Proof. By the construction of φ_n ($n \in \mathbb{N}$), the proof is evident. □

Notation 3.9. Let $x = (v_0, v_1, v_2, \dots) \in X$. Then, we define the *degree* of x as $\deg x := \min\{ \deg v_i \mid i \in \mathbb{N}^+ \}$. Note that $\deg x = +\infty$ implies that $x = p$.

Lemma 3.10. *For $p \ne x = (u_0, u_1, u_2, \dots) \in X$, there exists an $N > 0$ such that* $\deg u_N = \deg u_{N+1} = \cdots = \deg x$.

Proof. The proof is obvious. □

The next lemma is not explicitly used in this paper, but we believe it helps to clarify our argument.

Lemma 3.11. *The degree of each orbit is constant.*

Proof. Let $x = (v_0, v_1, v_2, \dots) \in X$. We must show that $\deg f(x) = \deg x$. If $x = p$, then the conclusion is obvious. Let $x \ne p$ and $\deg(x) = l$. Then, there exists $n \in \mathbb{N}$ such that $v_n \ne v_{n,0}$. By Lemma 3.10, we can assume that $\deg v_n = \deg v_{n+1} = \cdots = l$. Then, for $k \ge 0$, we get $v_{n+k} \in c_{n+k,l}$. Thus, for $k \ge 1$, it follows that $\operatorname{remn}(v_{n+k}) > 2l(c_{n+k-1,l+1}) + 2l(c_{n+k-1,l+2}) + \cdots + 2l(c_{n+k-1,n+k-1}) + 1$. Thus, $\operatorname{remn}(v_{n+k}) \to +\infty$ as $k \to +\infty$. Let (v_{n+k}, v'_{n+k}) be an edge of the circuit $c_{n+k,l}$. For $k \ge 1$, it follows that $v'_{n+k} \ne v_{n+k,0}$. Because $f(x) = (\dots, v'_{n+k}, v'_{n+k+1}, v'_{n+k+2}, \dots)$, it is clear that $\deg f(x) = l$. □

Lemma 3.12. *For $x \ne p$, it follows that* $\limsup_{k \to +\infty} d(f^k(x), p) > 0$.

Proof. Let $\deg x = l$ and $x = (u_0, u_1, u_2, \dots)$. Because $p \ne x$, we have that $l < +\infty$. By Lemma 3.10, there exists an $N \in \mathbb{N}$ such that $\deg u_n = l$ for all $n \ge N$. Let $n \ge N$. Then, $u_n \in c_{n,l}$. Let p_n be the path of $c_{n,l}$ from u_n to $v_{n,0}$. By the definition of N, it follows that $\varphi_{n,N}(u_n) \in V(c_{N,l})$. For $0 \le i \le \operatorname{remn}(u_n)$, $\varphi_{\infty,n}(f^i(x))$ follows the path from u_n to $v_{n,0}$. We get $\varphi_{n-1}(c_{n,l}) = e + 2c_{n-1,l} + 2c_{n-1,l+1} + \cdots + 2c_{n-1,n-1} + e$. Thus, p_n follows the totality of $2c_{n-1,l+1}$, and $\varphi_{n-1,N}(2c_{n-1,l+1})$ winds around $c_{N,l+1}$ exactly 2^n times. Therefore, $\varphi_{n,N}(p_n)$ winds around $c_{N,l+1}$ at least 2^n times. Fixing $\tau \in c_{N,l+1}$ such that $\tau \ne v_{N,0}$, $\varphi_{\infty,N}(f^i(x)) = \tau$ at least 2^n times. Because $n > 0$ is arbitrary, the conclusion is now obvious. □

Lemma 3.13. *Let $x \ne y$ be distinct from p. Then, it follows that*

$$\liminf_{k \to +\infty} d(f^k(x), f^k(y)) = 0.$$

Proof. Let $x \ne y \in X$ be distinct from p, $x = (u_0, u_1, u_2, \dots)$, and $y = (v_0, v_1, v_2, \dots)$. It follows that $\deg x, \deg y < +\infty$. Let $l = \deg x$ and $l' = \deg y$. By Lemma 3.10, there exists $N > 0$ such that $\deg u_N = \deg u_{N+1} = \cdots = l$ and

$\deg v_N = \deg v_{N+1} = \cdots = l'$. Note that $l, l' \le N$. For all $n > N$, $u_n \in c_{n,l}$ and $v_n \in c_{n,l'}$. Because

$$\begin{aligned}\varphi_{n-1}(c_{n,l}) &= e_{n-1,0} + 2c_{n-1,l} + 2c_{n-1,l+1} + 2c_{n-1,l+2} \\ &\quad + \cdots + 2c_{n-1,n-1} + e_{n-1,0},\end{aligned}$$

it follows that

$$\begin{aligned}\varphi_{n,N}(c_{n,l}) &= e_{N,0} + 2\varphi_{n-1,N}(c_{n-1,l}) + 2\varphi_{n-1,N}(c_{n-1,l+1}) + 2\varphi_{n-1,N}(c_{n-1,l+2}) \\ &\quad + \cdots + 2\varphi_{n-1,N}(c_{n-1,N}) \\ &\quad + 2\varphi_{n-1,N}(c_{n-1,N+1}) + \cdots + 2\varphi_{n,N}(c_{n-1,n-1}) + e_{N,0}.\end{aligned}$$

We write the first two lines as

$$\begin{aligned}p_n &= e_{N,0} + 2\varphi_{n-1,N}(c_{n-1,l}) + 2\varphi_{n-1,N}(c_{n-1,l+1}) + 2\varphi_{n-1,N}(c_{n-1,l+2}) \\ &\quad + \cdots + 2\varphi_{n-1,N}(c_{n-1,N})\end{aligned}$$

and the last line as $q_n = 2\varphi_{n-1,N}(c_{n-1,N+1}) + \cdots + 2\varphi_{n,N}(c_{n-1,n-1}) + e_{N,0}$. Note that we can write $q_n = L(N,n)e_{N,0}$ for the positive integer $L(N,n)$. It is clear that $L(N,n) \to +\infty$ as $n \to +\infty$. Because $\varphi_{\infty,N}(x) \in V(c_{N,l}) \setminus \{ v_{N,0} \}$, the sequence $\varphi_{\infty,N}(f^i(x))$ $(i \ge 0)$ starts from within $2\varphi_{n-1,N}(c_{n-1,l})$. Therefore, this sequence lies in the p_n for small i, and enters into q_n for larger i, eventually reaching the end of q_n. Note that p_n is a repetition of e_0 and $c_{N,j}$ with $l \le j \le N$. Let $M(N) := \max_{1 \le j \le N} l(c_{N,j}) < +\infty$. Take n to be sufficiently large such that $M(N) < L(N,n)$. The same situation occurs for y, and at least one of $\varphi_{\infty,N}(f^i(x))$ or $\varphi_{\infty,N}(f^i(y))$ enters into q_n; in the period that is less than or equal to $M(N)$, the other takes the value $v_{N,0}$. Let $L(n) := \min\{ \operatorname{remn}(\varphi_{\infty,n}(x)), \operatorname{remn}(\varphi_{\infty,n}(y) \}$. Then, there exists an i such that both $L(n) - M(N) \le i \le L(n)$ and $\{ f^i(x), f^i(y) \} \subset U(v_{N,0})$ are satisfied. Because $L(n) \to +\infty$ as $n \to +\infty$, we get $\liminf_{i \to +\infty} d(f^i(x), f^i(y)) \le \operatorname{diam} U(v_{N,0})$. Because we can take N to be arbitrarily large, we have $\liminf_{i \to +\infty} d(f^i(x), f^i(y)) = 0$. □

Notation 3.14. Let $x \ne y \in X$, $x = (u_0, u_1, u_2, \dots)$, and $y = (v_0, v_1, v_2, \dots)$. Let $n \in \mathbb{N}$. Suppose that there exists some $1 \le i \le n$ such that $u_n, v_n \in V(c_{n,i})$, $u_n = v_{n,i,j}$, and $v_n = v_{n,i,j'}$. Then, we denote $\operatorname{gap}(u_n, v_n) := j' - j$.

Lemma 3.15. *Let $x \ne y \in X$, $x = (u_0, u_1, u_2, \dots)$, and $y = (v_0, v_1, v_2, \dots)$. Suppose that $\deg x = \deg y < +\infty$ and $\limsup_{n \to +\infty} |\operatorname{gap}(u_n, v_n)| < +\infty$. Then, there exists a $d \in \mathbb{Z}$ such that $f^d(x) = y$.*

Proof. Because there exists an integer d and a subsequence n_k $(k \in \mathbb{N}^+)$ such that $\operatorname{gap}(u_{n_k}, v_{n_k}) = d$ for all k, the conclusion is obvious. □

Lemma 3.16. *Let $x \ne y \in X$ be such that $f^d(x) = y$ for some $d \ne 0$. Then, it follows that*

$$\limsup_{k \to +\infty} d(f^k(x), f^k(y)) > 0.$$

Proof. We prove the statement by contradiction. Assume that

$$\limsup_{k \to +\infty} d(f^k(x), f^k(y)) = 0.$$

By Lemma 3.12, $\limsup_{k \to +\infty} d(p, f^k(x)) > 0$. Thus, there exists a point $z \ne p$ and a subsequence k_i $(i \in \mathbb{N}^+)$ such that $f^{k_i}(x) \to z$ as $i \to +\infty$. By the assumption, we have $f^{k_i}(y) \to z$ as $i \to +\infty$. Thus, we get $f^d(z) = z$. Because $z \ne p$, this contradicts Lemma 3.4. □

Notation 3.17. Let $m > d \geq 1$. For $n > m$, we denote

$$r_{n,d,m} := 2\varphi_{n,m}(c_{n,d}) + 2\varphi_{n,m}(c_{n,d+1}) + \cdots + 2\varphi_{n,m}(c_{n,n}) + e_{m,0}.$$

Note that in $r_{n,d,m}$, there is no occurrence of $c_{m,d'}$ with $d' < d$.

Lemma 3.18. *Let $n \gg N \gg l \geq 1$. In $\varphi_{n+2,N}(c_{n+2,l})$, let $c_{N,l} + s + c_{N,l}$ be two occurrences of $c_{N,l}$ with no occurrence of $c_{N,l}$ in s. Then, $l(s) = m - N + 1 + \sum_{k=N}^{m} l(r_{k,l+1,N})$ for some m with $N \leq m \leq n$. Besides such appearances, we have those of the form $c_{N,l} + c_{N,l}$. Further, when we write $\varphi_{n+1,N}(c_{n+1,l}) = \cdots + c_{N,l} + s$ with no occurrence of $c_{N,l}$ in walk s, it follows that $l(s) = \sum_{k=N}^{n} r_{k,l+1,N}$.*

Proof. We abbreviate $e_{m,0}$ as e for all $m \in \mathbb{N}$. It follows that

$$\varphi_{n+1}(c_{n+2,l}) = e + 2c_{n+1,l} + 2c_{n+1,l+1} + \cdots + 2c_{n+1,n+1} + e.$$

Thus, we obtain the following:

$$\varphi_{n+1}(c_{n+2,l}) = e + 2c_{n+1,l} + r_{n+1,l+1,n+1}.$$

Therefore, it is sufficient to consider the gap between occurrences of $c_{N,l}$ in $2c_{n+1,l}$. We shall calculate the largest gap in $2c_{n+1,l} = c_{n+1,l} + c_{n+1,l}$, and show that it is between the last occurrence of $c_{N,l}$ in the first $c_{n+1,l}$ and the first occurrence of it in the last $c_{n+1,l}$. We calculate

$$\varphi_{n+2,n}(c_{n+2,l}) = e + 2(e + 2c_{n,l} + r_{n,l+1,n}) + r_{n+1,l+1,n}.$$

In the last expression, both $2c_{n,l}$ and $c_{n,l} + r_{n,l+1,n} + e + c_{n,l}$ occur. Thus, the largest gap is in $c_{n,l} + r_{n,l+1,n} + e + c_{n,l}$, between the last $c_{N,l}$ in $c_{n,l}$ and the first one in $c_{n,l}$, as stated above. This gives

$$\begin{aligned}\varphi_{n-1}(c_{n,l} + r_{n,l+1,n} + e + c_{n,l}) \quad = \quad & e + 2c_{n-1,l} + r_{n-1,l+1,n-1} + r_{n,l+1,n-1} + e \\ & +e + 2c_{n-1,l} + r_{n-1,l+1,n-1}.\end{aligned}$$

Thus, by induction, if we project the above expression by $\varphi_{n-1,N}$, the last occurrence of $c_{N,l}$ in the first $2c_{n-1,l}$ and the first occurrence of $c_{N,l}$ in the last occurrence of $2c_{n-1,l}$ can be bridged as:

$$c_{N,l} + r_{N,l+1,N} + r_{N+1,l+t,N} + \cdots + r_{n,l+1,N} + (n - N + 1)e + c_{N,l}.$$

Therefore, if we write $s = r_{N,l+1,N} + r_{N+1,l+t,N} + \cdots + r_{n,l+1,N} + (n - N + 1)e$, then $l(s) = n - N + 1 + \sum_{k=N}^{n} l(r_{k,l+1,N})$. This also shows that it is the largest gap in $2c_{n+1,l}$. We have seen that the gap between occurrences of $c_{N,l}$ appears as the largest gap in $2c_{m,l}$ with $N < m \leq n + 1$. Besides these gaps, of course there exist occurrences of the form $2c_{N,l}$. It remains to demonstrate the last statement. As in the above calculation, it follows that $\varphi_n(c_{n+1,l}) = e + c_{n,l} + c_{n,l} + r_{n,l+1,n}$. Consequently, we have

$$\begin{aligned}\varphi_{n+1,n-1}(c_{n+1,l}) \quad &= \quad \cdots + \varphi_{n-1}(c_{n,l}) + r_{n,l+1,n-1} \\ &= \quad \cdots + e + 2c_{n-1,l} + r_{n-1,l+1,n-1} + r_{n,l+1,n-1}.\end{aligned}$$

In this way, we get $\varphi_{n+1,N}(c_{n+1,l}) = \cdots + c_{N,l} + r_{N,l+1,N} + r_{N+1,l+1,N} + \cdots + r_{n,l+1,N}$. Thus, we have the desired conclusion. $\square$

From the proof of the above lemma, we get the following:

Lemma 3.19. *In $\varphi_{n,N}(c_{n,l}+c_{n,l})$, let A be the last occurrence of $c_{N,l}$ in the first $c_{n,l}$, and B the first occurrence in the last $c_{n,l}$. If we write $\varphi_{n,N}(c_{n,l}+c_{n,l}) = \cdots + A + s + B + \cdots$, then*

$$s = r_{N,l+1,N} + r_{N+1,l+t,N} + \cdots + r_{n-1,l+1,N} + (n-N)e.$$

In particular, $l(s) = n - N + \sum_{k=N}^{n-1} l(r_{k,l+1,N})$.

Notation 3.20. We denote $g_{n,l,N} = n - N + \sum_{k=N}^{n-1} l(r_{k,l+1,N})$. Then, $g_{n,l,N}$ is the largest gap between occurrences of $c_{N,l}$ in $2c_{n,l}$.

From this point, for $x \neq y$ with $x, y \in X \backslash \{p\}$, we present successive lemmas to check that $\limsup_{k\to+\infty} d(f^k(x), f^k(y)) > 0$.

Lemma 3.21. *Let $x \neq y \in X$ be such that $\deg x = \deg y < +\infty$. Then, $\limsup_{k\to+\infty} d(f^k(x), f^k(y)) > 0$.*

Proof. Let $x \neq y \in X$ be such that $\deg x = \deg y < +\infty$. Let $\deg x = \deg y = l$, $x = (u_0, u_1, u_2, \dots)$, and $y = (v_0, v_1, v_2, \dots)$. By Lemma 3.10, there exists an $N \in \mathbb{N}$ such that $\deg u_n = \deg v_n$ for all $n \geq N$. We can take N such that $N \gg l$, and a $\mathrm{gap}(u_n, v_n)$ is defined for every $n \geq N$. Suppose that $\limsup_{n\to+\infty} |\mathrm{gap}(u_n, v_n)| < +\infty$. Then, by Lemma 3.15 and Lemma 3.16, it follows that $\limsup_{k\to+\infty} d(f^k(x), f^k(y)) > 0$. Therefore, we assume that $\limsup_{n\to+\infty} |\mathrm{gap}(u_n, v_n)| = +\infty$. Broadly, we shall show that one of the two orbits enters a domain of degree larger than l, and, after a long time, another orbit still takes the vertices of degree l. Let $n \gg N$. By the definition of N, it follows that $\varphi_{n,N}(u_n) \in V(c_{N,l})$ and $\varphi_{n,N}(v_n) \in V(c_{N,l})$. By Lemma 3.18, both $\mathrm{remn}(u_n) \to +\infty$ and $\mathrm{remn}(v_n) \to +\infty$ hold. Let $K(n) = \min\{\mathrm{remn}(u_n), \mathrm{remn}(v_n)\}$. For $0 \leq i \leq K(n)$, both $\varphi_{\infty,n}(f^i(x))$ and $\varphi_{\infty,n}(f^i(y))$ follow the path on $c_{n,l}$ until one of them reaches the end. Without loss of generality, we assume that $\mathrm{gap}(u_n, v_n) > 0$ for infinitely many n, and we assume that we can take an arbitrarily large n with arbitrarily large $\mathrm{gap}(u_n, v_n) > 0$. Thus, $\varphi_{\infty,n}(f^i(y))$ follows the last $c_{N,l}$ first. To catch the timing of this last $c_{N,l}$, we take an $L(n) > 0$ such that $\deg(\varphi_{\infty,n}(f^{L(n)-1}(y))) = l$ and, for $L(n) \leq i \leq K(n)$, $\deg(\varphi_{\infty,n}(f^i(y))) \geq l+1$. Let $A(n) := \sum_{k=N}^{n-1} l(r_{k,l+1,N}) = K(n) - L(n)$. By Lemma 3.18, the gap between occurrences of $c_{N,l}$ is at most $B(n) := n - N - 1 + \sum_{k=N}^{n-2} l(r_{k,l+1,N})$. Therefore, $\varphi_{\infty,N}(f^i(x))$ follow $c_{N,l}$ for at least one i with $L(n) \leq i \leq K(n)$. We have to show that we can take i with $L(n) \leq i \leq K(n)$ such that $\deg \varphi_{\infty,N}(f^i(x)) = l$ is arbitrarily large. We have

$$\begin{aligned} A(n) - B(n) &= \textstyle\sum_{k=N}^{n-1} l(r_{k,l+1,N}) - \Big(n - N - 1 + \sum_{k=N}^{n-2} l(r_{k,l+1,N})\Big) \\ &= l(r_{n-1,l+1,N}) - n + N + 1 \\ &\to +\infty \text{ as } n \to +\infty. \end{aligned}$$

Let $L(n) \leq i_n \leq K(n)$ be the largest i with $L(n) \leq i \leq K(n)$ and $\deg \varphi_{\infty,N}(f^i(x)) = l$. Then, it follows that $i_n + B(n) > K(n) > A(n)$. Thus, $i_n \geq A(n) - B(n)$ is unbounded as $n \to +\infty$. Therefore, $\deg \varphi_{\infty,N}(f^i(x)) \neq \deg \varphi_{\infty,N}(f^i(y))$ for infinitely large $i > 0$. This concludes the proof. $\square$

By Notation 3.9, for $x \neq p$, we have $\deg x < +\infty$.

Lemma 3.22. *Let $x \neq y \in X$ be distinct from p and $\deg x + 2 \leq \deg y < +\infty$. Then, it follows that*

$$\limsup_{k\to+\infty} d(f^k(x), f^k(y)) > 0.$$

Proof. Let $x = (u_0, u_1, u_2, \dots)$ and $y = (v_0, v_1, v_2, \dots)$. Let $\deg x = l$ and $\deg y = l'$. Then, it follows that $l+2 \leq l'$. As before, fix a large $N > 0$ such that $\deg u_n = l$ and $\deg v_n = l'$ for all $n \geq N$. First, we show that the sequence $\varphi_{\infty,N}(f^i(x))$ with $i \geq 0$ treads $c_{N,l+1}$ infinitely many times. For each $n > N$, we get

$$\varphi_{n-1}(c_{n,l}) = e + 2c_{n-1,l} + 2c_{n-1,l+1} + r_{n-1,l+2,n-1}.$$

In the above expression, for large enough n, u_{n-1} lies in $2c_{n-1,l}$. Therefore, $\varphi_{\infty,N}(f^i(x))$ with $i \geq 0$ passes $2c_{n-1,l+1}$; it follows that it passes $c_{N,l+1}$ at least 2^{n-N} times. Because n is arbitrarily large, it passes $c_{N,l+1}$ infinitely many times. Next, note that $\varphi_{\infty,N}(f^i(y))$ with $i \geq 0$ passes only e or $c_{N,m}$ with $l + 1 < l' \leq m \leq N$. The conclusion is obvious. □

As in Notation 3.20, in $\varphi_{m,N}(2c_{m,l})$, the largest gap between occurrences of $c_{N,l}$ is calculated. In the following lemmas, we also consider the pattern of *the occurrence of gaps*.

Lemma 3.23. *Let $n \gg N > l$. In $\varphi_{n,N}(c_{n,l})$, whenever there exist two occurrences of a gap in $c_{N,l}$ with length $g_{m,l,N}$, there exists a gap in $c_{N,l}$ of length $g_{m',l,N}$ between them, where $m' > m$.*

Proof. We first make the following calculation:

$$\begin{aligned}
\varphi_{n,n-2}(c_{n,l}) &= \varphi_{n-2}(e + 2c_{n-1,l} + r_{n-1,l+1,n+1}) \\
&= e + 2\varphi_{n-2}(c_{n-1,l}) + \varphi_{n-2}(r_{n-1,l+1,n-1}) \\
&= e + 2(e + 2c_{n-2,l} + r_{n-2,l+1,n-2}) + r_{n-1,l+1,n-2} \\
&= e + e + 2c_{n-2,l} + r_{n-2,l+1,n-2} + e + 2c_{n-2,l} + r_{n-2,l+1,n-2} \\
&\quad + r_{n-1,l+1,n-2}.
\end{aligned}$$

Let us project the above expression by $\varphi_{n-2,N}$. Then, we find the occurrence of gaps as follows:

$$\cdots(\text{gap of } g_{n-2,l,N})\cdots(\text{gap of } g_{n-1,l,N})\cdots(\text{gap of } g_{n-2,l,N})\cdots.$$

By an easy induction, we obtain the conclusion. □

Lemma 3.24. *Let $n \gg N \gg l$. In $\varphi_{n,N}(c_{n,l})$, even after all occurrences of $c_{N,l}$, there exists an occurrence of $c_{N,l+1}$. We write $\varphi_{n,N}(c_{n,l}) = \cdots + c_{N,l} + s$ when there is no occurrence of $c_{N,l}$ in s. Then, in s, there exist two gaps in $c_{N,l+1}$ of length $g_{n-1,l+1,N}$ such that all gaps in $c_{N,l+1}$ between them have lengths of less than $g_{n-1,l+1,N}$. Furthermore, if we take n to be sufficiently large, after the last occurrence of $c_{N,l}$, there exists an arbitrarily large interval before the last occurrence of two such gaps of length $g_{n-1,l+1,N}$.*

Proof. We can calculate that:

$$\begin{aligned}
\varphi_{n,n-2}(c_{n,l}) &= \varphi_{n-2}(e + 2c_{n-1,l} + 2c_{n-1,l+1} + r_{n-1,l+2,n+1}) \\
&= \cdots + \varphi_{n-2}(c_{n-1,l}) + 2\varphi_{n-2}(c_{n-1,l+1}) + \varphi_{n-2}(r_{n-1,l+2,n-1}) \\
&= \cdots + c_{n-2,l} + 2c_{n-2,l+1} + r_{n-2,l+2,n-2} \\
&\quad + 2(e + 2c_{n-2,l+1} + r_{n-2,l+2,n-2}) + r_{n-1,l+2,n-2} \\
&= \cdots + c_{n-2,l} + 2c_{n-2,l+1} + r_{n-2,l+2,n-2} \\
&\quad + e + 2c_{n-2,l+1} + r_{n-2,l+2,n-2} \\
&\quad + e + 2c_{n-2,l+1} + r_{n-2,l+2,n-2} + r_{n-1,l+2,n-2} \\
&= \cdots + c_{n-2,l} + c_{n-2,l+1} \\
&\quad + c_{n-2,l+1} + r_{n-2,l+2,n-2} + e + c_{n-2,l+1} \quad \cdots\cdots \quad (1) \\
&\quad + c_{n-2,l+1} + r_{n-2,l+2,n-2} + e + c_{n-2,l+1} \quad \cdots\cdots \quad (2) \\
&\quad + c_{n-2,l+1} + r_{n-2,l+2,n-2} + r_{n-1,l+2,n-2}.
\end{aligned}$$

We consider the projection by $\varphi_{n-2,N}$ of the above expression. Then, in lines (1) and (2), there exists a gap in $c_{N,l+1}$ of length $g_{n-1,l+1,N}$, and the lengths of the gaps in $c_{N,l+1}$ between them are at most $g_{n-2,l+1,N}$. This concludes the first part of the claim. Because $l(c_{n-2,l+1}) \to +\infty$ as $n \to +\infty$, the last claim is obvious from the above expression. □

Lemma 3.25. *Let $x \neq y \in X$ be distinct from p, and* $\deg x + 1 = \deg y < +\infty$. *Then, it follows that*

$$\limsup_{k \to +\infty} d(f^k(x), f^k(y)) > 0.$$

Proof. Let $x \neq y \in X$ be distinct from p. Let $\deg x = l$. Then, $\deg y = l + 1$. Let $x = (u_0, u_1, u_2, \dots)$ and $y = (v_0, v_1, v_2, \dots)$. As we have already shown, there exists an $N > 0$ such that $\deg u_n = l$ and $\deg v_n = l + 1$ for all $n \geq N$. The sequence $\varphi_{\infty,N}(f^i(x))$ with $i \geq 0$ passes through only e or $c_{N,m}$ with $l \leq m \leq N$, and the sequence $\varphi_{\infty,N}(f^i(y))$ with $i \geq 0$ passes through only e or $c_{N,m}$ with $l + 1 \leq m \leq N$. Therefore, if $\varphi_{\infty,N}(f^i(x))$ with $i \geq 0$ passes $c_{N,l}$ infinitely many times, then the conclusion is obvious. Therefore, we assume that $\varphi_{\infty,N}(f^i(x))$ with $i \geq 0$ treads $c_{N,l}$ only finitely many times. Note that $\varphi_{\infty,N}(f^i(x))$ with $i \geq 0$ passes $c_{N,l+1}$ infinitely many times. On the other hand, if $\varphi_{\infty,N}(f^i(y))$ with $i \geq 0$ only take values on $c_{N,l+1}$ a finite number of times, then the conclusion is again obvious. Therefore, we assume that $\varphi_{\infty,N}(f^i(y))$ with $i \geq 0$ treads on $c_{N,l+1}$ infinitely many times. Because $\varphi_{n,N}(c_{n,l})$ contains $c_{N,l}$, there is some fixed $i_0 \in \mathbb{Z}$ for which $\varphi_{\infty,N}(f^{i_0-1}(x))$ becomes the last passage on $c_{N,l} \setminus \{ v_{N,0} \}$ and $\varphi_{\infty,N}(f^{i_0}(x)) = v_{N,0}$. Therefore, shifting x and y by f^{i_0}, we assume that $\varphi_{\infty,N}(f^i(x))$ with $i \geq 0$ does not pass $c_{N,l}$. For arbitrarily large $K > 0$, taking a large $n > N$, $\varphi_{\infty,n}$ projects the orbit $f^i(x)$ with $0 \leq i \leq K$ onto a path of $c_{n,l}$. Therefore, for every gap in $c_{N,l+1}$, there exists an $n > N$ such that the gap is seen in $\varphi_{n,N}(c_{n,l})$. By Lemma 3.24, as $\varphi_{\infty,N}(f^i(x))$ $(i = 0, 1, \dots)$ proceeds, there exist a couple of gaps in $c_{N,l+1}$ of length $g_{n-1,l+1,N}$, between which no larger gaps occur. Furthermore, this occurs for arbitrarily large $i > 0$ if n is large enough. On the other hand, for arbitrarily large $K > 0$, taking a large $n > N$, $\varphi_{\infty,n}$ projects the orbit $f^i(y)$ with $0 \leq i \leq K$ onto a path of $c_{n,l+1}$. By Lemma 3.23, if $\varphi_{\infty,N}(f^i(y))$ with $i > 0$ encounters a couple of gaps in $c_{N,l+1}$ of length $g_{n-1,l+1,N}$, a gap in $c_{N,l+1}$ of larger length must exist between them. Therefore, we obtain the desired conclusion. □

By proving the lemma above, we have shown that (X, f) is completely scrambled. The next lemma proves that (X, f) is topologically transitive.

Lemma 3.26. *There exists an x_0 such that $\{ f^i(x_0) \mid i \in \mathbb{N} \}$ is dense in X.*

Proof. Fix an arbitrary $N > 0$. In our notation, $c_{N,1} = (v_{N,1,0}, v_{N,1,1}, \dots, v_{N,1,l(N,1)} = v_{N,0})$. Let $u_N = v_{N,1,1}$. It follows that $\varphi_{N-1}(u_N) = v_{N-1,0}$. Because $\varphi_n(c_{n+1,l}) = e + 2c_{n,l} + \cdots$ for all $n > 0$, we get $\varphi_N(v_{N+1,1,2}) = v_{N,1,1} = u_N$. In this way, if u_n is defined, then u_{n+1} is defined as the first occurrence of $u \in V(c_{n+1,1})$ such that $\varphi_n(u) = u_n$. We define $x_0 := (v_{0,0}, v_{1,0}, \dots, v_{N-1,0}, u_N, u_{N+1}, \dots)$. Let $n \gg N$ be arbitrarily large. Then, $\varphi_{\infty,n+1}(f^i(x_0))$ with $i > 0$ follows a path $(u_{n+1}, \dots, v_{n+1,0})$ in $c_{n+1,1}$. Because $\varphi_n(c_{n+1,1})$ winds around $c_{n,1}$ twice, $\varphi_{\infty,n+1}(f^i(x_0))$ with $i > 0$ passes all vertices of $c_{n,1}$. It is obvious that $\varphi_{n-1}(c_{n,1})$ passes all vertices of G_{n-1}. Because n is arbitrary, the conclusion is obvious. □

The following lemma shows that (X, f) is not locally equicontinuous, and every $x \neq p$ is not a point of local equicontinuity.

Lemma 3.27. *Let $x \in X$ with $x \neq p$. Then, for every sufficiently large $n > N > 0$, we have some $v \in V(G_n)$ and $i_n \in \mathbb{Z}$ such that, for $y_n = f^{i_n}(x)$, it follows that $x, y_n \in U(v)$ and there exists an $i > 0$ with $\varphi_{\infty,N}(f^i(x)) \neq \varphi_{\infty,N}(f^i(y_n))$. Consequently, $x \neq p$ is not a point of local equicontinuity.*

Proof. Let $x \neq p$. Let $\deg x = l < +\infty$, and write $x = (u_0, u_1, u_2, \dots)$. As before, there exists an $N > 0$ such that $\deg u_n = l$ for all $n \geq N$. Let $n \gg N \gg l$. It is sufficient to show that there exists a $y \in U(u_n)$ on the orbit of x such that $\varphi_{n,N}(f^i(x)) \neq \varphi_{n,N}(f^i(y))$ for infinitely many $i > 0$. Because $c_{n+1,l}$ winds around $c_{n,l}$ twice, there exists a v_{n+1} with $u_{n+1} \neq v_{n+1} \in V(c_{n+1,l})$ such that $\varphi_n(v_{n+1}) = u_n$. Then, we can construct $y = (v_0, v_1, v_2, \dots)$ with $v_i = u_i$ $(0 \leq i \leq n)$ such that $\deg y = l$ and $\mathrm{gap}(u_i, v_i)$ is equal to some constant $i_n \neq 0$ for all $i \geq n$. Therefore, we have constructed a $y_n \in U(u_n)$ with $f^{i_n}(x) = y_n$. We must consider two cases:

Case 1. Suppose that both $\varphi_{\infty,N}(f^i(x))$ and $\varphi_{\infty,N}(f^i(y))$ with $i > 0$ trace $c_{N,l}$ only finitely many times. Then, after tracing all $c_{N,l}$'s, they only trace $c_{N,l+k}$ with $k > 0$ and trace $c_{N,l+1}$ infinitely many times. By Lemma 3.24, these occurrences are not periodic. Therefore, we have the desired conclusion.

Case 2. Suppose that both $\varphi_{\infty,N}(f^i(x))$ and $\varphi_{\infty,N}(f^i(y))$ with $i > 0$ trace $c_{N,l}$ infinitely many times. By Lemma 3.23, these occurrences are not periodic. Therefore, we also obtain the conclusion.

This completes the proof. □

We have finished the proof of Theorem 3.1. We suggest that, in defining the sequence of graph covers, the expression $\varphi_n(c_{n+1,i}) = e_{n,0} + n_{n,i}c_{n,i} + n_{n,i+1}c_{n,i+1} + \cdots + n_{n,n}c_{n,n} + e_{n,0}$, with $n_{n,i} \geq 2$ for all $n \in \mathbb{N}$ and $1 \leq i \leq n$ can be used. The above proofs may also be applicable in this case.

After the author submitted the first version of this manuscript, he was notified by the referee(s) that Foryś, Huang, Li, and Oprocha [3] have presented two methods for the construction of completely scrambled systems that are weakly mixing, proximal, and uniformly rigid. According to a personal communication with Oprocha, their systems are completely scrambled systems on compact continua. Therefore, their examples are different from that presented in this paper. This means, unexpectedly for us, that the completely scrambled compacta have a variety of systems. In any case, it seems that this opens a new topic in this area. Finally, because all completely scrambled zero-dimensional homeomorphisms are essentially simple, they have Bratteli–Vershik representations (see Herman, Putnam, and Skau [6]). Our construction by graph covers is easily translated to the method of Bratteli diagrams–we refer readers to [11, Section 7.1], which describes a simple link. A more satisfactory link is now being prepared.

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